Optimal Dynamic Matching under Local Compatibility: An Application to Kidney Exchange

Ömer Faruk Şahin — Stanford
Duygu Sili — Carlos III Madrid
M. Utku Ünver — Boston College
Özgür Yılmaz — Koç

Department of Economics Seminar
Bilkent University
September 9, 2026

Kidney Exchange

- Kidney transplantation is the best treatment for end-stage renal disease, particularly when the kidney comes from a living donor.
- Patients often arrive with a willing donor who is incompatible.
  - Compatibility = ABO blood-type *and* tissue-type (cross-match).
- **Kidney exchange**: incompatible patient–donor pairs swap donors so that each patient gets a compatible kidney.
- Proposed by Rapaport (1986). Starting with S. Korea, US, Netherlands, ...

Kidney Exchange Volume by Country

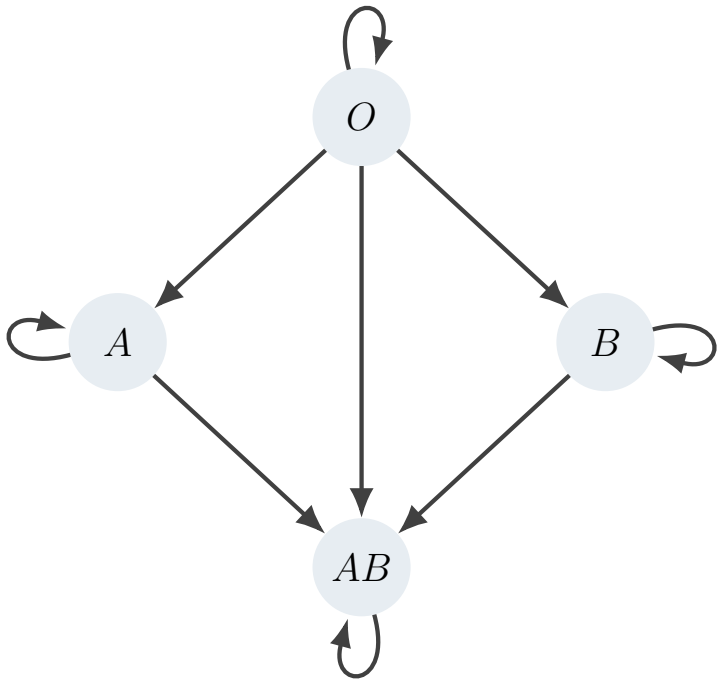
Country / Program	KPD transplants/yr	Notes
United States†	1,100–1,400	Single centers/NKR/OPTN/APD
India††	high	Single centers; no national reg.
Turkey	200–300	Single centers
United Kingdom	∼125	UKLKSS (NHSBT)
Spain	50–60	2nd largest in Europe
Canada	50–90	LDPE (since 2009)
Australia / NZ	50–90	ANZKX (since 2010)
Netherlands	40–60	First national KEP (2004)
Czech Republic	50–60	Long 6–7-way chains
Other EU†††	<30 each	AT, BE, IT, PL, PT, CH, FR
Saudi Arabia	new (2024)	SCOT; launched Aug. 2024

†Mostly 2–3-way, then NDD-chains
††No national figure; single-center series only
†††Mostly 2–3-way; some pools internationally linked

Sources: Agarwal et al. (2019); OPTN (2022); NHSBT (2024/25); Turkish Soc. Nephrology 2012–23 (via Sönmez, 2024); SCOT (2024).

Compatibility

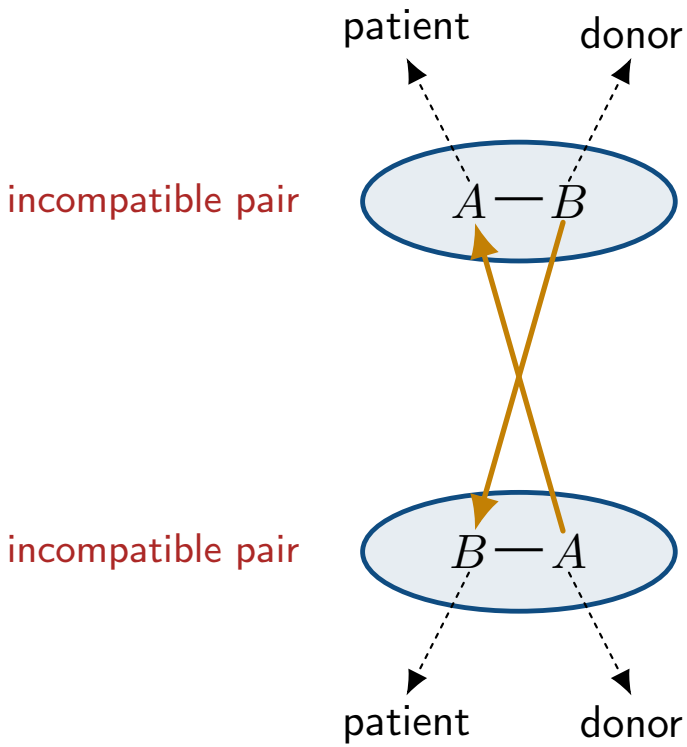
- Blood-type Compatibility:



$O \succ A, B \succ AB$ in donor-receiver compatibility.

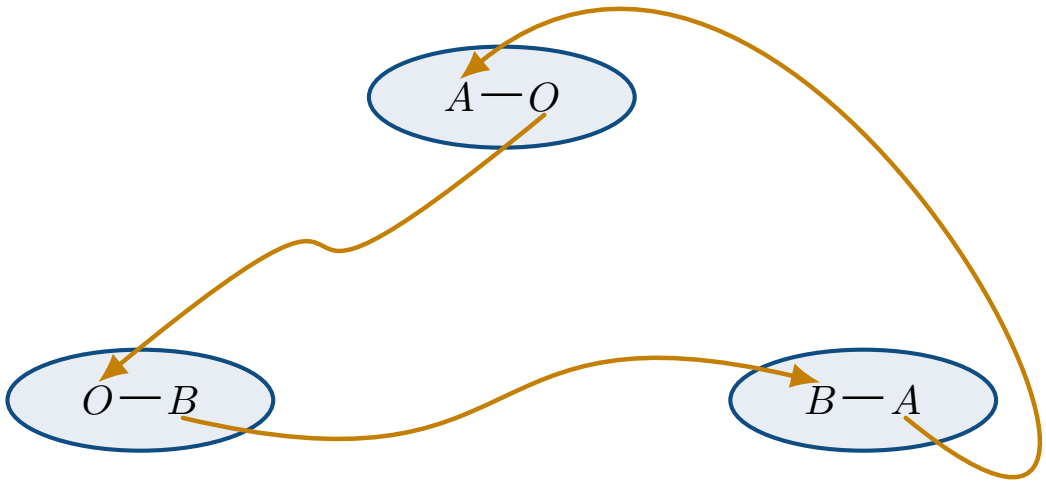
- Tissue-type Compatibility: Patient does not have pre-formed antibodies against donor's tissue-type DNA.

Two-way Exchange



Denoted $(A-B,\ B-A)$. Surplus: 2 transplants.

Three-way Exchange



Denoted $(A-O,\ O-B,\ B-A)$. Surplus: 3 transplants.

Surgeries are simultaneous (no reneging) $\Longrightarrow$ exchange size is bounded: mostly 2-way and 3-way.

Dynamic Nature of Kidney Exchange

- Busy programs match continuously as pairs arrive, not in periodic batches.
- Pairs arrive with a Poisson process, different rates for each type.
- Patients incur a flow waiting cost c.
- Continuous discount rate ρ.
- Question. How should a centralized clearinghouse match dynamically to maximize total expected discounted surplus?

Core trade-off: match *now* for a smaller, sooner surplus vs. wait for a *larger but later* (discounted) surplus.

An Illustrative Trade-off

Suppose the pool contains a $B-A$ pair and an $A-O$ pair just arrived. A future $A-B$ arrives stochastically.

Option	Exchange now	Exchange later	Surplus
1	$(B-A,\ A-O,\ O-B)$	—	3 now
2	$(A-O,\ O-A)$	$(B-A,\ A-B)$	2 now + 2 later

Intuition. Hold the $B-A$ if and only if the marginal value of an additional $B-A$ in the pool exceeds 1. But that marginal value depends on the entire state.

Same trade-off for $B-O$ vs. $A-B$; the two are coupled through the 2-way $(A-B,\ B-A)$.

Matching Environment

- $O-A$, $O-B$ are abundant (underdemanded pairs — matched on the spot at surplus 2, treated as use-it-or-lose-it).
- Simplification: no AB donor/patient (rare); no self-demanded pair $(A-A, B-B, O-O)$.

Four pair types control the state:

- $A-B$ and $B-A$: reciprocal pairs.
- $\mathbb{O}_{A-B} = \{B-O\}$: 3-way $(A-B,\ \mathbb{O}_{A-B},\ O-A)$, surplus 3.
- $\mathbb{O}_{B-A} = \{A-O\}$: 3-way $(B-A,\ \mathbb{O}_{B-A},\ O-B)$, surplus 3.

