Two-Sided Matching via Balanced Exchange

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We introduce a new matching model to mimic two-sided exchange programs such as tuition and worker exchanges, in which export-import balances are required for longevity of programs. These exchanges use decentralized markets, making it difficult to achieve this goal. We introduce the two-sided top trading cycles, the unique mechanism that is balanced-efficient, worker-strategy-proof, acceptable, individually rational, and respecting priority bylaws regarding worker eligibility. Moreover, it encourages exchange, because full participation induces a dominant-strategy equilibrium for firms. We extend it to dynamic settings permitting tolerable yearly imbalances and demonstrate that its regular and tolerable versions perform considerably better than models of current practice.

I. Introduction

We introduce and model a new class of two-sided matching markets without explicit transfers, in which there is an additional fundamental con-

We thank John Duffy for initial discussions and Eric Budish, Michael Grubb, Scott Kominers, Robert Lay, Jacob Leshno, and Konrad Mierendorff for useful comments. We thank coeditor Ali Hortaçsu and three anonymous referees for comments. We thank Thayer Morrill for helping us in coding the simulation programs. We also acknowledge various comments we got from the audience during presentations at Texas, Rice, Zurich, Queen's Belfast, Microsoft Research at New England, Sabanci, North Carolina State, Frontiers of Market Design 2012, Spring Midwest Theory Conference 2012, Stony Brook Game Theory Conference 2012, Match-Up 2012, Games 2012, Social Choice and Welfare 2012, Texas The-

Electronically published April 10, 2019
[*Journal of Political Economy*, 2019, vol. 127, no. 3]
 0022-3808/2019/12703-0004$10.00

straint.[1] The eventual market outcome is linked to an initial status quo matching, which may give participants certain rights that constrain how future activity can play out. Since market outcome is typically different from the status quo, such activities loosely resemble an exchange in which one side of the agents are changing or acquiring new partners in addition to the two-sided matching market structure. In such markets, a fundamental balancedness condition needs to be sustained with respect to the status quo matching. The motivation for such a balancedness constraint can be different depending on the features of the market. Two such examples are labor and higher-education markets, where workers and colleges provide services to be compensated, respectively. In *worker exchange*, a worker needs to be replaced with a new one at her home firm so that this firm can function properly, and thus, the market needs to clear in a balanced manner. In *student exchange*, the college that is matched with an exchange student should be able to send out a student as well so that its education costs do not increase, and thus, the market needs to clear in a balanced manner. There are several prominent examples of such exchanges, such as national and international teacher exchange programs, clinical exchange programs for medical doctors, worker exchange programs within or across firms, and student exchange programs among colleges. This balancedness constraint induces preferences for firms/colleges not only over whom they get matched with (i.e., *import*) but also over whom they send out (i.e., *export*). The most basic kind of such preferences requires the firm/college to have a preference for balanced matchings, that is, for import and export numbers to be equal. We analyze our model over two explicit market applications: (permanent) tuition exchange and temporary worker exchange (see Sec. II for details).

In *tuition exchange*, the two sides are colleges and students. Each student who is a dependent of a faculty member at a college may attend another institution for free, if admitted as part of a tuition exchange program. Colleges have preferences over matchings. We assume only a weak structure for these preferences. Colleges' rankings over the incoming class are assumed to be responsive to their strict rankings over individual students.

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ory Workshop, INFORMS 2013, North American Econometric Society 2013, and Paris School of Economics and thank them. Ünver acknowledges the financial support of the National Science Foundation. The earlier drafts of this paper were entitled "Tuition Exchange." Simulated data are provided as supplementary material online.

[1] The theory and design of two-sided matching markets, such as entry-level labor markets for young professionals, online dating markets, or college admissions, have been one of the cornerstones of market design for more than 30 years (see Gale and Shapley 1962; Roth 1984; Roth and Peranson 1999; Hitsch, Hortaçsu, and Ariely 2010). Moreover, the theory of these markets has some important applications in allocation problems such as student placement and school choice (see Balinski and Sönmez 1999; Abdulkadiroğlu and Sönmez 2003).

Moreover, their preferences over matchings are determined through their rankings over the incoming class and how balanced the eventual matching is.[2] We start by showing, through a simple example, that individual rationality and nonwastefulness, standard concepts in two-sided matching markets, and balancedness are in general conflicting requirements (proposition 1). For this reason, we restrict our attention to the set of balanced-efficient mechanisms. Unfortunately, there exists no balanced-efficient and individually rational mechanism that is immune to preference manipulation for colleges (theorem 2).

We propose a new two-sided matching mechanism that is balanced-efficient, student-group-strategy-proof, acceptable, respecting internal priorities, individually rational, and immune to quota manipulation by colleges (theorems 1, 3, and 4).[3] We also show that it is the unique mechanism satisfying the first four properties (theorem 5). To our knowledge, this is one of the first papers using axiomatic characterization in the context of practical market design.

