Applications of Image Space Reconstruction Algorithms to Ionospheric Tomography

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Introduction

- We have been applying Image Space Reconstruction Algorithms (ISRAs) to the solution of large-scale ionospheric tomography problems

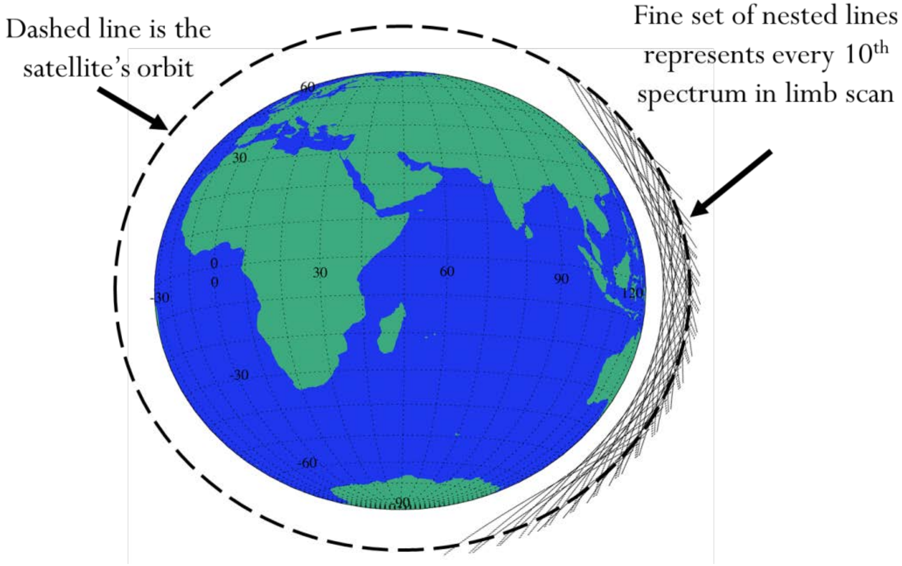
- Desirable features of ISRAs
  - *Positive definite ➔ more physical solutions*
  - *ISRAs are amenable to spare-matrix formulations*
  - *Fast, stable, and robust*
  - *Easy to add between iteration physicality constraints*

- We present the results of our studies of two types of ISRA
  - *Least-Squares Positive Definite (LSPD): iterative non-negative least-squares generalization*
  - *Richardson-Lucy: applicable to measurements that follow Poisson statistics*

- We compare their performance to the Multiplicative Algebraic Reconstruction (MART) and the Conjugate Gradient Least Squares algorithms

Overview

- What are we trying to do?
  - *Specific application: improve on-orbit specification of the ionosphere or thermosphere*
  - *Approach: Use aggregates of limb scan information to infer the 2-D (or 3-D) distribution of O^+ ions in the ionosphere*
- Brightness measurements are linear in the volume emission rate
  - *Analogous to Computerized Ionospheric Tomography ➔ linear in the electron density*
  - *Noise on brightness measurements obeys Poisson statistics – not the Normal Distribution*

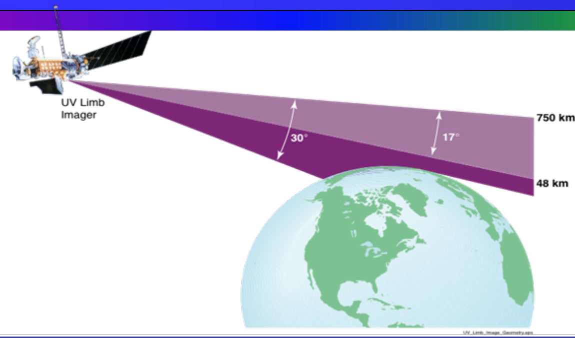


$$4\pi I = 10^{-6}\int_0^\infty \varepsilon\left(s,z,\lambda,\phi\right)\, ds\left(z,\lambda,\phi\right)$$

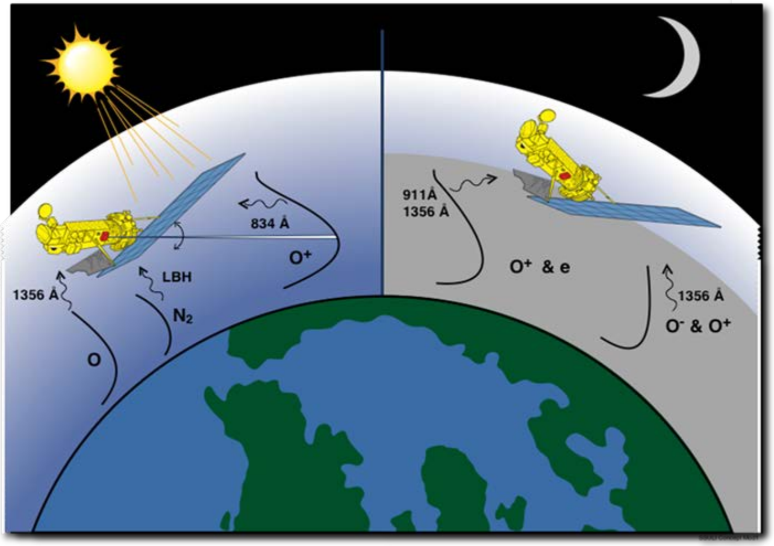
Volume emission rate, ε:

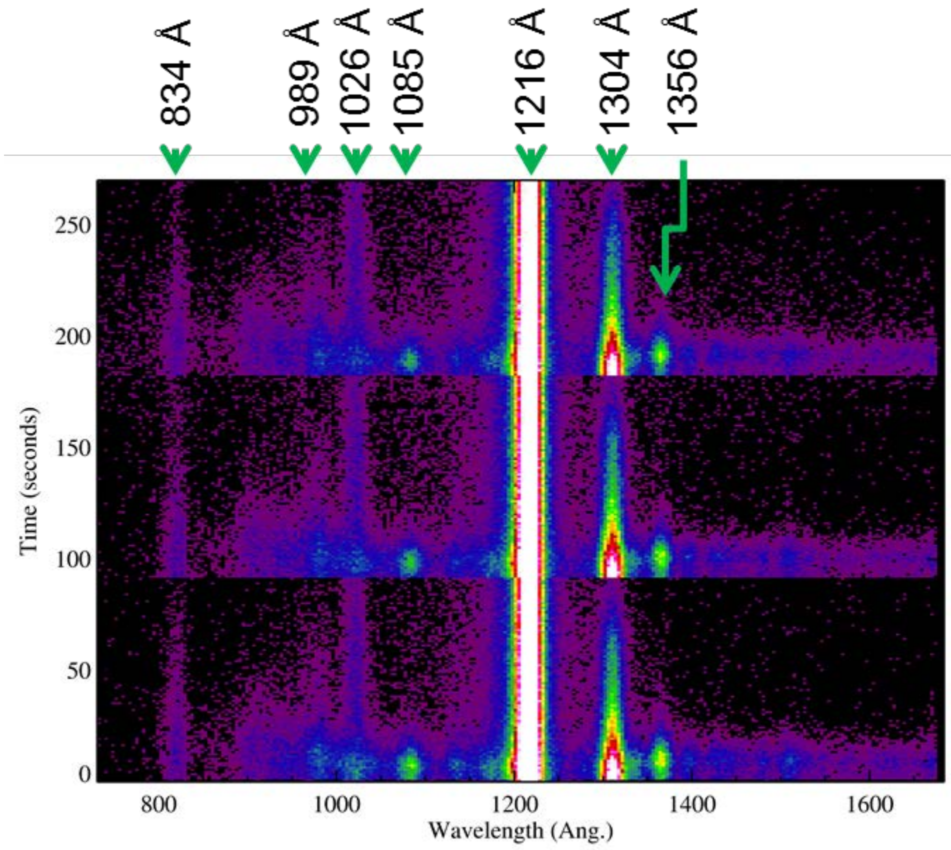
$$\varepsilon\left(z,\lambda,\phi\right) = \alpha n_e\left(z,\lambda,\phi\right) n_{O^+}\left(z,\lambda,\phi\right)$$

SSULI Measurement Scenario



SSULI Processing Algorithm and Data Products		
Solar Zenith Angle	**Algorithm**	**EDR Data Products**
SZA < 85°	Dayside Ionosphere Algorithm (Requires 834 SDRs)	O+ Density Profile. hmF2, nmF2
SZA >= 108°	Nightside Ionosphere Algorithm (Required 1356 and 911 SDRs)	O+, O Density Profiles, hmF2, nmF2
SZA < 85°	Dayside Neutral Density Algorithm (Requires 1356 and LBH SDRs)	O, N_2, O_2 Density Profiles. Temperature Profile.





3 Daytime Limb Scans

Ionospheric Tomography & Current Algorithms

- Line-of-sight integrals are replaced by summations assuming constant volume emission rate in a voxel
- The result is a large sparse linear system of equations
- To solve this in the Least-Squares sense, we minimize the Chi-squared statistic
- This system is solved by
  - *Multiplicative Algebraic Reconstruction Technique (MART)*
  - *Conjugate Gradient Methods (for example Conjugate Gradient Least Squares – CGLS)*
  - *And others…*

$$4\pi I = 10^{-6} \sum_i \varepsilon(z,\lambda,\phi)\, \Delta s_i(z,\lambda,\phi)$$

$$Ax = b$$

$$\chi^2 = (Ax-b)^T \Sigma_D^{-1} (Ax-b)$$

$$\left(A^T \Sigma_D^{-1} A\right) x = A^T \Sigma_D^{-1} b$$

$$\Sigma_D^{-1} = \begin{pmatrix} 1/\sigma_i^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n^2 \end{pmatrix} =$$

inverse data covariance matrix

The Problem

- How can we produce accurate, physical solutions in the presence of measurement noise?
  - *Want to weight solutions using signal-to-noise ratio using Weighted Least Squares approach*
  - *Solutions must be physical and ideally smooth*
    - Noise introduces high frequency components to the solution ➔ often results in non-physical negative density or volume emission rate values and undesirable solution roughness
    - Smoothness: Current regularization schemes are *ad hoc* – can we introduce a physicality constraint?
  - *Account for the type of measurement statistics*
    - Current methods can approximate Poisson solutions: Is there an exact method?

- Our solution: Image Space Reconstruction Algorithms
  - *Richardson-Lucy (RL): non-negative, naturally handles Poisson statistics*
  - *Least-Squares, Positive-Definite (LSPD): non-negative, naturally handles Gaussian statistics*

CGLS Inversion, Noise-free
-Non-physicality-

- Right: IRI-2007 input ionosphere
- Center: LSPD reconstruction, showing
  - *Reconstruction is imperfect due to limited instrument sampling*
  - *But is non-negative*
- Right: CGLS reconstruction
  - *Parts of image show negative, non-physical values*