State

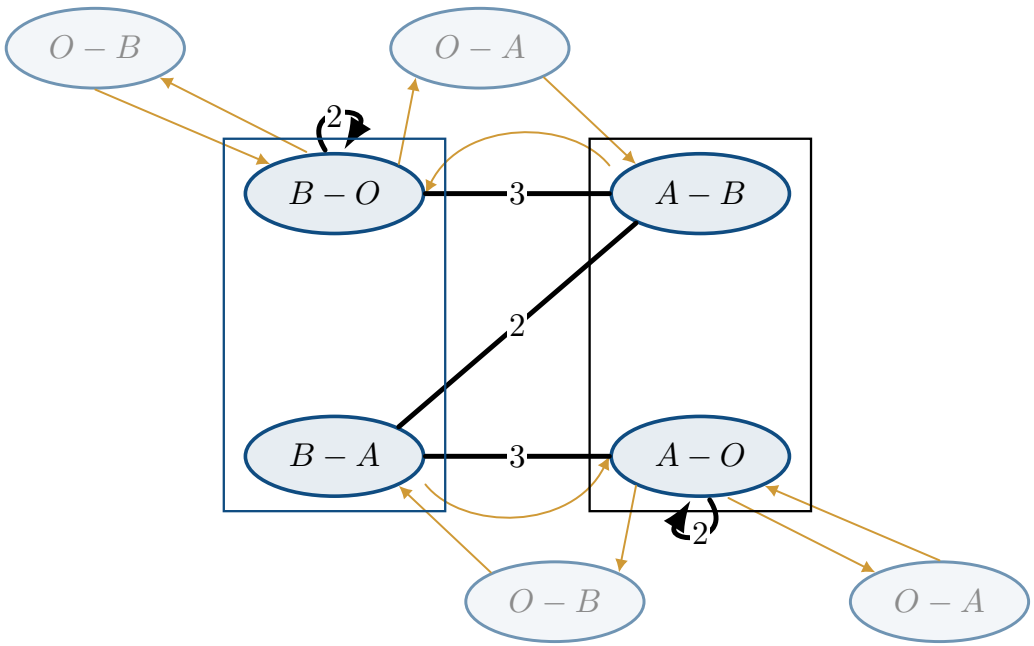
$s = (s_1, s_2, s_3, s_4) \in \mathbb{N}^4 = (|\mathbb{O}_{A-B}|,\ |A-B|,\ |B-A|,\ |\mathbb{O}_{B-A}|)$.

Assumptions

- Limit assumption: no tissue-type incompatibility across pairs. (Very highly sensitized patients accumulate in any pool; we leave them out.)
- Long-run assumption: arbitrarily many underdemanded pairs of each type in the pool at all times, under any mechanism (Roth, Sönmez & Ünver 2007; Ünver 2010).
- No self-demanded pairs: an $X - X$ pair can always be inserted into some exchange at no cost.
- Rare AB: an exchange with an $\mathbb{O}_{A-B}$ pair has size 3 with an $A - B$ pair, 2 without; same for $\mathbb{O}_{B-A}$.

Compatibility is deterministic here $\implies$ exact characterization. (Probabilistic compatibility, as in most of the recent literature $\implies$ approximations with error bounds.)

Linear Compatibility Structure



A local linear compatibility graph with surplus weights $(2, 3, 2, 3, 2)$.

1–d Was Easier

1–d case (Ünver, 2010): overdemanded pairs *assumed* matched on arrival, so $s_1 = s_4 = 0$ and the state collapses to $s_2 - s_3 \in \mathbb{Z}$.

- A threshold mechanism is optimal. Driver: concavity in 1–d state.
- The assumption was motivated by Prop. 3 there; its proof has an error (see the paper). Without it, overdemanded pairs become a decision.

4–d case: all four state variables matter; state space is $\mathbb{N}^4$.

- Concavity alone is not propagated by the optimality operator.
- Even concavity plus sub-/supermodularity / strong superconcavity is not propagated.
- Prior queueing-theory methods (Koole, 1998) don't apply: their *queueing basis* $e_j - e_{j-1}$ has no economic meaning in matching.

Our Contributions

- Methodology: a tailored notion of $\mathcal{D}^M$-multimodularity (with a new "matching basis" $\mathcal{D}^M$) that
  - is propagated by the optimality operator,
  - is just strong enough to imply all required second-order properties.
- Structural result: characterize the second-order shape of V^* and which exchanges the optimal mechanism conducts at each state, for any n.
- Optimal mechanism: in the (numerically generic) unbalanced regime, the optimal policy is a multi-dimensional threshold mechanism with three monotone thresholds.
- Extension: methodology applies to any dynamic matching with a local linear compatibility graph.

Related Literature

- **Kidney exchange & market design.** Rapaport (1986); Roth, Sönmez & Ünver (2004–2007); Anderson et al. (2015), Andersson & Kratz (2020), Biró et al. (2021) . . . MANY; *Surveys:* Sönmez & Ünver (2023, 2024), Ashlagi & Roth (2021).
- **Dynamic kidney exchange.** Originates with Ünver (2010). Prevailing view: near-greedy matching is almost optimal (Anderson et al. 2017). Blood- & tissue-type fluid model: Sönmez, Ünver & Yenmez (2020). Altruistic-donor/deceased-donor chains (U.S.): Roth et al. (2006), Agarwal et al. (2019), Furian et al. (2020), Akbarpour et al. (2024).
- **Optimal dynamic matching with complementarities.** Baccara, Lee & Yariv (2020, adoption, two types); Leshno (2022, incentives in queues); dynamic stability: Kurino (2020), Doval (2022); . . . MANY.
- **Methodology.** Markov decision processes / queueing theory and multimodularity. OR tools handle *substitutable* decisions: Hajek (1985), Koole (1998), Altman, Gaujal & Hordijk (2003).

This paper: As matching adds *complementarities*, we have exact characterization of the optimal mechanism with *arbitrarily many* types under local compatibility; greedy can be *strictly* suboptimal—overdemanded pairs are sometimes optimally held (multidimensional threshold structure).

1 Introduction

2 Dynamic Model

3 Second-order Properties

4 The Propagation Puzzle and $\mathcal{D}^M$-Multimodularity

5 Structure of the Optimal Mechanism

6 The Optimal Kidney Exchange Mechanism

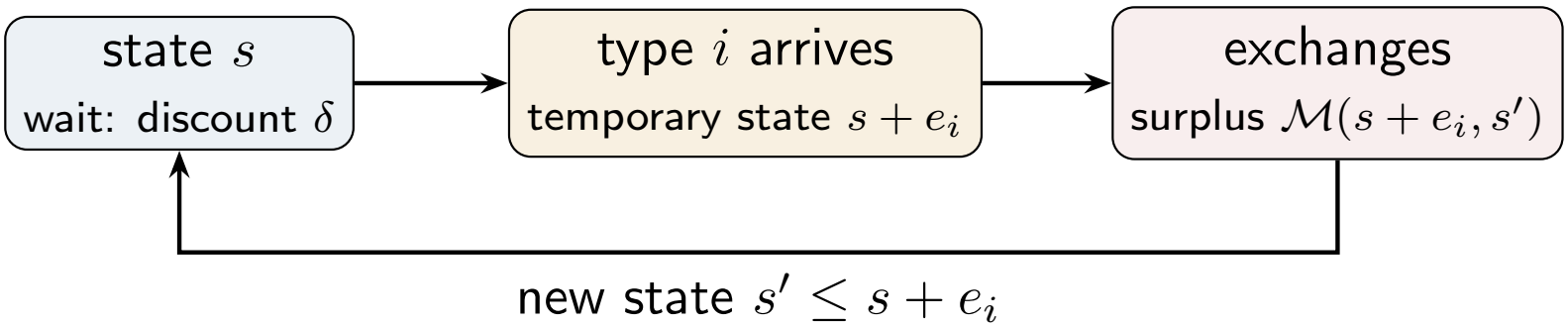
7 Concluding Remarks

Model

- Types $\mathcal{T} = \{1, \ldots, n\}$,
  local linear compatibility:
  type i can match $i \pm 1$ & endpoints with themselves.
- Poisson arrival rates λ_i in continuous time; let
  overall arrival rate $\lambda := \sum \lambda_i$,
  prob. for next pair's type i $p_i := \lambda_i / \lambda$,
  (discrete time $\equiv$) discount $\delta := \lambda / (\lambda + \rho) \in (0, 1)$.
- Surplus a_{ij}
  $\in \{2, 3\}$ for kidney exchange,
  $\in \mathbb{R}$ for other problems.
- State $s \in \mathbb{N}^n$. At most q exchanges of each type per period (will never bind).

Timing

Poisson arrivals are memoryless $\implies$ Markov in the pool composition.



- $s \to s'$: some feasible exchange sequence (at most q of each type) takes s to s'.
  $\mathcal{M}(s, s')$: max total surplus over such sequences, $\mathcal{M}(s, s) := 0$.

- Kidney exchange: $(1,1,0,0) \to (0,0,0,0)$ by one 3-way;
  $(1,1,1,1) \to (1,0,0,1)$ by the 2-way $(A-B, B-A)$;
  $(1,1,1,0) \not\to (1,0,1,0)$.
  Any sequence: $\mathcal{M}(s,s') = 2(s_1 + s_4 - s'_1 - s'_4) + (s_2 + s_3 - s'_2 - s'_3)$.

Bellman Equation

Dynamically optimal: maximizes expected discounted surplus $\mathbb{E}\sum_t M(t)e^{-\rho t}$.