The outcome of this mechanism can be computed with a variant of David Gale's top trading cycles (TTC) algorithm (Shapley and Scarf 1974). In the school choice problem (Abdulkadiroğlu and Sönmez 2003) and the house allocation problem with existing agents (Abdulkadiroğlu and Sönmez 1999), variants related to Gale's TTC have been introduced and their properties have been extensively discussed (also see Pápai 2000). In all of these problems, one side of the market is considered to be objects to be consumed that are not included in the welfare analysis. Moreover, they are not strategic agents. In two-sided matching via exchange, in contrast to school choice and house allocation, both sides of the market are strategic agents and must be included in the welfare analysis. On the basis of these variants of Gale's TTC, we formulate our algorithm, and thus, we refer to the induced mechanism as the *two-sided top trading cycles* (2S-TTC). As far as we know, this is the first time a TTC-variant algorithm has been used to find the outcome of a two-sided matching mechanism.[4]

[2] We do not rule out colleges having more complex preferences over which students they send out.

[3] A mechanism *respects internal priorities* if, after a college increases the number of sponsored students, every student who was initially sponsored by that college is not hurt.

[4] Ma (1994) had previously characterized the core of a house exchange market, which can be found by Gale's TTC algorithm, when each house has a unit quota through Pareto efficiency, individual rationality, and strategy-proofness for students. Our characterization uses not only a proof technique different from that in Ma's study but also subsequent simpler proofs of this prior result by Sönmez (1995) and Svensson (1999). There are a few other TTC-related characterization results in the literature: Dur (2012), Morrill (2013), and Abdulkadiroğlu et al. (2017) characterize school choice TTC à la Abdulkadiroğlu and Sönmez (2003); Pycia and Ünver (2017) characterize general individually rational TTC

Although 2S-TTC is balanced-efficient, it may not match the maximum possible number of students while maintaining balance. We show that if the maximal-balanced solution is different from the 2S-TTC outcome for some preference profile, it can be manipulated by students (theorem 6).

Some tuition exchange programs require keeping a balance in a moving 3-year window for their member colleges. For this reason, we extend our model to a dynamic setting, where colleges can have tolerable yearly imbalances. We propose an extension, a *two-sided tolerable top trading cycle* (2S-TTTC) mechanism, which allows one to keep the imbalance of each college between some upper and lower bounds, and these bounds can be adjusted over the years. Once the bound-setting and adjustment processes are externally set, we show that 2S-TTTC keeps good properties of 2S-TTC: it is student-strategy-proof, acceptable, and respecting internal priorities; moreover, no acceptable matching within the balance limits can Pareto dominate its outcome.[5]

As the last part of our analysis of tuition exchange programs, in appendix I, we compare the performances of 2S-TTC and 2S-TTTC with that of the best-case scenario of the current practice of tuition exchange with a wide range of simulations.[6] By considering different degrees of correlation among students' and also colleges' preferences, and different yearly imbalance tolerance levels, we show that 2S-TTC and its variant match considerably more students to colleges and increase students' welfare over the naive student-proposing deferred acceptance outcome, the best-case scenario for the current market. This is a best-case scenario for the decentralized market as it minimizes coordination failures and

rules à la Pápai (2000) when there are more objects than agents; and Sönmez and Ünver (2010) characterize TTC rules à la Abdulkadiroğlu and Sönmez (1999) for house allocation with existing tenants. Kesten (2006) provides the necessary structure on the priority order to guarantee fairness of school choice TTC. Besides these characterizations, a mechanism related to ours was proposed by Ekici (2011) in an object allocation problem for temporary house exchanges with unit quotas.

[5] The closest in the literature to 2S-TTTC's algorithm is the top trading cycles and chains (TTCC) algorithm proposed by Roth, Sönmez, and Ünver (2004); however, the use and facilitations of "chains" are substantially different in this algorithm than in 2S-TTTC.

[6] We also develop a model of current semi-decentralized practice in tuition exchange in app. A. We show that balancedness is not in general achieved through decentralized market outcomes, jeopardizing the continuation and success of such markets. We define stability for particular externalities in college preferences. We show that stable matchings exist when colleges have plausible preferences over matchings (proposition 3). Moreover, proposition 4 implies that stability and balancedness are incompatible. Then we show that stability discourages exchange and can prevent the market from extracting the highest gains from exchange (see theorems 10 and 11).

ignores possible college incentives to underreport their certification quotas (see app. A).[7] Moreover, Combe, Tercieux, and Terrier (2016) conducted an empirical study using teacher assignment data from France using a model related to ours. Compared to deferred-acceptance-based current practice, they show that a TTC-based approach doubles the number of teachers moving from their initial assignment. Additionally, when the distribution of the ranks of teachers over the schools is considered, the outcome of the TTC-based approach stochastically dominates that of the current practice. Thus, there exist real-life settings in which our proposals can lead to significant welfare improvements.