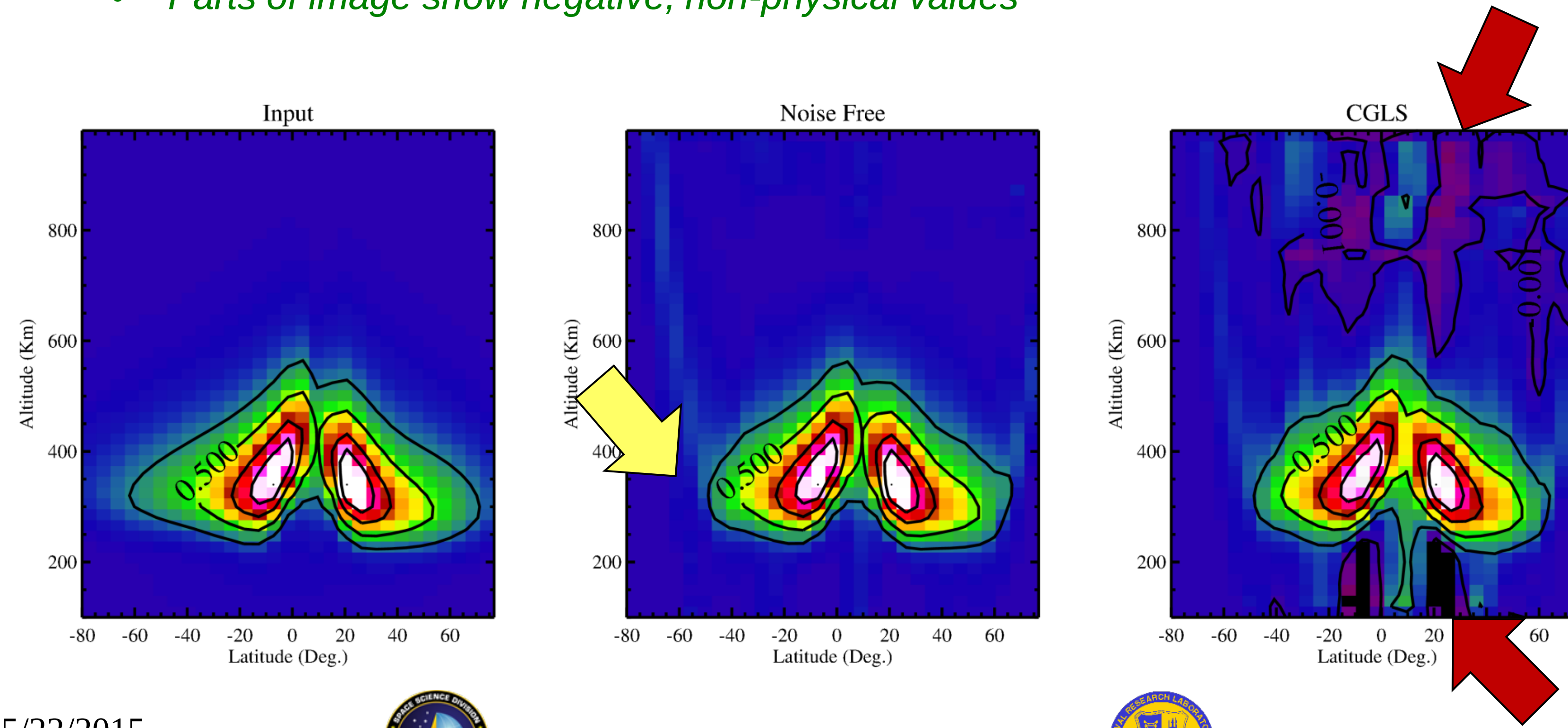


Image Space Reconstruction Algorithms

Least-squares Positive Definite

$$\chi^2 = (Ax - b)^T \Sigma_D^{-1} (Ax - b)$$

$$\left(A^T \Sigma_D^{-1} A\right) x = A^T \Sigma_D^{-1} b$$

Ensure Karush-Tucker-Kuhn conditions are met:

$$x \otimes \left(A^T \Sigma_D^{-1} A\right) x = x \otimes A^T \Sigma_D^{-1} b$$

$$x_{j+1} = x_j \otimes \frac{A^T \Sigma_D^{-1} b}{\left(A^T \Sigma_D^{-1} A x_j\right)}$$

Richardson-Lucy

$$J = 1^T \left(Ax - b \otimes \log Ax\right)$$

$$\nabla J = A^T \left(1 - \frac{b}{Ax}\right) = 0$$

Ensure Karush-Tucker-Kuhn conditions are met:

$$x \otimes A^T \left(\bar{1}\right) = x \otimes A^T \left(\frac{b}{Ax}\right)$$

$$x_{j+1} = x_j \otimes \frac{A^T}{A^T \left(\bar{1}\right)} \left(\frac{b}{Ax_j}\right)$$

What About Measurements With Poisson Noise?

- CGLS, MART, and LSPD approaches work well for random variables that follow Normal/Gaussian distributions
  - *But when used on Poisson distributed data can result in biases*
  - *For following comparisons, we use adjusted error bars for those approaches*

- Mighell suggested modifications to Gaussian-based approaches that will work for Poisson distributed data
  - *Adjust the count rates for non-zero values upward by one count:*