Bellman equation for optimal value function V^*:

$$V^*(s) \;=\; \delta \sum_i p_i \max_{s+e_i \to s'} \Big\{ \mathcal{M}(s+e_i, s') + V^*(s') \Big\}.$$

Compact, but analytically hard: $\max$ over all s' reachable from $s + e_i$.

Event-based DP (Koole, 1998): write T^* as a composition of event operators, one per event, and study V^* through them. Koole's operators are for queues; matching has no antecedent there.

Matching Vectors

A single feasible exchange transitions s by $-w$ where

$$w \in \mathcal{D}^* := \{ e_1,\ e_1 + e_2,\ e_2 + e_3,\ e_3 + e_4,\ e_4 \}.$$

- $w = e_1$: $\mathbb{O}_{A-B}$ matched with itself (surplus 2).
- $w = e_1 + e_2$: 3-way $(\mathbb{O}_{A-B}, A - B)$ (surplus 3).
- $w = e_2 + e_3$: 2-way $(A - B, B - A)$ (surplus 2).
- $w = e_3 + e_4$: 3-way $(B - A, \mathbb{O}_{B-A})$ (surplus 3).
- $w = e_4$: $\mathbb{O}_{B-A}$ matched with itself (surplus 2).

General n: $\mathcal{D}^* = \{e_1,\ e_1 + e_2,\ \ldots,\ e_{n-1} + e_n,\ e_n\}$, $n + 1$ matching vectors.

The Matching Operator

$\mathcal{V}$: bounded functions $\mathbb{N}^n \to \mathbb{R}$. $\#_w$: how many times matching vector w is used.

Joint matching operator $T_M : \mathcal{V} \to \mathcal{V}$:

$$T_M f(s) = \max_{\substack{(\#_w)_{w\in\mathcal{D}^*}\in\{0,\dots,q\}^{n+1} \\ s-\sum_w \#_w\, w \geq 0}} f\Big(s - \sum_{w\in\mathcal{D}^*} \#_w\, w\Big) + \sum_{w\in\mathcal{D}^*} \#_w\, a_w.$$

- One operator picks the whole set of exchanges at a state.
- Not a composition of single-exchange operators $T_w f(s) = \max\{f(s), f(s-w) + a_w\}$: the T_w's don't commute (one exchange changes whether another one is still optimal).

Other Event Operators and the Decomposition

- Discount: $(T_\delta f)(s) := \delta f(s)$.
- Arrival of i: $(T_{\text{arr},i} f)(s) := f(s + e_i)$.
- Uniformization: $T_p(f_1, \ldots, f_m)(s) := \sum_i p_i f_i(s)$.

Observation (decomposition of dynamic programming operator T^*)

$$\begin{aligned} V^*(s) &= T^* V^*(s) \\ &= \Big(T_\delta \circ T_p\big(T_M \circ T_{\text{arr},1} V^*, \ldots, T_M \circ T_{\text{arr},n} V^*\big)\Big)(s). \end{aligned}$$

Wait (discount) $\rightarrow$ type drawn from p $\rightarrow$ agent added $\rightarrow$ match.

Value Iteration and Propagation

Theorem (Fundamental Theorem of Dynamic Programming)

V^* exists and is the unique fixed point of T^*. For any $V_0 \in \mathcal{V}$, $V_k := T^* V_{k-1}$ converges uniformly to V^*.

Definition

T propagates property P if f satisfies $P \implies Tf$ satisfies P.

Our approach. To show V^* satisfies P:

- start from a V_0 with P (a constant, say),
- each event operator propagates $P \implies$ so does T^*,
- P preserved under uniform limits $\implies$ V^* has P.

Difficulty: the properties we want are too weak to be propagated one by one. Strengthen by intersection until the family is closed.

1 Introduction

2 Dynamic Model

3 Second-order Properties

4 The Propagation Puzzle and $\mathcal{D}^M$-Multimodularity

5 Structure of the Optimal Mechanism

6 The Optimal Kidney Exchange Mechanism

7 Concluding Remarks

From Concavity to a Family of Properties

Example

For a 1-d operator $(Tf)(x) = \max\{f(x), f(x - e_i) + 1\}$:

$$\text{match} \iff f(x) - f(x - e_i) \leq 1.$$

- LHS is the marginal value of a type-i agent (opportunity cost of matching), RHS the benefit.
- Concavity in $e_i \implies$ LHS is decreasing in $s_i \implies$ threshold: pool below, match above.

In higher dimensions we need more: substitution / complementarity between types, and "second-derivative" comparisons across coordinates.

Second-order Properties of a Discrete Function

$f : \mathbb{N}^n \to \mathbb{R}$; each condition for every x with all arguments in $\mathbb{N}^n$.

- Concavity in component i: $f(x + 2e_i) - f(x + e_i) \leq f(x + e_i) - f(x)$. Diminishing marginal value of type i.
- $\{i, j\}$-submodularity: $f(x + e_i + e_j) - f(x + e_j) \leq f(x + e_i) - f(x)$. j lowers the marginal value of i: substitutes.
- $\{i, j\}$-supermodularity: $f(x + e_i) - f(x) \leq f(x + e_i + e_j) - f(x + e_j)$. j raises the marginal value of i: complements. (Both symmetric in i, j, hence the set notation.)
- Strong ij-superconcavity: two clauses,

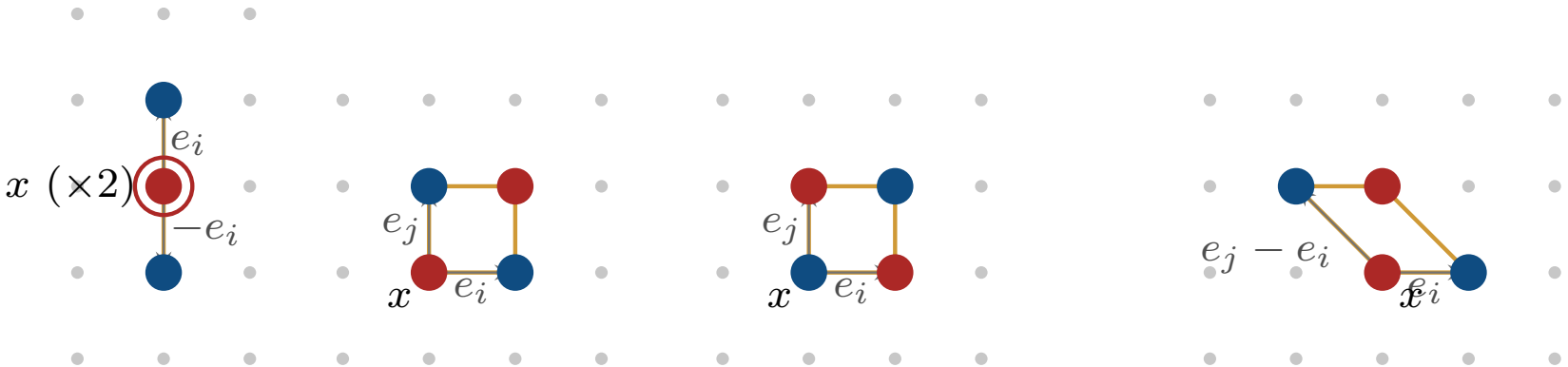
$$f(x + 2e_i) - f(x + e_i) \leq f(x + e_i + e_j) - f(x + e_j),$$
$$f(x + 2e_i) - f(x + e_i) \leq f(x + e_i - e_j) - f(x - e_j).$$

  Own effect of i on its marginal value beats the cross effect of j: 1st clause binds if j is a substitute, 2nd if a complement. Not symmetric in i, j.

Componentwise concave: all i. Strongly superconcave: all $i \neq j$.