We extend this model for temporary worker exchanges, such as teacher exchange programs. We tweak our model slightly and assume that the quotas of the firms are fixed at the number of their current employees, and, hence, firms would like to replace each agent who leaves. We also assume that firm preferences are coarser than those of colleges in tuition exchange because of the temporary nature of the exchanges. We assume they have weakly size-monotonic preferences over workers: larger groups of acceptable workers are weakly better than weakly smaller groups of acceptable workers when the balance of the matching with larger groups of acceptable workers is zero and the balance of the matching with smaller groups of worker is nonpositive.[8] In this model, we prove that 2S-TTC not only carries all of its previous properties through but also is strategy-proof for the firms, making it a very viable candidate (theorem 9). Our aforementioned characterization also holds in this model.

II. Applications

A. *Tuition Exchange*

Some of the best-documented matching markets with a balancedness requirement are tuition exchange programs in the United States. These are semi-decentralized markets, and some have failed over the years because of problems related to unbalanced matching activity.

It has been difficult for small colleges and universities to compete with bigger schools in trying to hire the best and brightest faculty. Colleges located farther away from major metropolitan areas face a similar chal-

[7] It should be noted that there could be other market structures not governed by our simulation generating distributions such that the results we find do not hold. Thus, these simulations are examples of domains in which 2S-TTC or its variant dominates the best-case outcomes of decentralized markets under a vast majority of parameters.

[8] Weakly size-monotonic preferences are weaker than dichotomous preferences (in the absence of externalities), which are widely used in the matching literature; see, e.g., Bogolomania and Moulin (2004), Roth, Sönmez, and Ünver (2005, 2007), Ekici (2011), and Sönmez and Ünver (2014).

lenge. Tuition exchange programs play a prominent role for these colleges in attracting and retaining highly qualified faculty.[9]

Many colleges give tuition waivers to qualified dependents of faculty. Through a tuition exchange program, they can use these waivers at other colleges and attend these colleges for free. The dependent must be admitted to the other college. Tuition exchange has become a desirable benefit that adds value to an attractive employment package without creating additional out-of-pocket expenses for colleges; that is, colleges do not transfer money to each other for accepting their faculty's dependents.

One of the prominent programs is The Tuition Exchange, Inc. (TTEI), which is also the oldest and largest of its kind.[10]

Each participating college to TTEI establishes its own policies and procedures for determining the eligibility of dependents for exchange and the number of scholarships it will grant each year. Each member college has agreed to maintain a balance between the number of students sponsored by that institution ("exports") and the number of scholarships awarded to students sponsored by other member colleges ("imports"). Colleges aim to maintain a one-to-one balance between the number of exports and the number of imports. In particular, if the number of exports exceeds the number of imports, then that college may be suspended from the tuition exchange program.[11] Colleges often set the maximum number of sponsored students in a precautionary manner. Many colleges explicitly mention in their application documents that in order to guarantee their continuation in the program, they need to limit the number of sponsored students.[12] As a result, in many cases not all qualified dependents are sponsored.

A tuition exchange program usually functions as follows: each college determines its quotas, which are the maximum number of students it will sponsor (its "eligibility quota") and the maximum number it will admit (its "import quota") through the program. Then the eligible students apply to colleges, and colleges make scholarship decisions based on preferences and quotas. A student can get multiple offers. She declines all but

[9] "Tuition Exchange enables us to compete with the many larger institutions in our area for talented faculty and staff. The generous awards help us attract and retain employees, especially in high-demand fields like nursing and IT" (Frank Greco, director of Human Resources, Chatham University, from the home page of The Tuition Exchange, Inc., http://www.tuitionexchange.org). Also see app. C about the results of a survey that we conducted detailing the importance of tuition exchange programs in job choice for faculty members.

[10] See http://www.tuitionexchange.org. Through TTEI, 7,000 scholarships were awarded in 2015–16, with an annual value of $34,000 per scholarship that is paid as a tuition reduction. Despite TTEI's large volume, other tuition exchange programs clear a significant number of all exchange transactions in the United States. In app. C, we describe the features of prominent tuition exchange programs.

[11] See http://www.sxu.edu/admissions/financial_aid/exchange.asp.

[12] Lafayette College, Daemen College, DePaul University, and Lewis University are just a few examples.

one, and, if possible, further scholarship offers are made in a few additional rounds. Students who are not sponsored cannot participate in the program and hence do not receive a tuition exchange scholarship. The admitting institution de facto awards a tuition waiver to the dependent of the faculty of another college.

B. *Temporary Worker Exchanges*

The balancedness requirement also matters in temporary worker exchange programs (such as those for teachers, students, academic staff, and medical doctors). The Commonwealth Teacher Exchange Programme (CTEP), Fulbright Teacher Exchange Program, Erasmus Student Exchange Program, and the exchange program of the International Federation of Medical Students' Associations (IFMSA) are just a few examples.[13] Some of these have been running for decades,[14] and thousands of participants benefit from these worker exchange programs annually. Every year more than 10,000 medical students and 200,000 college students around the world participate in IFMSA's and Erasmus's exchanges, respectively.[15] The main difference between these programs and tuition exchange is that (1) most exchange appointments are temporary, typically lasting 1 year, and (2) the workers are currently employed by their associated firms, so if they cannot be exchanged, they will continue to work in their current jobs.