$$b_a = b + 1 \quad where \quad b > 1$$

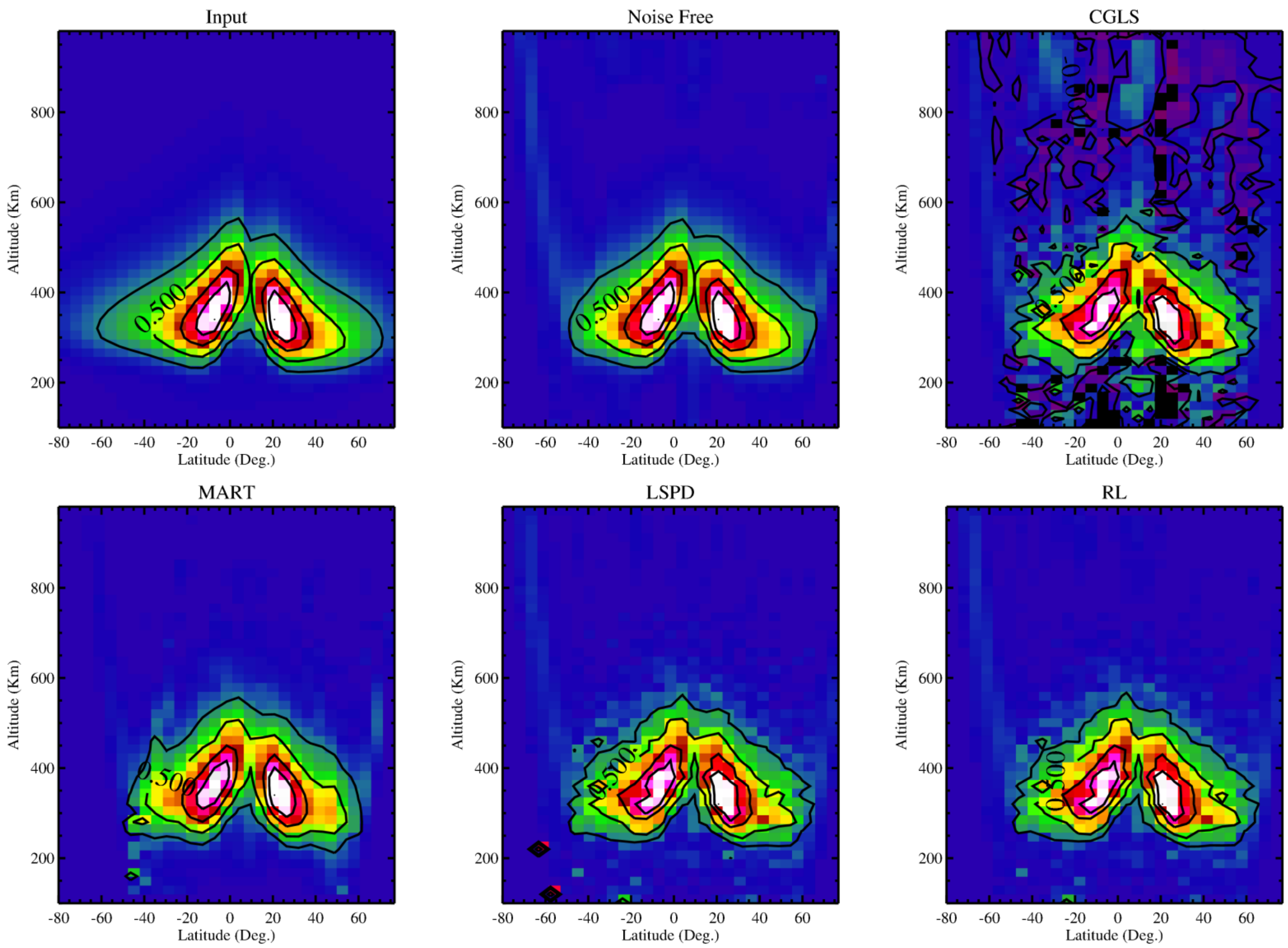
  - *Force the data to be greater than one and take the square-root to get the uncertainties:*

$$\sigma = \sqrt{b_a + 1}$$

Test Problems

- Used IRI-2007 to generate the test ionosphere
  - *Nighttime case at solar maximum*
- Simulated SSULI measurements using:
  - *Realistic instrument viewing information*
  - *Varying sensitivity → varied signal-to-noise ratio of "data"*
- Realistic photon shot noise was added based on the instrument sensitivity
  - *Sensitivities: 1000, 100, 10, 1, 0.1, 0.01 ct/s/Rayleigh*
  - *SSULI sensitivity ~0.1 ct/s/Rayleigh*
- Studied the accuracy of the retrievals
  - *No Physicality Constraint applied*
  - *Adjusted/optimized the diffusion weight*
- Non-regularized CGLS solutions used as a "control"

Reconstructions with Noise
-Non-Constrained, S = 1ct/s/R-



Regularization

- Most common regularization scheme is Tikhonov, standard approach of introducing a penalty term to enforce smoothness

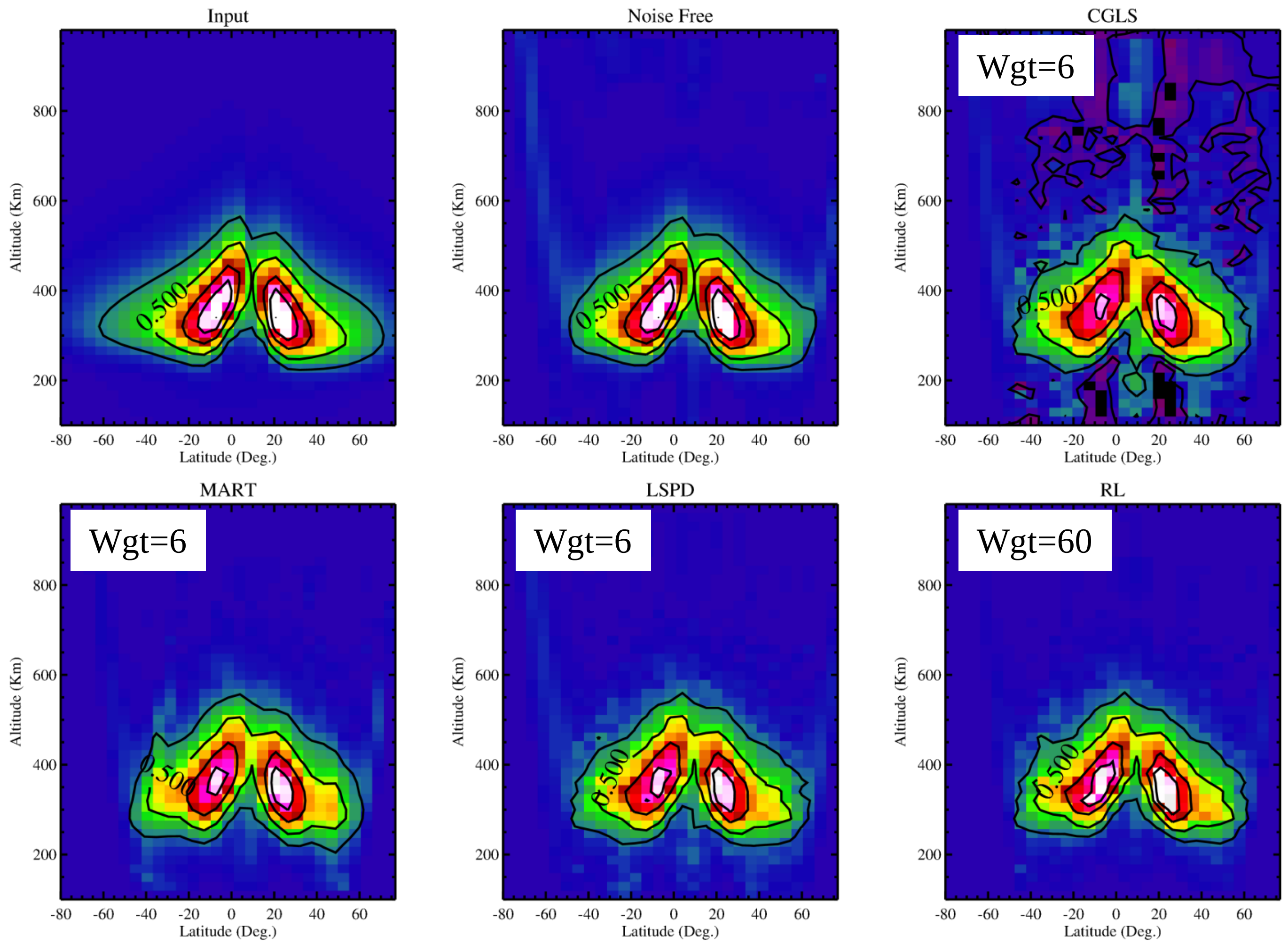
$$\left(A^T \Sigma_D^{-1} A + \lambda L\right) x = A^T \Sigma_D^{-1} b$$