The Four Properties on the Lattice

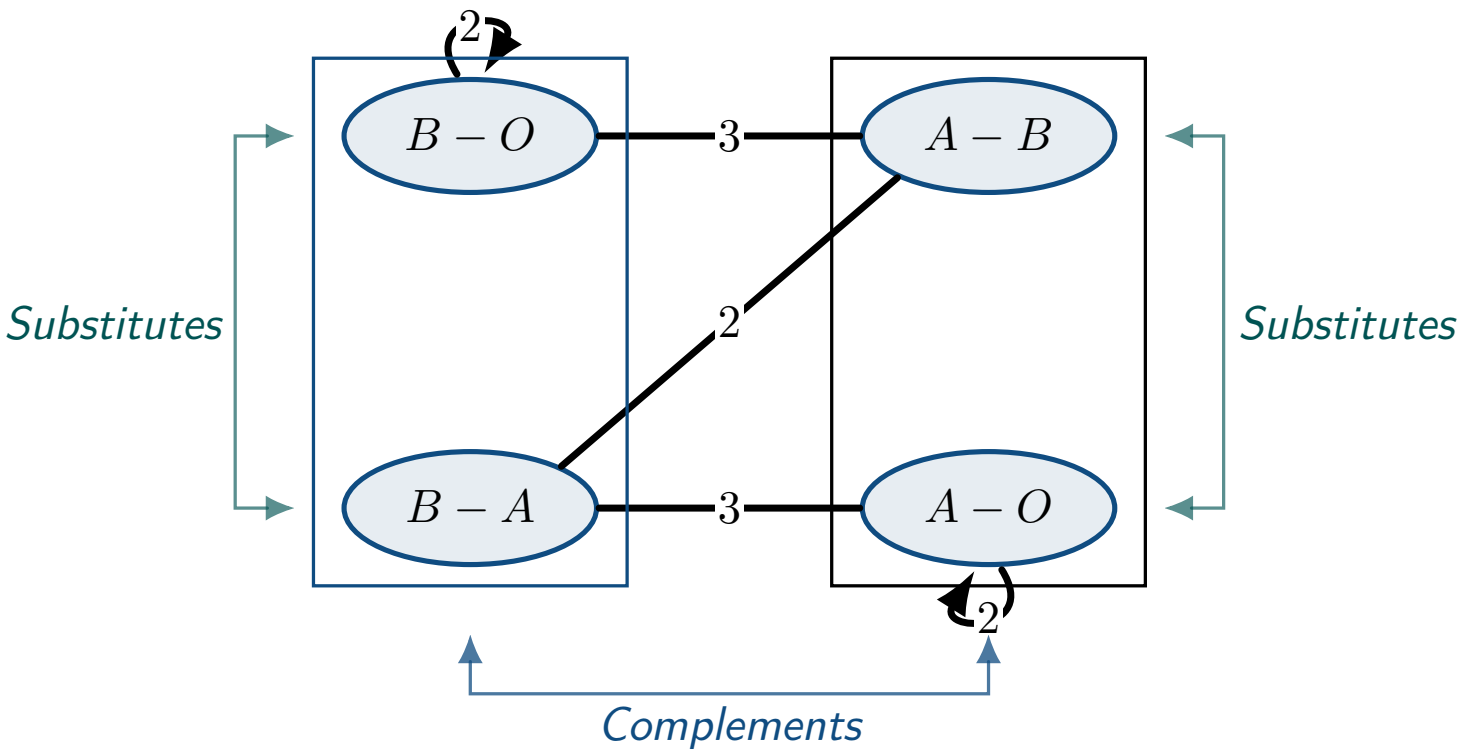
Each one: sum of f over the navy points $\leq$ sum of f over the maroon points.



concavity in i $\{i,j\}$-supermodular $\{i,j\}$-submodular strong ij-superconcave (1st clause)

Sub- and supermodularity: the same square, colors swapped. Strong superconcavity, 2nd clause: the parallelogram with $-e_j - e_i$ in place of $e_j - e_i$.

Substitutes and Complements in Kidney Exchange



Side B ('B' referring to B patients) **Side A** ('A' referring to A patients)

Same side: substitutes. Across: complements.

A Unified Framework: Property $P(u, v)$

Definition

$f \in \mathcal{V}$ satisfies $P(u, v)$ if, for every x with $x + u,\ x + v,\ x + u + v \in \mathbb{N}^n$,

$$f(x + u) + f(x + v) \leq f(x) + f(x + u + v).$$

u and v are complements. $P(u, v) = P(v, u)$; $P(u, v) \iff P(-u, -v)$. In the pictures: navy $x + u$, $x + v$; maroon x, $x + u + v$.

The four properties are special cases (set $u = e_i$):

- $v = -e_i$: concavity in i;
- $v = -e_j$: $\{i, j\}$-submodularity; $v = e_j$: $\{i, j\}$-supermodularity;
- $v = e_j - e_i$ and $v = -e_j - e_i$: the two clauses of strong ij-superconcavity.

Propagation will turn out to be a question about which patterns a family has to contain.

1 Introduction

2 Dynamic Model

3 Second-order Properties

4 The Propagation Puzzle and $\mathcal{D}^M$-Multimodularity

5 Structure of the Optimal Mechanism

6 The Optimal Kidney Exchange Mechanism

7 Concluding Remarks

Closedness

Propagating $P(u, v)$ through a single-exchange operator $T_w f(x) = \max\{f(x), f(x - w) + a_w\}$: when the two max's on the left pick different arguments, one needs $P(u, v + w)$ or $P(u + w, v)$ instead of $P(u, v)$.

Closedness condition: $\mathcal{P} = \{P(u, v) : (u, v) \in U\}$ is closed under T_w if

$$P(u, v) \in \mathcal{P} \implies P(u, v + w) \in \mathcal{P}$$

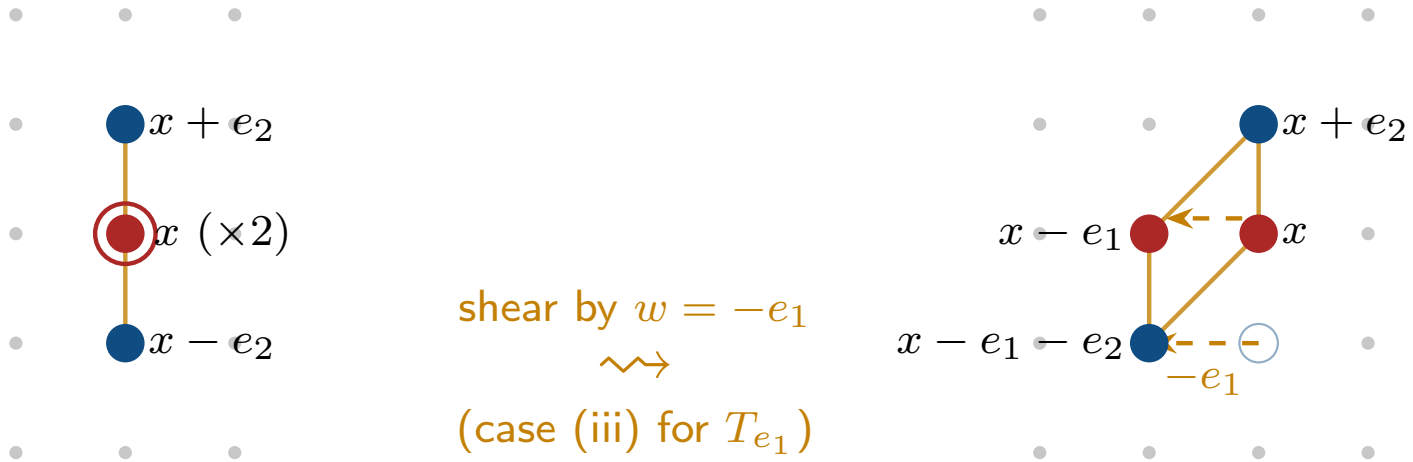
(by symmetry then also $P(u + w, v) \in \mathcal{P}$). Closed family $\implies$ propagated as a whole.

Geometrically: $P(u, v + w)$ is the pattern of $P(u, v)$ with the side $\{x + v,\ x + u + v\}$ translated by w.
T_w shears patterns along a matching vector. Closedness = closed under such shears.

Plan. 2–d first, then the general puzzle.

Two Dimensions

$n = 2$: $\mathcal{D}^* = \{e_1, e_1 + e_2, e_2\}$; single-exchange operators
$T_w f(x) = \max\{f(x), f(x - w) + a_w\}$.



concavity in 2: $P(e_2, -e_2)$ $P(e_2, -e_2 - e_1)$

Step 1. Sheared pattern = 2nd clause of strong 21-superconcavity, not implied by concavity: concavity alone is not closed under T_{e_1}, T_{e_2}.
Step 2. {componentwise concavity, strong superconcavity} is closed under T_{e_1}, T_{e_2}.
Step 3. Adding $T_{e_1+e_2}$ requires $\{1, 2\}$-supermodularity too; with it the family is closed (and minimal).

Beyond Three Dimensions

- $n \geq 3$: shear strong ij-superconcavity $P(e_i, e_j - e_i)$ by $-e_k$, $k \notin \{i, j\}$. The pattern $P(e_i,\, e_j - e_i - e_k)$ is none of concavity, sub-, super-, strong superconcavity.
- Puzzle: which U makes $\mathcal{P} = \{P(u, v) : (u, v) \in U\}$ closed, i.e. $P(u, v) \in \mathcal{P} \implies P(u, v + w) \in \mathcal{P}$ for every matching vector w?
- Moreover T_M picks a *subset* of exchanges, not one. Propagation by each single T_w is necessary but not sufficient for propagation by T_M. (This is why T_M is one joint maximization.)

Beyond 3 dimensions: closure requires properties that are not concavity, sub-, super-, or strong superconcavity.

Way out: take u, v from a common set $\mathcal{D}$ and require $P(u, v)$ for all pairs: $\mathcal{D}$-multimodularity (Hajek, 1985).

$\mathcal{D}$-Multimodularity (Hajek, 1985)

Definition

- A set $\mathcal{D} \subseteq \mathbb{Z}^n$ is a multimodular basis if $\sum_{v \in \mathcal{D}} v = 0$ and $\text{span}(\mathcal{D}) = \mathbb{Z}^n$.
- f is $\mathcal{D}$-multimodular if $P(u, v)$ holds for every pair of distinct $u, v \in \mathcal{D}$.

Lemma 1 (disjoint-sum lemma)

for any two disjoint $U, V \subseteq \mathcal{D}$ and admissible x,

$$f\big(x + \textstyle\sum_{u \in U} u\big) + f\big(x + \sum_{v \in V} v\big) \;\leq\; f(x) + f\big(x + \sum_{u \in U \cup V} u\big).$$

Sums of disjoint subsets of $\mathcal{D}$ are complements — exactly what closedness needs: $u, v, w \in \mathcal{D}$ give $P(u, v + w)$.