C. *Importance of Balancedness Requirement*

Although two-sided matching via exchange induces a two-sided matching market, workers (students) cannot participate in market activity unless their home firms (colleges) sponsor them. Hence, an import/export balance emerges as an important feature of sustainable outcomes, as there are no monetary transfers between parties and there are costs for colleges associated with providing students. Balance requirements are the most important feature of these markets that distinguishes them from the previously studied matching markets. We illustrate three cases in which the absence of a balanced exchange led to the failure of the exchange program in different contexts.[16]

[13] There are also small bilateral staff exchange programs. See app. D for details.

[14] CTEP, which allows participants to exchange teaching positions and homes with a colleague from the United Kingdom, Australia, or Canada, has been running for 100 years. See http://www.cyec.org.uk/exchanges/commonwealth-teacher-exchange.

[15] See http://ifmsa.org/professional-exchanges/ and http://europa.eu/rapid/press-release_IP-13-657_en.htm.

[16] When we talk about balancedness in this paper, we are not strictly talking about zero-balance conditions in which imports and exports even each other out. The idea can also be relaxed in static and dynamic manners to attain an approximate balance over time. In-

The Northwest Independent Colleges Tuition Exchange program was founded in 1982 and included five members. In contrast to TTEI, the colleges were not able to limit their exports. Because of sizable imbalances between imports and exports, members agreed to dissolve the program, and it stopped accepting new applicants after fall 2015 in its current form.[17] The Jesuit universities exchange program FACHEX is another one that is adversely affected. The program still does not have an explicitly embedded balancedness requirement. It includes all Jesuit universities but Georgetown, which is arguably the most prominent one.

The Erasmus student exchange program among universities in Europe is another example of a market in which a lack of balancedness has caused some exchange relationships to be terminated. Member colleges that want to exchange students with each other sign bilateral contracts that set the maximum number of students to be exchanged in certain years. The renewal decision of the contract depends on whether a reasonable balance is maintained between the incoming and outgoing exchange students between these colleges. In particular, if one of the colleges has more incoming students than outgoing students, then that college might not renew the contract.

Tuition and worker exchange markets are closely related to favor markets, also known as "time banks," where time spent doing a favor or the number of favors is used as the currency of exchange. Holding of the transaction currency in such markets corresponds to a positive imbalance in our model. If not enough currency is injected initially into the system and there is too much uncertainty, agents may shy away from using their currency. Babysitting co-ops are an example of such time banks. Such banks could be adversely affected by the lack of balanced clearing mechanisms that clear all favors through a well-defined schedule-matching scheme.[18]

III. Two-Sided Matching Via Exchange: Model

Let C and S be the finite sets of colleges and students, respectively.[19] Set S is partitioned into $|C|$ disjoint sets, that is, $S = \cup_{c\in C} S_c$, where S_c is the set

deed, there could be gains for intertemporal trades, and our proposals also address these issues in Sec. IV.B.

[17] See https://www.insidehighered.com/news/2012/02/15/tuition-exchange-program-northwest-colleges-coming-end.

[18] In the mid-1970s, at the Capitol Hill Baby-Sitting Coop in Washington, DC, negative-balance aversion of families resulted in imbalances between families and decreased the number of favor exchanges between families. For details see http://www.ft.com/cms/s/2/f74da156-ba70-11e1-aa8d-00144feabdc0.html. This fits our setting perfectly: if the matches could be done in a monthly schedule using a centralized method, then balancedness requirements could be easily addressed.

[19] We will keep tuition exchange in mind in naming our concepts. The minor differences in the temporary worker exchange model will be highlighted in Sec. V.

of students who are applying to be sponsored by $c \in C$. Let $q = (q_c)_{c\in C} \in \mathbb{N}^{|C|}$ be the *(scholarship) admission quota* vector, where q_c is the maximum number of students who will be admitted by c with tuition exchange scholarships, and $e = (e_c)_{c\in C} \in \mathbb{N}^{|C|}$ is the *(scholarship) eligibility quota* vector, where $e_c \leq |S_c|$ is the number of students in S_c certified as eligible students by c. Let $\rhd_C = (\rhd_c)_{c\in C}$ be the list of college *internal priority orders,* where $\rhd_c$ is a linear order over S_c based on some exogenous rule. We define the set of *eligible students* of c as $E_c = \{s \in S_c | r_c(s) \leq e_c\}$, where $r_c(s)$ is the rank of $s \in S_c$ under $\rhd_c$. Let $E = \cup_{c\in C} E_c$. The being unassigned option, named the *null college*, is denoted by $c_\varnothing$, and its quota is set as $q_{c_\varnothing} = |S|$.