  - *Where L is a regularization operator*
    - L = I ; the identity matrix ➔ ad hoc, provides simplest solution, but drives image to prior
    - L = variety of derivative operators ; smooth solution ➔ ad hoc, lower bias than using identity operator
    - L = Σ_x^{-1} ;the inverse model covariance matrix ➔ based on prior information, could bias solution to prior knowledge
  - *NO accepted best approach to estimate the optimal weighting value, λ*
    - Approaches: Truncate iterations, TSVD, GCV, L-curve, Draftsmen's license (chi-by-eye)…

- We opted for between iteration application of a physicality constraint
  - *This approach equally weights solution physicality and accuracy of the fit to the data*

CGLS Inversion with Noise
-Tikhonov Regularization, Identity Operator-

- S = 1 ct/s/R, Tikhonov regularization
  - *Weight estimates using "Draftsmen License"*
  - *Arc densities are too low*
  - *Arc asymmetry is not correct*
- Weight for RL is 10 times what is needed for weighted least squares approach



Physicality Constraint

- Regularization to a differential equation is an approach used in the computer graphics modeling community
  - *Improves computer rendering by generating a smooth surface from facet information*
- We use the time independent diffusion equation

$$\frac{\partial n}{\partial t} = \nabla \cdot \left(\overline{\overline{D}} \nabla n \right) \Rightarrow 0 = \nabla^2 n \quad (time \;\; independent)$$

- Currently, we assume uniform, isotropic transport
  - *Permits the algorithms to produce reasonable results during daytime and at night*
    - Will work for either ionospheric emissions (nighttime ionosphere) or for emission generated by neutral species (O and N_2 in the dayglow)
  - *However, some emissions, for example O I 1356 Å, have both ionospheric and thermospheric components during the daytime*
    - Drives eventual need for non-isotropic, non-uniform diffusion approximation
- Implemented using the Successive Over-Relaxation approximation
  - *Makes small steps to "relax" solution to the diffusion approximation*

Successive Over-Relaxation (SOR)

- We chose this iterative approach to solve the diffusion equation
  - *Desired a method with low computational overhead*
  - *Wanted a means to guide the algorithms to a physically meaningful solution*
- Approximating the diffusion equation at time step k+1 by finite difference equations (assuming $\Delta x = \Delta y$, i & j are cell indices):

$$n_{i,j}^{k+1} = n_{i,j}^{k} - \frac{D\Delta t}{(\Delta x)^2}\left(n_{i-1,j}^{k} + n_{i+1,j}^{k} + n_{i,j-1}^{k} + n_{i,j+1}^{k} - 4n_{i,j}^{k}\right)$$