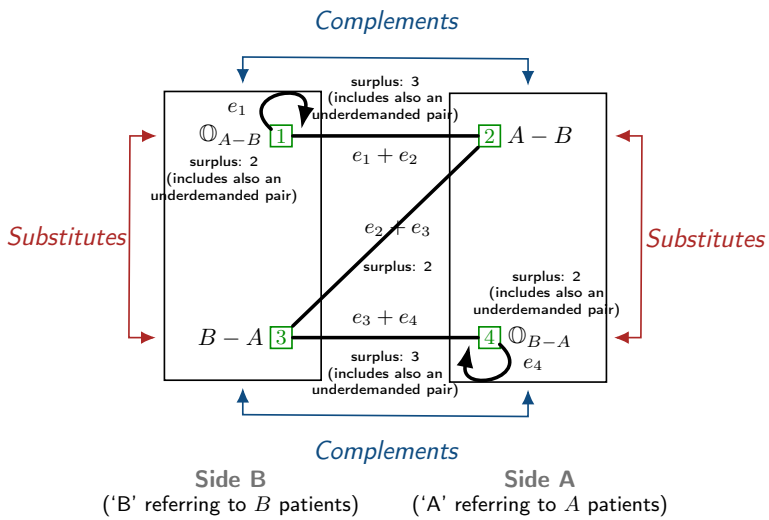
$\sum_{v \in \mathcal{D}} v = 0 \implies -u$ is the sum of $\mathcal{D} \setminus \{u\} \implies P(u, -u)$, concavity along each $u \in \mathcal{D}$, for free.

The Matching Basis $\mathcal{D}^M$

Definition (matching basis)

$$\mathcal{D}^M := \{ e_1,\ -(e_1+e_2),\ e_2+e_3,\ -(e_3+e_4),\ e_4 \}.$$

- Same vectors as $\mathcal{D}^*$, with alternating signs so they sum to 0. General n: $\{e_1,\ -(e_1+e_2),\ \ldots,\ (-1)^n e_n\}$.
- Each element corresponds to a feasible matching transition.
- Concavity, sub-/supermodularity and strong superconcavity all follow as corollaries.



Main Propagation Result

Proposition (Prop. 1, linear operators)

$f, f_1, \ldots, f_m$ $\mathcal{D}^M$-multimodular $\implies$ $T_\delta f$, $T_{\text{arr},i} f$, $T_p(f_1, \ldots, f_m)$ are $\mathcal{D}^M$-multimodular.

Proposition (Prop. 2, propagation under matching operator)

If $f \in \mathcal{V}$ is $\mathcal{D}^M$-multimodular, then so is $T_M f$.

The crux.

Properties of the Value Function

Theorem (Thm. 2)

The optimal value function V^* is $\mathcal{D}^M$-multimodular.

Corollary (Thm. 3)

V^* is

- *componentwise concave;*
- *$\{i, i+k\}$-supermodular for every odd k: types of different parity are complements;*
- *$\{i, i+k\}$-submodular for every even $k \neq 0$: types of the same parity are substitutes;*
- *strongly ij-superconcave for all $i \neq j$.*

Kidney exchange: $\{\mathbb{O}_{A-B}, A-B\}$, $\{A-B, B-A\}$, $\{B-A, \mathbb{O}_{B-A}\}$, $\{\mathbb{O}_{A-B}, \mathbb{O}_{B-A}\}$ complements; $\{\mathbb{O}_{A-B}, B-A\}$, $\{A-B, \mathbb{O}_{B-A}\}$ substitutes.

Proof of the Sub- and Supermodularity Part

Write $\mathcal{D}^M = \{d_1, \ldots, d_{n+1}\}$: $d_1 = e_1$, $\quad d_k = (-1)^{k+1}(e_{k-1} + e_k)$ for $k = 2, \ldots, n$, $\quad d_{n+1} = (-1)^n e_n$.

Partial sums telescope. $d_1 + \cdots + d_i = (-1)^{i+1} e_i$, and since all $n+1$ vectors sum to 0, $d_{i+1} + \cdots + d_{n+1} = (-1)^i e_i$.

$$\mathcal{D}^M = \{\ \underbrace{d_1, \ldots, d_i}_{\text{sum } (-1)^{i+1} e_i}\ ,\ d_{i+1}, \ldots, d_j,\ \underbrace{d_{j+1}, \ldots, d_{n+1}}_{\text{sum } (-1)^j e_j}\ \} \qquad (i < j).$$

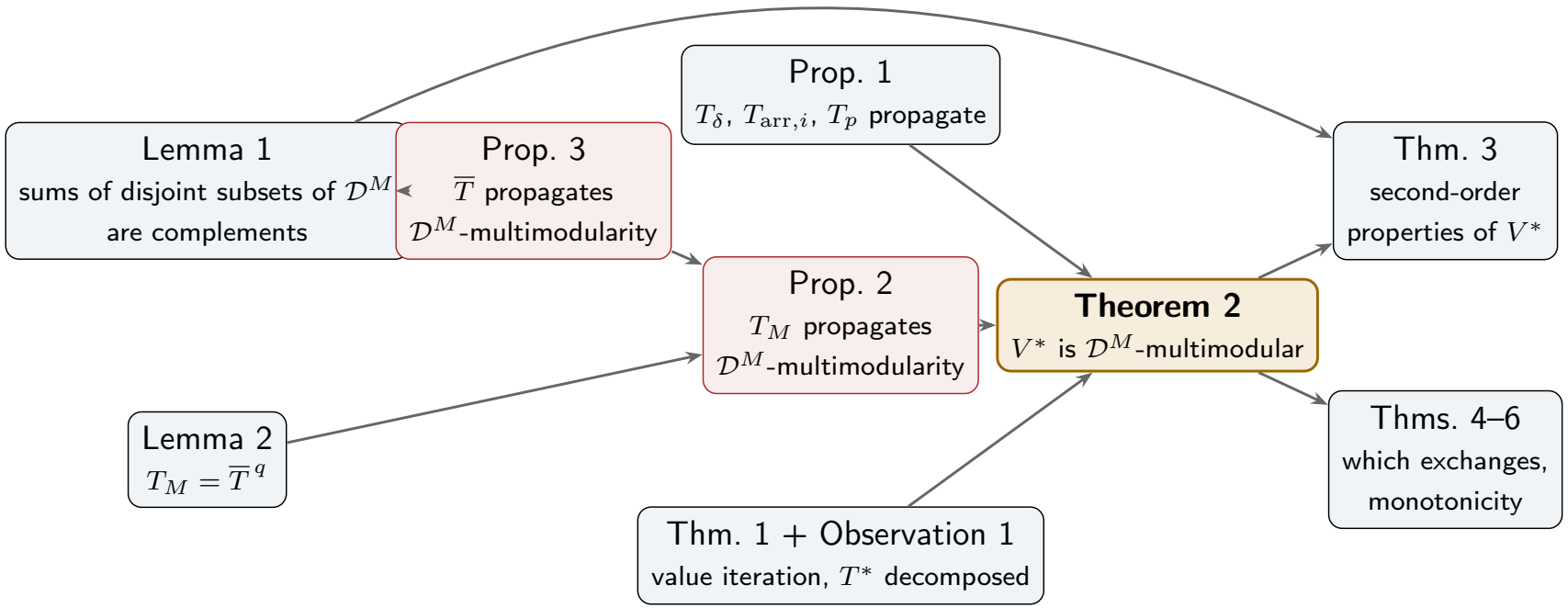
Take $i < j$. $U :=$ first i vectors, $V :=$ last $n + 1 - j$ vectors. Disjoint.
Lemma 1 $\implies$ $P\big((-1)^{i+1} e_i,\ (-1)^j e_j\big)$.

$P(u, v) \iff P(-u, -v)$, so only the parities matter:

- i, j same parity $\implies$ $P(e_i, -e_j)$, i.e. $\{i, j\}$-submodular;
- i, j different parity $\implies$ $P(e_i, e_j)$, i.e. $\{i, j\}$-supermodular. $\square$

Same with $U =$ first i, $V =$ last $n + 1 - i$: $P(e_i, -e_i)$, concavity in i. In general: $P(u, v)$ for any u, v that are sums over disjoint subsets of $\mathcal{D}^M$.
(Strong superconcavity: Appendix C of the paper.)

What Goes into Theorem 2



$\overline{T}$: the matching operator that uses each matching vector at most once. Maroon: the new part. The rest is standard MDP machinery plus Hajek's lemma.

1 Introduction

2 Dynamic Model

3 Second-order Properties

4 The Propagation Puzzle and $\mathcal{D}^M$-Multimodularity

5 Structure of the Optimal Mechanism

6 The Optimal Kidney Exchange Mechanism

7 Concluding Remarks

Reachable States

Definition

s is reachable if for every $s' \neq s$ with $s \to s'$:
$V^*(s) > \mathcal{M}(s, s') + V^*(s')$.

No exchange among the pairs already in the pool increases surplus. Unreachable states are never visited. $\mathcal{R}$: reachable states.

Decision at an arrival. s reachable, type i arrives: which exchanges to conduct, i.e. which matching vectors $w \in \mathcal{D}^*$ to use, and how many times each.

Question. Which exchange types can be optimal after a type-i arrival? How does this change with the state?