To define the preferences properly, we define an auxiliary concept first: An *unconstrained matching* is a correspondence $\lambda : C \cup S \twoheadrightarrow C \cup S \cup c_\varnothing$ such that (i) $\lambda(c) \subseteq S$ for all $c \in C$; (ii) $\lambda(s) \subseteq C \cup c_\varnothing$, where $|\lambda(s)| = 1$ for all $s \in S$; and (iii) $s \in \lambda(c)$ if and only if $\lambda(s) = c$ for all $c \in C$ and $s \in S$.[20] An outcome is a *matching*, which is an unconstrained matching μ satisfying $|\mu(c)| \leq q_c$ for all $c \in C$, and $\mu(s) = c_\varnothing$ for all $s \notin E$.[21]

Let $\mathcal{M}^u$ and $\mathcal{M}$ be the sets of unconstrained matchings and matchings, respectively. Given a fixed set of colleges C and students S, the set of unconstrained matchings is fixed across different admission and eligibility quotas, while the sets of matchings will change. Let $X_c^\mu = \{s \in S_c | \mu(s) \in C \setminus c\}$ be the *set of exports* for c in $\mu \in \mathcal{M}^u$.[22] Let $M_c^\mu = \{s \in S \setminus S_c | \mu(s) = c\}$ be the *set of imports* for c in $\mu \in \mathcal{M}^u$. We refer to $b_c^\mu \in \mathbb{Z}$ as the *net balance* of $c \in C$ in μ and define it as $b_c^\mu = |M_c^\mu| - |X_c^\mu|$. We say $c \in C$ has a zero (*negative*) [*positive*] net balance in μ if $b_c^\mu = 0$ ($b_c^\mu < 0$) [$b_c^\mu > 0$].

Let $\succsim = (\succsim_S, \succsim_C) = ((\succsim_s)_{s\in S}, (\succsim_c)_{c\in C})$ be the list of student and college preferences over unconstrained matchings, where $\succsim_i$ is the preference relation of agent $i \in S \cup C$. We denote the strict preference of $i \in S \cup C$ by $\succ_i$ and her indifference relation by $\sim_i$.

Each $s \in S$ cares only about her own match in an unconstrained matching and has a strict preference relation P_s on $C \cup c_\varnothing$. Let R_s denote the at-least-as-good-as relation associated with P_s for any student $s \in S$: cR_sc' if cP_sc' or $c = c'$ for all $c, c' \in C \cup c_\varnothing$. Student s's preference over unconstrained matchings $\succsim_s$ is defined as follows: if $\mu(s) R_s \mu'(s)$, then $\mu \succsim_s \mu'$.

Each college potentially cares not only about its admitted class of (scholarship) students but also about its net balance. Each preference relation for a college is related to some strict ranking over sets of admitted

[20] We may refer to singleton $\{x\}$ as x with a slight abuse of notation. The only exception is $\{\varnothing\}$.

[21] In tuition exchange, only the students who are certified eligible can be assigned to other institutions. Therefore, if s is not certified eligible, i.e., if $s \in S \setminus E$, then she will be assigned to the null college.

[22] When we say $s \in S$ is matched to $c \in C$, we mean that s receives a tuition exchange scholarship from c.

students, that is, subsets of S. Given a college c and preference $\succsim_c$, suppose that P_c^* is this ranking. In turn, P_c^* is responsively induced through a linear order P_c over $S \cup \emptyset$. The ranking P_c^* is *responsive* to P_c if for all $T \subseteq S$ and $s, s' \in S \setminus T$, (1) $sP_c\emptyset \Leftrightarrow (T \cup s)P_c^* T$ and (2) $sP_cs' \Rightarrow (T \cup s)P_c^*(T \cup s')$.[23] Note that P_c^* is not the preference relation of c, but is a ranking over sets of admitted students. Let R_c^* be the weak ranking over the subsets of students induced by P_c^*. Throughout the paper, the relationship between preferences of c and this ranking is assumed as follows: between any two unconstrained matchings in which c has the same net balance, it prefers the one with the higher-ranked set of admitted students according to P_c^*, that is, for any $\mu, \nu \in \mathcal{M}^u$, if $b_c^\mu = b_c^\nu$ and $\mu(c)R_c^*\nu(c)$, then $\mu \succsim_c \nu$. The domain of preferences for c includes all such possible preferences $\succsim_c$.

Throughout the paper, C, S, and $\rhd_C$ are fixed; a quota vector, an eligibility vector, and a preference profile define a *tuition exchange market*—or simply a *market*—as $[q, e, \succsim]$.