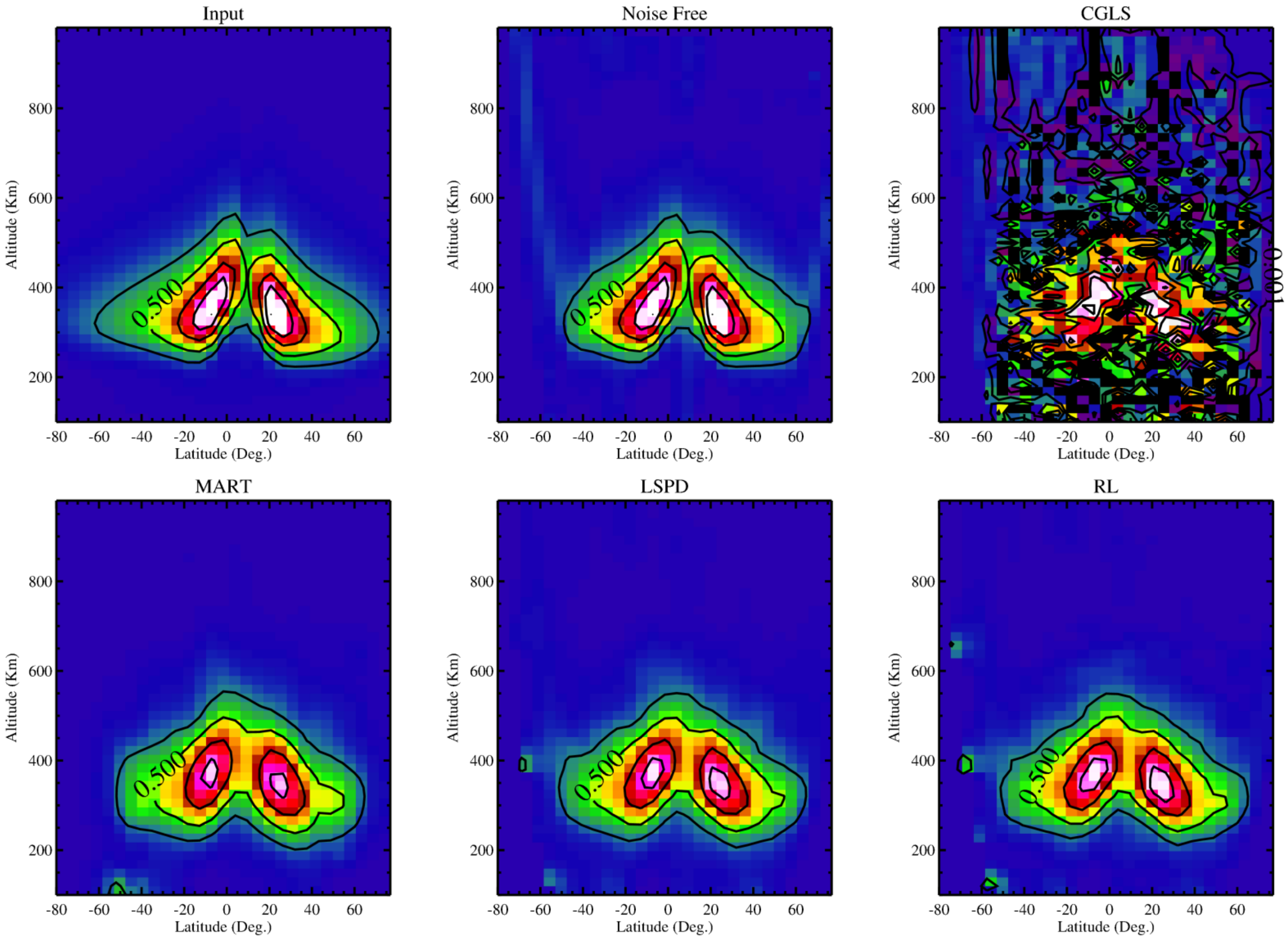
- To ensure a stable solution, the maximum time step size allowed is limited by the diffusion time across the cell:

$$W \equiv \frac{D\Delta t}{(\Delta x)^2} \leq \frac{1}{4}$$

  - *We refer to W as the diffusion weight and use it to tune the weighting of the physicality constraint*

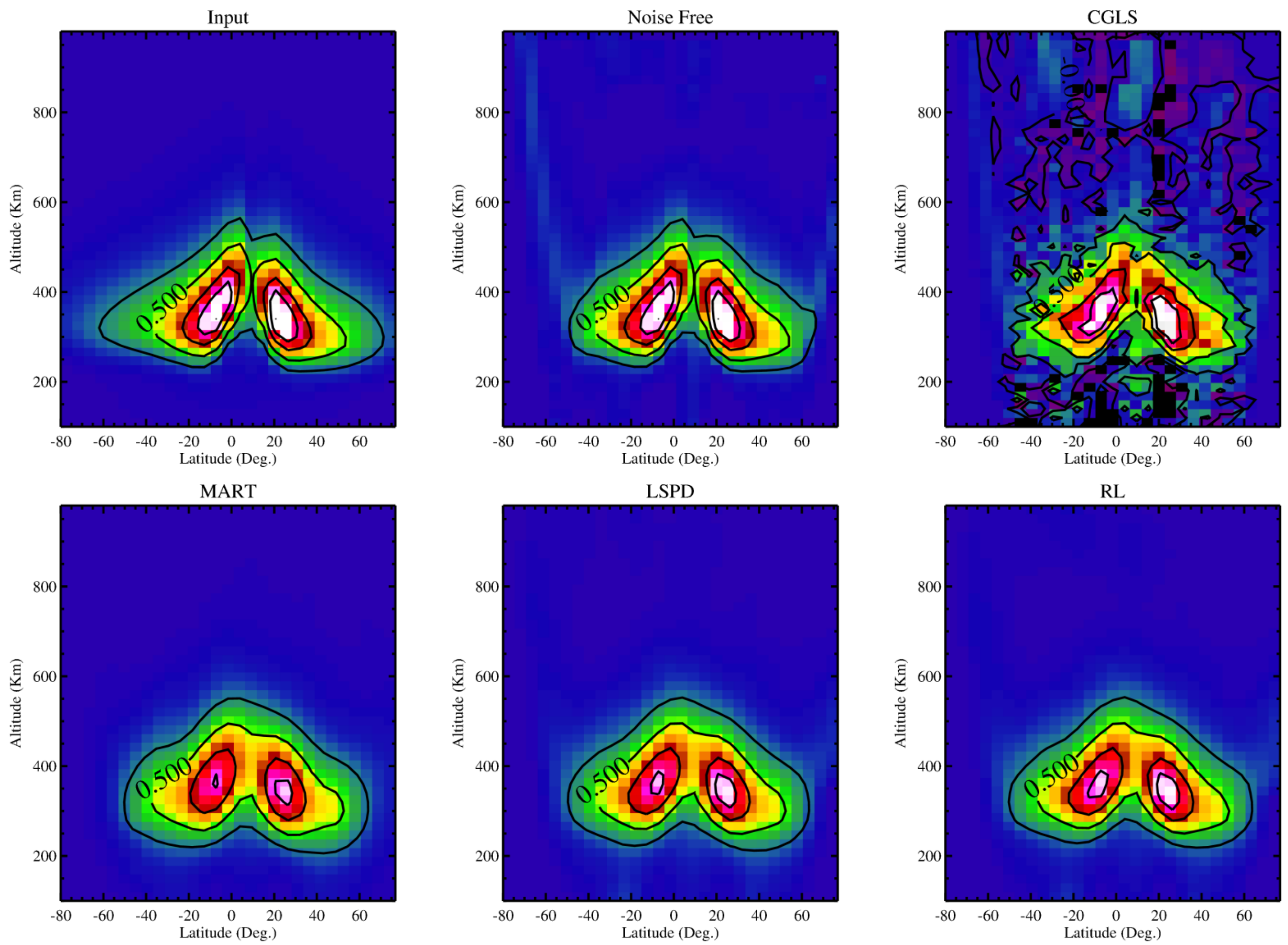
Reconstructions with Noise
-Physicality Constrained-