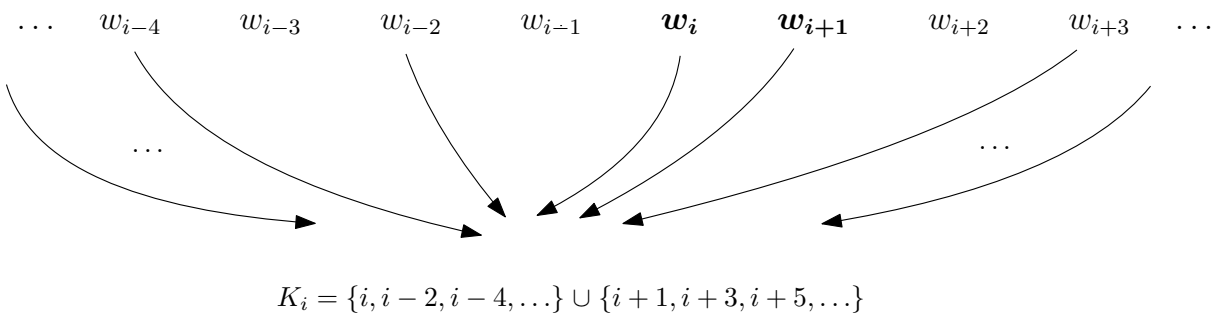
Skip to kidney exchange

The Index Sets K_i

Number the matching vectors: $w^1 = e_1$, $w^k = e_{k-1} + e_k$ for $k = 2, \ldots, n$, $w^{n+1} = e_n$ (so $\mathcal{D}^M = \{w^1, -w^2, w^3, \ldots, (-1)^n w^{n+1}\}$).

Definition

For each type i:
$K_i := \{i,\, i-2,\, i-4, \ldots\} \cup \{i+1,\, i+3,\, i+5, \ldots\} \subseteq \{1, \ldots, n+1\}$.



Observation 3 (by Lemma 1)

$k \in K_i$: marginal value of exchange w^k falls when a type-i agent arrives.
$\ell \notin K_i$: marginal value of exchange w^ℓ rises.

Kidney Exchange: The Sets K_i

$w^1 = e_1$ ($\mathbb{O}_{A-B}$ alone), $w^2 = e_1 + e_2$ ($\mathbb{O}_{A-B}$ with $A-B$), $w^3 = e_2 + e_3$ ($A-B$ with $B-A$), $w^4 = e_3 + e_4$ ($B-A$ with $\mathbb{O}_{B-A}$), $w^5 = e_4$ ($\mathbb{O}_{B-A}$ alone).

arrival	K_i	exchanges whose marginal value falls
$\mathbb{O}_{A-B}$ $(i=1)$	$\{1,2,4\}$	$\mathbb{O}_{A-B}$ alone; $(\mathbb{O}_{A-B}, A-B)$; $(B-A, \mathbb{O}_{B-A})$
$A-B$ $(i=2)$	$\{2,3,5\}$	$(\mathbb{O}_{A-B}, A-B)$; $(A-B, B-A)$; $\mathbb{O}_{B-A}$ alone
$B-A$ $(i=3)$	$\{1,3,4\}$	$\mathbb{O}_{A-B}$ alone; $(A-B, B-A)$; $(B-A, \mathbb{O}_{B-A})$
$\mathbb{O}_{B-A}$ $(i=4)$	$\{2,4,5\}$	$(\mathbb{O}_{A-B}, A-B)$; $(B-A, \mathbb{O}_{B-A})$; $\mathbb{O}_{B-A}$ alone

A $B-A$ arrival makes matching an $\mathbb{O}_{A-B}$ pair away more attractive ($1 \in K_3$, substitutes) and does not trigger the 3-way $(\mathbb{O}_{A-B}, A-B)$ ($2 \notin K_3$).

Which Exchanges Are Conducted

Theorem (Thm. 4)

s reachable, type i arrives: the optimal mechanism conducts only exchanges $k \in K_i$, each at most once. Decisions are sets $A(s,i) \subseteq K_i$: transition to $s + e_i - \sum_{k \in A(s,i)} w^k$, immediate reward $\sum_{k \in A(s,i)} a_{w^k}$.

- The capacity q never binds.

Comparative Statics and Monotonicity

Theorem (Thm. 5)

s and $s + e_j$ reachable $\implies$ for every arrival type i,

$$A(s,i) \cap K_j \subseteq A(s+e_j,i) \subseteq A(s,i) \cup K_j.$$

One more type-j agent can only *add* exchanges from K_j (marginal value fell) and *drop* exchanges from K_j^c (marginal value rose).

Theorem (Thm. 6)

s reachable, $s + e_j$ not $\implies$ $s + ke_j$ not reachable for any $k \geq 1$.

Marginal value changes are monotone: $\mathcal{R}$ is an interval along each coordinate.

$n = 4$: these give an explicit threshold mechanism.

1 Introduction

2 Dynamic Model

3 Second-order Properties

4 The Propagation Puzzle and $\mathcal{D}^M$-Multimodularity

5 Structure of the Optimal Mechanism

6 The Optimal Kidney Exchange Mechanism

7 Concluding Remarks

Reachable States in Kidney Exchange

$n = 4$, $s = (s_1, s_2, s_3, s_4) = (|\mathbb{O}_{A-B}|, |A - B|, |B - A|, |\mathbb{O}_{B-A}|)$.

Observation 4

(i) $s_2 > 0$ and $s_3 > 0 \implies s$ is not reachable.

(ii) $s_1 = s_3 = s_4 = 0$ or $s_1 = s_2 = s_4 = 0 \implies s$ is reachable.

- (i): all other types are overdemanded, can be matched any time $\implies$ matching $A - B$ with $B - A$ now has no cost, earlier is better.
- (ii): no overdemanded pair, one reciprocal type $\implies$ no feasible exchange.

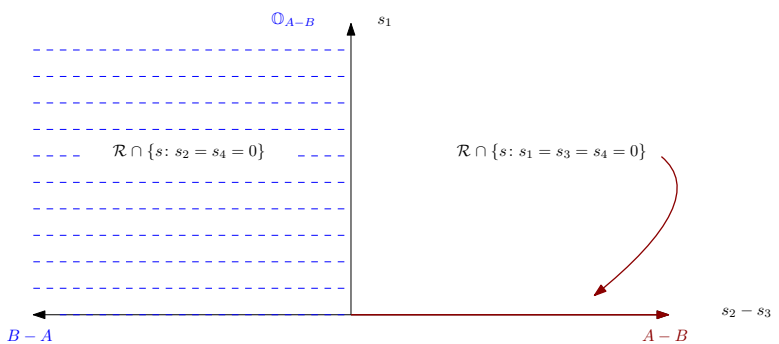
$A - B$ and $B - A$ pairs are never held at the same time $\implies$ one axis for reciprocal pairs, $s_2 - s_3 \in \mathbb{Z}$.
$\mathcal{R}$ is 3–d after this collapse; a numerically generic case brings it down to 2–d.

Unbalanced Dynamic Exchange

Unbalanced: at every reachable state s, (a) $s_4 = 0$, (b) $V^*(s) - V^*(s - e_2) < 1$ whenever $s - e_2 \geq 0$.

$A - B$ arrives sufficiently more often than $B - A$: few $B - A$ pairs wait $\implies$ holding $\mathbb{O}_{B-A}$ has little value (a); $A - B$ pairs are in excess $\implies$ marginal value below 1 (b).

- Numerically generic: holds whenever $p_{A-B} \neq p_{B-A}$ in the simulations of Ünver (2010). Field: $p_{A-B}/p_{B-A} = 5/3$ (Terasaki et al., 1998).
- $\mathcal{R} \subseteq \{s : s_2 = s_4 = 0\} \cup \{s : s_1 = s_3 = s_4 = 0\}$. Reachable state space collapses to a 2–d picture in $(s_2 - s_3,\ s_1)$.



Four Decisions, Three Thresholds

$s_2 > 0$ (trivial): arriving $\mathbb{O}_{A-B}$ or $B-A$ pairs are matched with an existing $A-B$; arriving $\mathbb{O}_{B-A}$ matched 2-way; arriving $A-B$ pooled.

$s_2 = s_4 = 0$: pool has $\mathbb{O}_{A-B}$ and $B-A$ pairs; the optimal mechanism faces four decisions, governed by three thresholds $t^{1,3}$, t^2, t^4:

- $\mathbb{O}_{A-B}$ arrives. Keep if $s_1 + 1 \leq t^{1,3}(s_3)$, else match 2-way.
- $B-A$ arrives. Keep if $s_1 \leq t^{1,3}(s_3+1)$, else match an existing $\mathbb{O}_{A-B}$ (keeping the $B-A$).
- $\mathbb{O}_{B-A}$ arrives. 2-way if $s_3 \leq t^4(s_1)$, else 3-way with $B-A$.
- $A-B$ arrives with $s_1, s_3 > 0$. Match $\mathbb{O}_{A-B}$ if $s_3 \leq t^2(s_1)$, else match $B-A$. (Only one of s_1, s_3 positive: no decision.)

$t^{1,3}$: threshold number of $\mathbb{O}_{A-B}$ pairs as a function of s_3. t^2, t^4: threshold number of $B-A$ pairs as a function of s_1.