We now introduce the properties of desirable matchings in a given market. A matching μ *Pareto dominates* $\nu \in \mathcal{M}$ if $\mu \succsim_i \nu$ for all $i \in C \cup S$ and $\mu \succ_j \nu$ for some $j \in C \cup S$. A matching μ is *Pareto efficient* if it is not Pareto dominated by any other $\nu \in \mathcal{M}$. A student s is acceptable for a college c if $sP_c\emptyset$ and c is acceptable for s if $cP_sc_\emptyset$. A matching μ is *acceptable* if it matches every agent with only acceptable partners. A matching μ is *balanced* if $b_c^\mu = 0$ for all $c \in C$.[24] Balancedness is the key property in tuition exchange. We say a balanced matching μ is *balanced-efficient* if it is not Pareto dominated by any other balanced matching.

We say $\mu \in \mathcal{M}$ is *blocked by a college* c if there exists some $\mu' \in \mathcal{M}$ such that $\mu' \succ_c \mu$, $\mu'(s) = \mu(s)$ for all $s \in S \setminus \mu(c)$, and $\mu'(c) \subset \mu(c)$. A matching μ is *blocked by a student* s if $c_\emptyset P_s \mu(c)$. A matching μ is *individually rational* if it is not blocked by any individual agent. A matching μ is *nonwasteful* if there does not exist a college-student pair (c, s) such that $|\mu(c)| < q_c$, $cP_s\mu(s)$ and $\mu' \succ_c \mu$ for some matching μ', where $\mu'(s) = c$ and $\mu'(s') = \mu(s')$ for all $s' \in S \setminus s$. In appendix A, we provide an analysis of the decentralized practice of tuition exchange and "stability," defined there for a market with externalities.

Tuition exchange mechanisms.—The current practice of tuition exchange is implemented through indirect semi-decentralized market mechanisms. Although our new proposal can also be implemented indirectly, it will be useful to discuss it as a direct mechanism to analyze its properties. A (*direct*) *mechanism* is a systematic way of selecting a matching for each market. Let φ be a mechanism; then the matching selected by φ in market $[q, e, \succsim]$ is denoted by $\varphi[q, e, \succsim]$, and the assignment of agent

[23] In the literature, property 1 is originally referred to as separability and 2 is the original responsiveness condition due to Roth (1985). We refer to the collection of both as responsiveness.

[24] Note that $b_c^\mu \geq 0$ for all $c \in C$ or $b_c^\mu \leq 0$ for all $c \in C$ each implies $b_c^\mu = 0$ for all $c \in C$.

$i \in S \cup C$ is denoted by $\varphi[q, e, \succsim](i)$. A mechanism satisfies a specific property (e.g., balanced efficiency) if its outcome for any market satisfies this property.

In a revelation game, students and colleges report their preferences; additionally, colleges report their admission and eligibility quotas.[25] A mechanism φ is *immune to preference manipulation for students* (or *colleges)* if for all $[q, e, \succsim]$, there exist no $i \in S$ (or $i \in C$) and $\succsim'_i$ such that $\varphi[q, e, (\succsim'_i, \succsim_{-i})](i) \succ_i \varphi[q, e, \succsim](i)$. A mechanism φ is *immune to preference manipulation* if it is immune to preference manipulation for both students and colleges. A mechanism φ is *immune to quota manipulation* if for all $[q, e, \succsim]$, there exist no $c \in C$ and (q'_c, e'_c) with $q'_c \leq q_c$ such that $\varphi[(q'_c, q_{-c}), (e'_c, e_{-c}), \succsim](c) \succ_c \varphi[q, e, \succsim](c)$. A mechanism φ is *strategy-proof for colleges* if for all $[q, e, \succsim]$, there exist no $c \in C$ and $(q'_c, e'_c, \succsim'_c)$ with $q'_c \leq q_c$ such that $\varphi[(q'_c, q_{-c}), (e'_c, e_{-c}), (\succsim'_c, \succsim_{-c})](c) \succ_c \varphi[q, e, \succsim](c)$. A mechanism is *strategy-proof for students* if it is immune to preference manipulation for students. A mechanism is *strategy-proof* if it is strategy-proof for both colleges and students.[26] A mechanism φ is *group strategy-proof for students* if for all $[q, e, \succsim]$, there exist no $S' \subseteq S$ and $\succsim'_{S'} = (\succsim'_s)_{s \in S'}$ such that $\varphi[q, e, (\succsim'_{S'}, \succsim_{-S'})](s) \succsim_s \varphi[q, e, \succsim](s)$ for all $s \in S'$ and $\varphi[q, e, (\succsim'_{S'}, \succsim_{-S'})](s') \succ_{s'} \varphi[q, e, \succsim](s')$ for some $s' \in S'$.

One distinctive feature of tuition exchange is the existence of internal priorities for each $c \in C$, $\rhd_c$. The internal priority order is used to determine which students will be certified eligible. This priority order is usually based on the seniority of faculty members. We incorporate this priority-based fairness objective into our model by introducing a new property. Formally, a mechanism φ *respects internal priorities* if whenever a student $s \in S_c$ is assigned to a college in market $[q, e, \succsim]$, then s is assigned to a weakly better college in $[q, (\tilde{e}_c, e_{-c}), \succsim]$, where $\tilde{e}_c > e_c$.[27] Respect for internal priorities is a fairness notion rather than efficiency.