- S = 0.01 ct/s/R, W=1/4
  - *Solution is too smooth*
  - *Arc densities are too low*
  - *Arc asymmetry is not correct*
- Able to reconstruct incredibly noisy data



Reconstructions with Noise -Physicality Constrained-

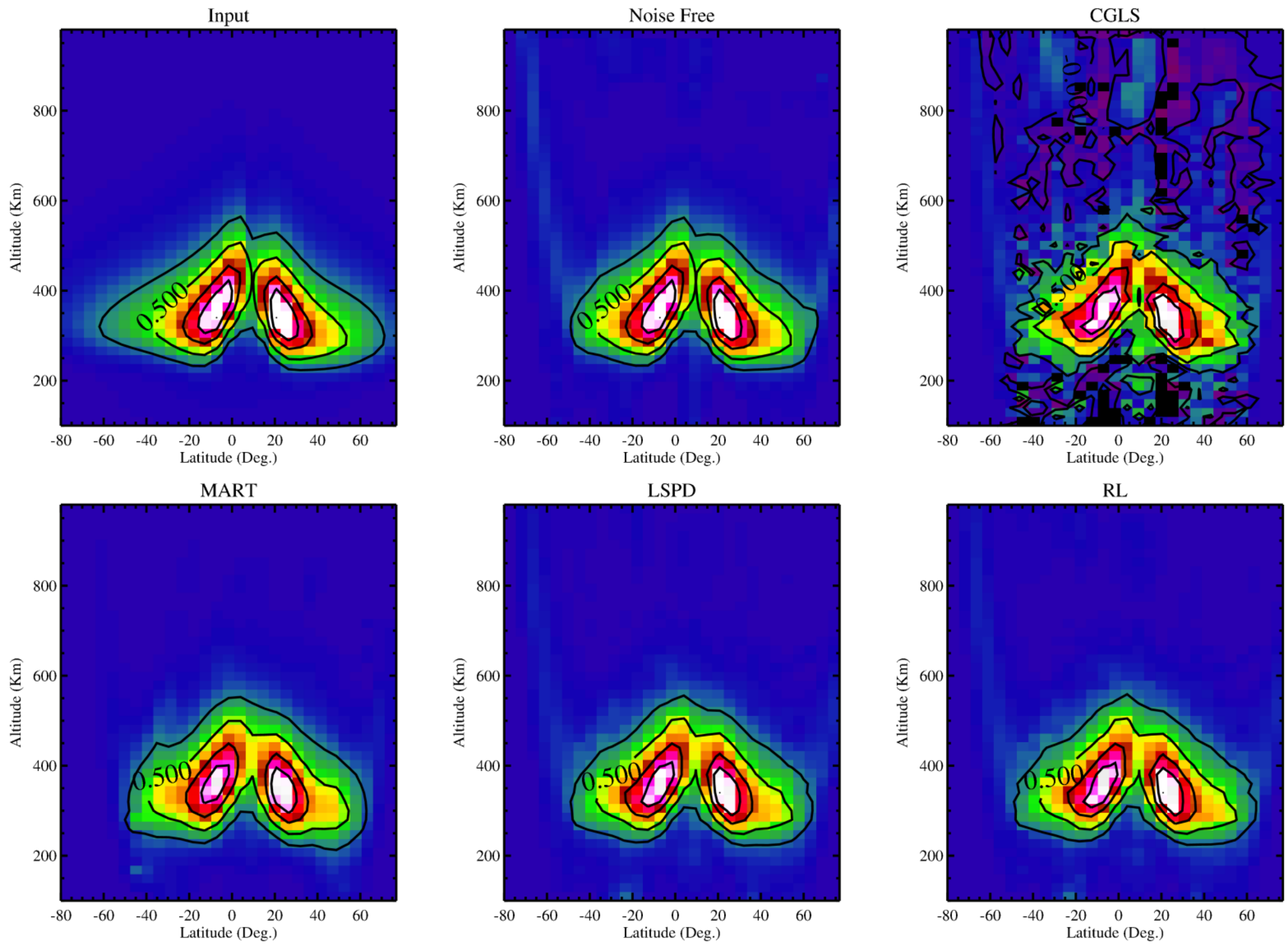
- S = 1 ct/s/R, W=1/4
  - *Solution is too smooth*
  - *Arc densities are too low*
  - *Arc asymmetry is not correct*
- Need to reduce diffusion weight, W



Reconstructions with Noise
-Physicality Constrained-

- S = 1 ct/s/R, estimated best diffusion weight
  - *Solution is smooth, but not too smooth*
  - *Arc densities are in good agreement*
  - *Arc asymmetry is more correct*
- Best diffusion weight estimated from Signal-to-Noise Ratio of measurements:

$$W \approx \sqrt{2}\; mean(SNR)$$



Speed Comparison

- Test problem had:
  - *1820 lines of sight*
  - *1305 density cells*
- Measured execution speed versus accuracy of convergence, ε:
  - *All algorithms use same stopping criteria*
  - *Fractional change in the volume emission rate and the chi-squared of the fit to the data both change by < ε between steps*
- During each set of tests, data mean signal-to-noise ratio fixed at:
  - *Top: 2.7*
  - *Bottom: 283*