Main Theorem (Optimal Mechanism)

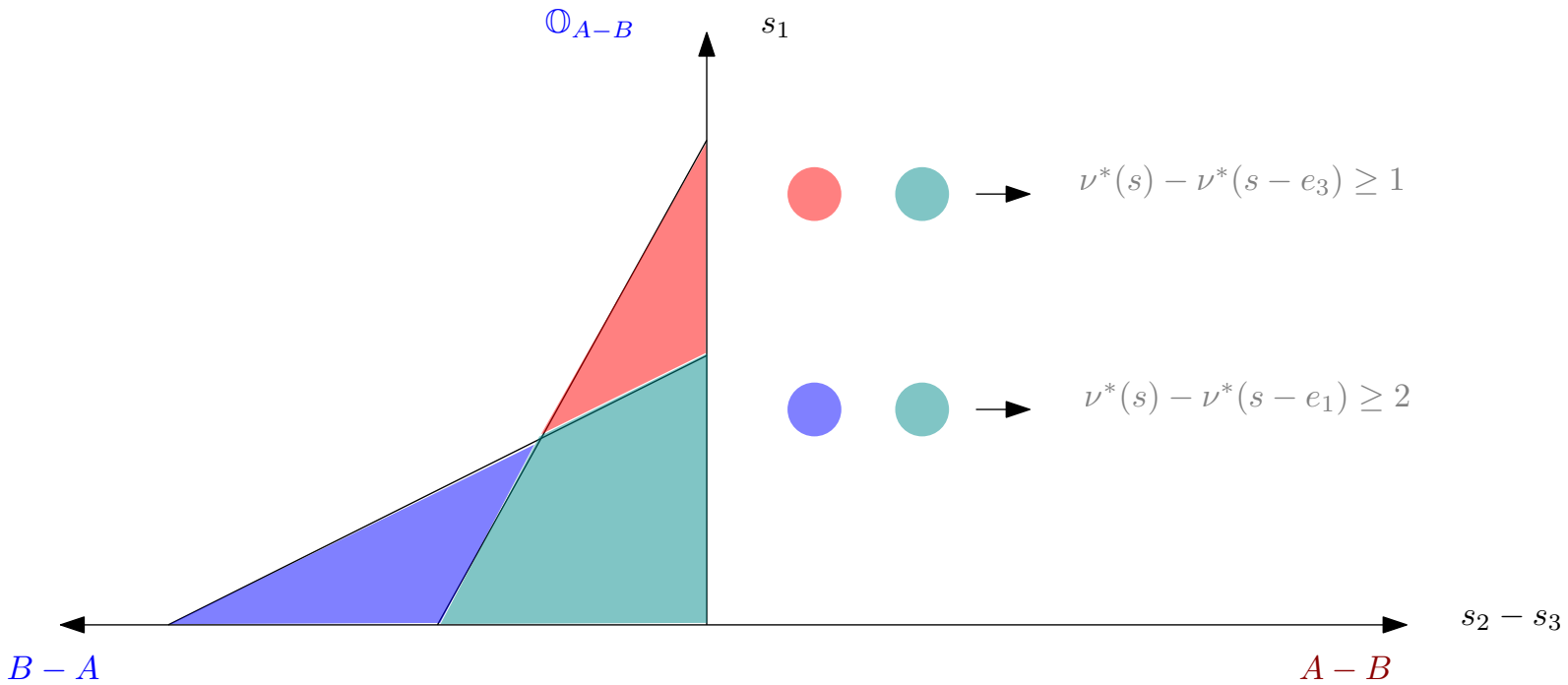
Theorem (Thm. 7)

For any unbalanced kidney exchange there exist threshold functions $t^{1,3}, t^2, t^4 : \mathbb{N} \to \mathbb{N}$ *such that the induced multi-dimensional threshold mechanism is dynamically optimal. Moreover,*

- $t^{1,3}$ *is non-increasing, with* $t^{1,3}(k+1) \geq t^{1,3}(k) - 1$;
  $V^*(s) - V^*(s - e_1) \leq 2 \iff s_1 > t^{1,3}(s_3)$;
- t^2 *is non-decreasing;*
  $V^*(s - e_3) - V^*(s - e_1) \leq 1 \iff s_3 < t^2(s_1)$;
- t^4 *is non-increasing, with* $t^4(k+1) \geq t^4(k) - 1$;
  $V^*(s) - V^*(s - e_3) \leq 1 \iff s_3 > t^4(s_1)$.

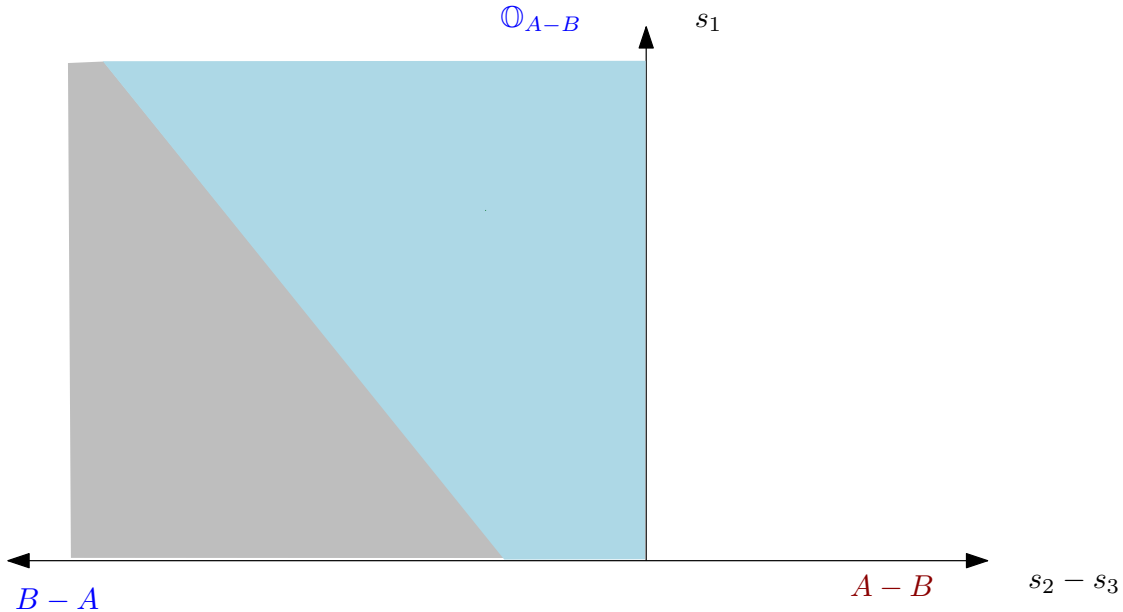
Each shape property maps to a second-order property of V^*, hence to $\mathcal{D}^M$-multimodularity.

Optimal Mechanism: Picture



- Triangular reachable region: existence follows from concavity.
- Boundary slope decreases: $\{1, 3\}$-submodularity.
- Slope in $(-1, 0)$: strong superconcavity.

The Third Threshold

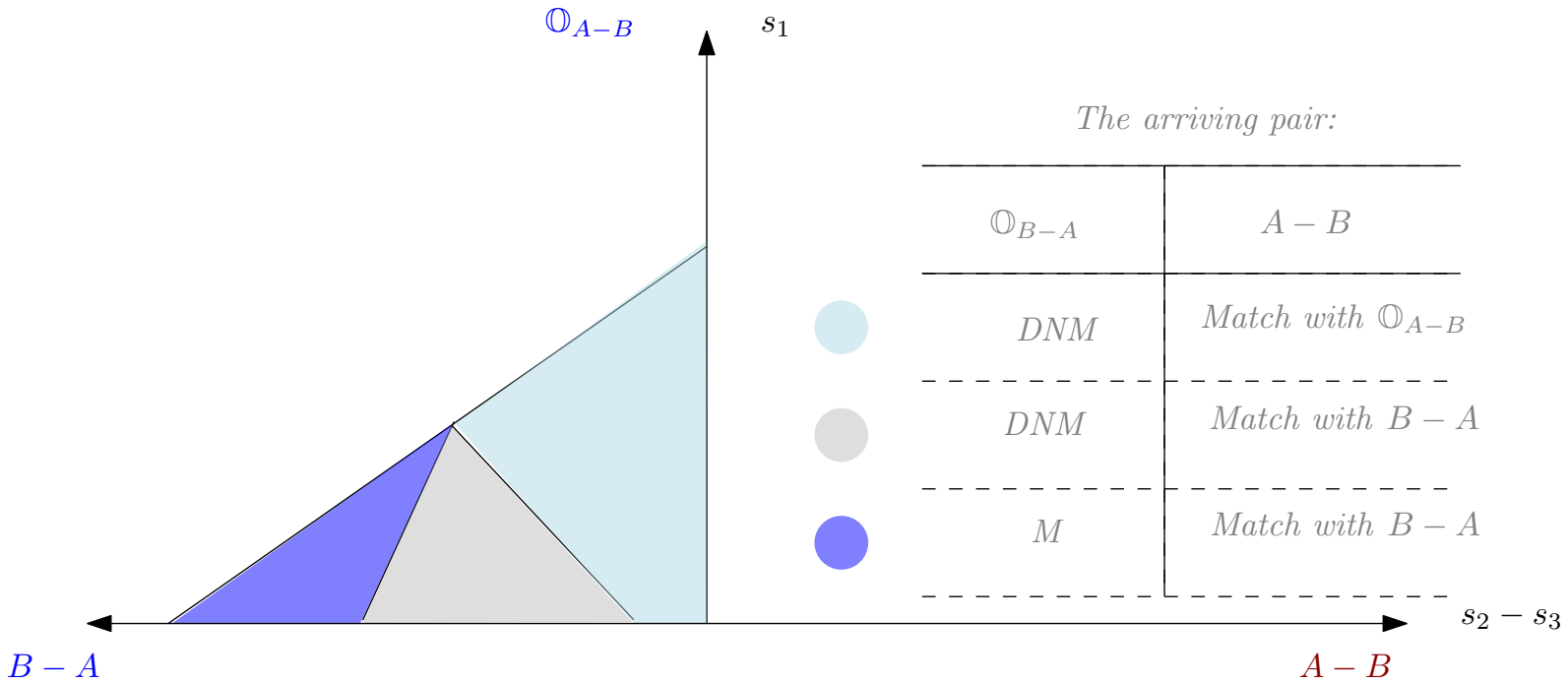


Observation 5

$s_1, s_3 > 0$: $V^*(s) - V^*(s-e_1) < 2$ and $V^*(s) - V^*(s-e_3) > 1$ $\Longrightarrow$ $V^*(s-e_3) - V^*(s-e_1) < 1$; both reversed $\Longrightarrow$ conclusion reversed.