IV. Two-Sided Top Trading Cycles

In this section, we propose a mechanism that is individually rational, acceptable, balanced-efficient, and strategy-proof for students. Moreover,

[25] Since the internal priority order is exogenous, the set of eligible students can be determined by the eligibility quota.

[26] Since students care only about the colleges they are matched with, it will be sufficient for them to report their preferences over colleges. Under an additional assumption, our proposal in Sec. IV can also be implemented by having colleges report individual students as only "acceptable" or "unacceptable."

[27] This property is used in our characterization in Sec. IV, where we show that this axiom does not bring additional cost to our proposed mechanism (theorem 6). Moreover, this property can be weakened as follows at no cost: a sponsored student who is matched to a college better than her outside option continues to be matched with a (possibly different) college better than her outside option when her home college increases the number of its sponsored students. The outside option for tuition exchange and worker exchange is null college and home firm, respectively.

it respects colleges' internal priorities. Throughout our analysis, we impose a weak restriction on college preferences. Assumption 1 states that a college prefers a better scholarship class with zero net balance to an inferior scholarship class with a nonpositive net balance.

ASSUMPTION 1. For any $\mu, \nu \in \mathcal{M}^u$ and $c \in C$, if $b_c^\mu = 0$, $b_c^\nu \le 0$, and $\mu(c) P_c^* \nu(c)$, then $\mu \succ_c \nu$.

We start with the following proposition, which shows the incompatibility between balancedness and individual rationality, and nonwastefulness.

PROPOSITION 1. Under assumption 1, there may not exist an individually rational and nonwasteful matching that is also balanced.[28]

It will be useful to denote a matching as a directed graph, as we will find the outcome of our mechanism through an algorithm over directed graphs. In such graphs, colleges and students are nodes; a directed edge is between a college and a student, and it points to either the college or the student, but not both. Given a matching μ, let each $s \in S$ point to $\mu(s)$ and each $c \in C$ point to all its matched students, that is, those in $S_c \setminus \mu(c_\varnothing)$; moreover, let $c_\varnothing$ point to students assigned to it. In this graph, we define the following subgraph: A *trading cycle* consists of an ordered list of agents $(c_1, s_1, c_2, s_2, \dots, c_k, s_k)$ such that c_1 points to s_1, s_1 points to $c_2, \dots, c_k$ points to s_k, and s_k points to c_1.

In the following remark, we state that if a matching is balanced, then we can decompose it into a finite number of trading cycles. We skip its proof for brevity.

REMARK 1. A matching μ is balanced if and only if each student is in a trading cycle in the graph of the matching.

We are ready to propose a new two-sided matching mechanism. We will find its outcome using an algorithm inspired by top trading cycles (TTC) introduced for one-sided resource allocation problems, such as for school choice (by Abdulkadiroğlu and Sönmez 2003) and dormitory room allocation (by Abdulkadiroğlu and Sönmez 1999). These TTC algorithms were inspired by Gale's TTC algorithm (Shapley and Scarf 1974), which was used to find the core allocation of a simple exchange economy, referred to as the *housing market*, a subclass of one-sided matching problems. Most common mechanisms in one-sided matching problems function through algorithms that mimic agents exchanging objects that are initially allocated to them either through individual property rights or through the mechanism's definition of the agents (see also Pápai 2000; Pycia and Ünver 2017). In contrast, in our market, college slots are not objects. Therefore, our definition of a mechanism and the properties of matchings and mechanisms (except strategy-proofness for students) do not have any analogous translation in such problems. However,

[28] All proofs are in app. B.

because we use a variant of the TTC algorithm to find the outcome, we refer to our mechanism as *two-sided (student-pointing) top trading cycles* (2S-TTC). Its outcome is found for any given $[q, e, \succsim]$ as follows:[29]

2S-TTC ALGORITHM.

Round 0. Assign two counters, for admission and eligibility, for each college $c \in C$, and set them equal to q_c and e_c, respectively.

Round $k \geq 1$. Each available student points to her favorite among available colleges, which consider her acceptable, and $c_\varnothing$. Each available college c points to the highest-priority available student in S_c according to $\rhd_c$. Null college $c_\varnothing$ points to all students pointing to it. Owing to the finiteness of C and S, there exists at least one cycle. Each agent can be part of at most one cycle. Every student in each cycle is assigned a seat at the option she is pointing to and removed. If the cycle does not contain $c_\varnothing$, then the counters of each college in that cycle are reduced by one. If the cycle contains $c_\varnothing$ and an eligible student from an available college $c \in C$, then we reduce only the eligibility counter of c by one. If any counter of a college reaches zero, then that college is removed.

The algorithm terminates when all students are removed.