Low SNR = 2.7

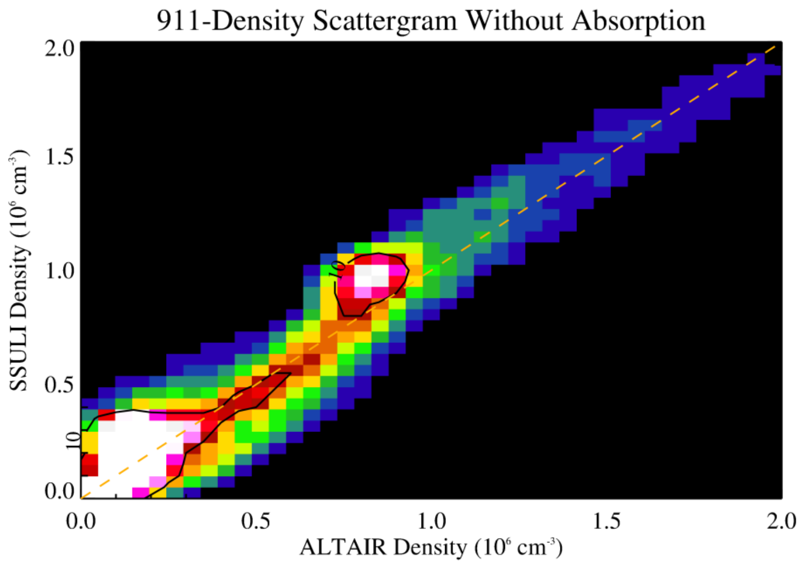
ε	CGLS	MART	LSPD	RL
10^{-2}	0.14	2.1	0.14	0.15
10^{-3}	0.20	4.8	0.17	0.18
10^{-4}	0.60	8.9	0.23	0.24
10^{-5}	4.9	16.3	0.34	0.35

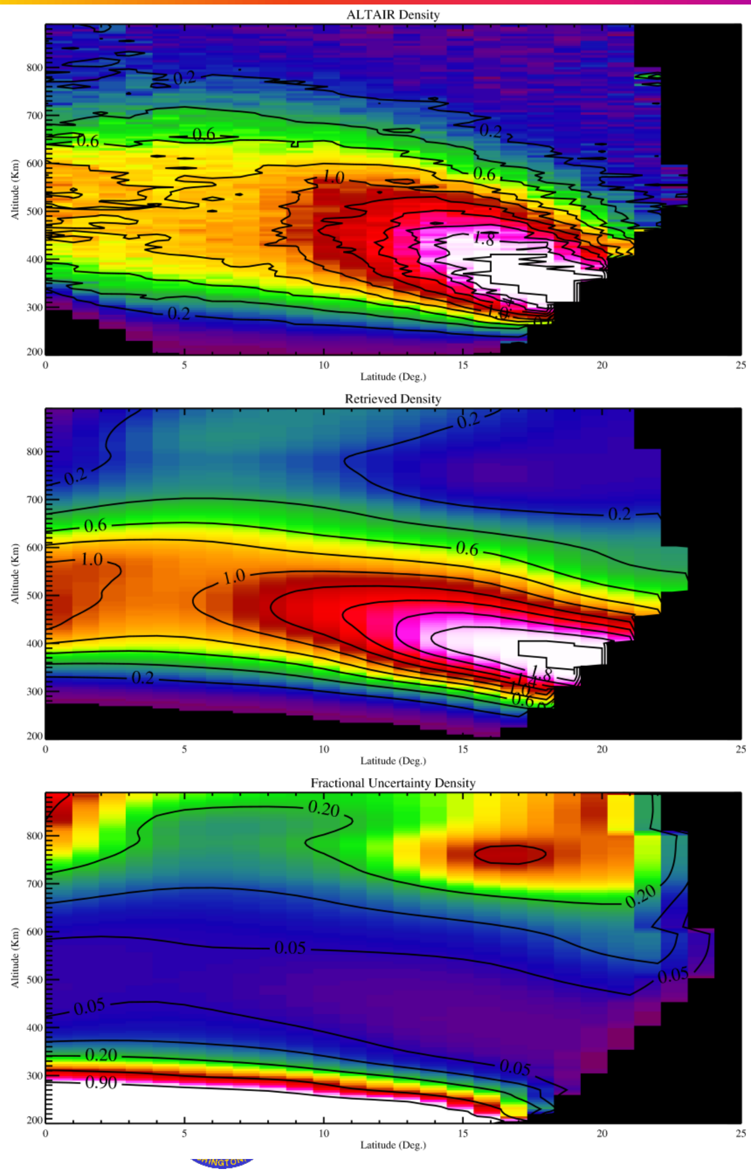
High SNR = 283

ε	CGLS	MART	LSPD	RL
10^{-2}	0.14	2.3	0.14	0.14
10^{-3}	0.20	6.5	0.19	0.20
10^{-4}	0.41	21.8	0.34	0.40
10^{-5}	3.72	226.7	3.07	3.71

Does it really work?

- Comparison of SSULI tomography versus ALTAIR radar measurements using Richardson-Lucy algorithm and physicality constraint
  - *Agreement is very good*
  - *Scatterplot below shows high degree of correlation*
  - *Diffusion weight estimated from SNR of measurements*





Summary

- We now have the means to rapidly and accurately invert spaceflight limb-scan data
  - *Routine, automated processing is possible*
  - *Can now derive 2D structure along the orbit plane*
  - *Approach is being extended to 3D*
  - *Also works with other applications*

- Our approach entails
  - *New iterative Image Space Reconstruction Algorithms*
  - *Physicality constraint using regularization to a partial differential equation*

- Advantages of our approach:
  - *The algorithms are both fast and robust*
  - *The Richardson-Lucy algorithm handles Poisson nose explicitly*
    - Can work on data with very low signal-to-noise ratio
  - *Regularization approach is somewhat "vanilla", in that minimal tuning is required*

Acknowledgements

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