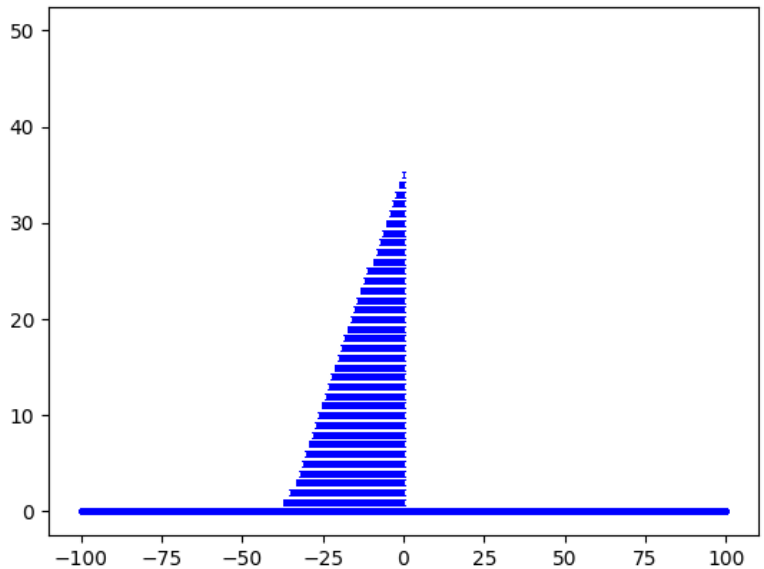
$\Longrightarrow$ t^2 passes through the intersection of the $t^{1,3}$ and t^4 boundaries. Above $t^{1,3}$: unreachable, drops out.

All Three Thresholds in One Picture



Keep vs. match an arriving $\mathbb{O}_{A-B}$, and 2-way of an existing $\mathbb{O}_{A-B}$ when a $B-A$ arrives: given by $\mathcal{R}$ (boundary $t^{1,3}$). The two colored regions: M vs. DNM for an $\mathbb{O}_{B-A}$ arrival (t^4), $\mathbb{O}_{A-B}$ vs. $B-A$ for an $A-B$ arrival (t^2).

Numerically

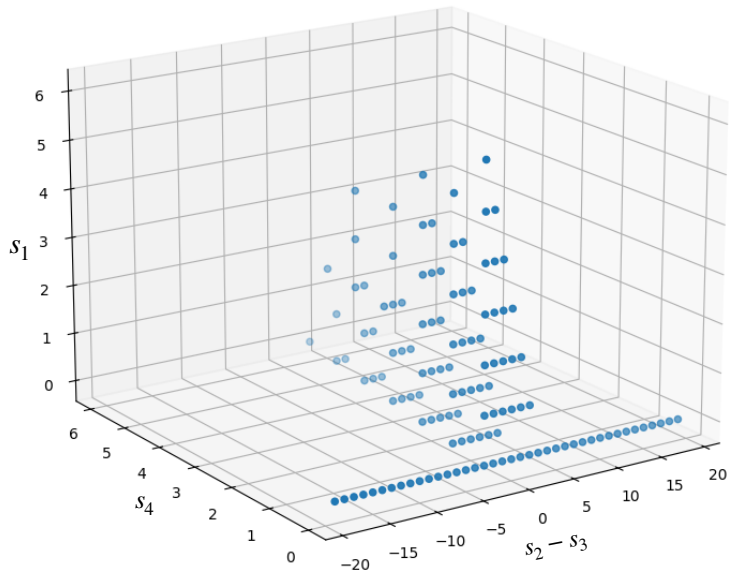


Reachable states, $p_{A-B} = 0.0635$, $p_{B-A} = 0.0381$, $p_{\mathbb{O}_{A-B}} = 0.00626$, $p_{\mathbb{O}_{B-A}} = 0.021$, $\delta = 0.999$. Horizontal axis $s_2 - s_3$, vertical s_1.

- Blue and gray regions of the previous picture are empty: with an $\mathbb{O}_{A-B}$ in the pool, incoming $A - B$ pairs are matched with it; the DNM region has all reachable states with $s_1 > 0$.
- So: the triangle $\mathcal{R}$ plus one number $\bar{s}$ for M vs. DNM. Numerical, no proof yet.
- Anti-greedy: thresholds above 30 pairs for both $B - A$ and $\mathbb{O}_{A-B}$ at realistic rates.

Skip the balanced case

Balanced Dynamic Exchange



Reachable states, $p_{A-B} = p_{B-A} = 0.0508$,
$p_{\mathbb{O}_{A-B}} = p_{\mathbb{O}_{B-A}} = 0.001346$, $\delta = 0.999$.
Symmetric in $s_2 - s_3 = 0$.

- $p_{A-B} \approx p_{B-A}$: $\mathbb{O}_{B-A}$ pairs are held too $\implies$ $\mathcal{R}$ is 3–d, a polyhedral region (the triangle was a slice).
- Up to three thresholds active at once, larger decision sets (organized by Thms. 4–6), thresholds $t : \mathbb{N}^2 \to \mathbb{N}$.
- Counter-intuitive: reachable states with $s_1 > 0$ and $s_2 > 0$. An $\mathbb{O}_{A-B}$ pair is held next to an $A - B$ pair it could match.

Why Hold an Overdemanded Pair?

Example. Only $A-B$ and $B-A$ pairs arrive, each w.p. $1/2$; one $\mathbb{O}_{A-B}$ pair in the pool at time 0.

- Match it 2-way now (surplus 2). If the next three arrivals are $A-B$: excess $A-B$ pairs wait long; the $\mathbb{O}_{A-B}$ could have taken one of them in a 3-way.
- Match it with the first $A-B$ (surplus 3). If three $B-A$ pairs come next: that $A-B$ was worth more matched with one of them.
- Hold it as a buffer. Release in a 3-way once $A-B$ pairs are in excess by some margin, 2-way once $B-A$ pairs are. Not dominated by either; this is what the numerical solution does.

Overdemanded pairs are a buffer against future imbalance: the "buffer zone" in the 3–d picture.

Balanced arrivals $\Longrightarrow$ imbalances are rare $\Longrightarrow$ small buffers; greedy is a good approximation (cf. Anderson et al., 2017). Dynamic optimization matters in the unbalanced case.

1 Introduction

2 Dynamic Model

3 Second-order Properties

4 The Propagation Puzzle and $\mathcal{D}^M$-Multimodularity

5 Structure of the Optimal Mechanism

6 The Optimal Kidney Exchange Mechanism

7 Concluding Remarks

Concluding Remarks

- Methodological contribution: $\mathcal{D}^M$-multimodularity — a closed family of second-order properties for dynamic matching with a local linear compatibility graph.
- Core technical insight: define T_M as a single joint maximization, not a composition of T_w's.
  $\Longrightarrow$ propagation of all required second-order properties *simultaneously*.
- Structure for any n: the sets K_i tell which exchanges an arrival can trigger; each at most once; monotone comparative statics.
- Economic structure: in the (empirically generic) unbalanced regime, the optimal mechanism is a multi-dimensional threshold mechanism with three monotone thresholds.
  Overdemanded pairs are not matched on arrival; thresholds above 30 pairs at realistic rates.
- Generality: framework extends to any dynamic matching with a local linear compatibility graph.

Appendix

Marginal Value Calculations

For the unbalanced regime, the threshold $t^{1,3}(s_3)$ is the largest s_1 such that

$$V^*\big((s_1, 0, s_3, 0)\big) - V^*\big((s_1 - 1, 0, s_3, 0)\big) > 2.$$

Concavity in $1 \implies$ such a threshold exists.
$\{1, 3\}$-submodularity $\implies$ $t^{1,3}$ is non-increasing in s_3.
Strong superconcavity $\implies$ $t^{1,3}(s_3 + 1) \geq t^{1,3}(s_3) - 1$.

Finite: with infinitely many $\mathbb{O}_{A-B}$ pairs an extra one has no future value, so its marginal value is at most 2.

Back