In theorem 1, we show that 2S-TTC is balanced-efficient, acceptable, and individually rational, and it respects internal priorities.

THEOREM 1. Under assumption 1, 2S-TTC is an individually rational, balanced-efficient, and acceptable mechanism that also respects internal priorities.

It should be noted that balanced efficiency of 2S-TTC is not directly implied by (Pareto) efficiency of TTC in a one-sided market. Here, colleges are players with multiple seats. Observe that by assigning a college at least one highly preferred student and some unacceptable ones, some acceptable, individually rational, and balanced matchings can potentially be (weakly) improved for everyone while keeping balancedness intact (and even the number of students who are assigned to a college can go up). In this theorem, through an iterative approach, we show that it is

[29] The converse of this process, using an algorithm originally introduced for two-sided matching markets in one-sided matching markets, has already been utilized in market design. For certain real-life one-sided problems regarding student placement and school choice, Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003) introduced the student-optimal stable mechanism, whose algorithm was originally introduced to find stable matchings in two-sided matching markets by Gale and Shapley (1962). Later on, many school districts in the United States adopted this mechanism for public school admissions (see Abdulkadiroğlu, Pathak, and Roth 2005; Abdulkadiroğlu et al. 2005).

not possible to improve over 2S-TTC's outcome in such a fashion. Also consider the following concept: For any $I \subseteq S \cup C$, a balanced matching is *balanced-efficient for I* if there is no other balanced matching that makes each agent in I weakly better off and at least one agent in I strictly better off. The 2S-TTC mechanism is neither balanced-efficient for students nor balanced-efficient for colleges. However, it is balanced-efficient overall (when all agents' welfare is taken into account). Thus, it is a compromise between the welfare of both sides, slightly favoring students by construction. Since side-balanced efficiency is not satisfied by 2S-TTC in general (even under strict preferences), we need a new proof to prove its overall balanced efficiency. To illustrate that 2S-TTC is not balanced-efficient for any side, we provide a simple example (example 2) in appendix G and further explanation regarding why previously known results do not immediately imply our efficiency result.

Under a centralized mechanism, incentives for participants to truthfully reveal their preferences are desirable. Unfortunately, we show that balanced efficiency, individual rationality, and immunity to preference manipulation for colleges are incompatible.

Theorem 2. There does not exist an individually rational (or acceptable) and balanced-efficient mechanism that is also immune to preference manipulation for colleges, even under assumption 1.

We prove this theorem by constructing several markets and showing that it is not possible to satisfy all three properties in one of these markets.

Theorems 1 and 2 imply that the 2S-TTC mechanism is not strategy-proof for colleges.

The following theorem shows that it is group strategy-proof for students. This result is a consequence of TTC being group strategy-proof in a housing market (see Pápai 2000).

Theorem 3. 2S-TTC is group strategy-proof for students.

The 2S-TTC mechanism can be run as an indirect mechanism in which colleges report only their acceptable incoming students. Hence, the strategy space for the colleges is very simple in using 2S-TTC in the field: their strategy is to report their admission and eligibility quotas and their set of acceptable students based on their preferences, set of own students, and internal priority order.

Moreover, if we focus on the game played by the tuition exchange offices of colleges, when admissions preferences are fixed, truthful admission quota revelation and certification of all their own students induce a dominant-strategy equilibrium under 2S-TTC.[30]

[30] On their websites, colleges explain that the main reason for certifying a limited number of students is maintaining a balanced exchange. The 2S-TTC mechanism removes the need for this rightful caution associated with the current market practices (see app. A).

Theorem 4. Under assumption 1 and when true eligibility quotas satisfy $e_c = |S_c|$ for all $c \in C$, 2S-TTC is immune to quota manipulation.

We prove the theorem with a lemma showing that as the quotas of a college increase, the import and export sets and the admitted class of students of this college also (weakly) expand under 2S-TTC.[31]

Theorems 3 and 4 point out that only colleges can benefit from manipulation, and they can manipulate by misreporting their preferences. Moreover, the only way to manipulate preferences is to report an acceptable student as unacceptable. Suppose we take all the admitted students in the regular admission procedure as acceptable for a tuition exchange scholarship. Then, to manipulate 2S-TTC, a college needs to reject a student who satisfies the college admission requirements. Usually college admission decisions are made before the applicants are considered for scholarships.[32]

Proposition 2 implies that colleges do not benefit from misreporting their ranking over incoming classes.

Proposition 2. Under assumption 1, colleges are indifferent among strategies that report preferences in which the same set of students is acceptable with the same quota report under the 2S-TTC mechanism.

We have shown that 2S-TTC has appealing properties. In the following theorem, we show that it is the unique mechanism satisfying a subset of these properties.

Theorem 5. Under assumption 1, 2S-TTC is the unique student-strategy-proof, acceptable, and balanced-efficient mechanism that also respects internal priorities.

In the proof of our characterization theorem, we use a technique different from what is usually employed in elegant single quota characterization proofs such as Sönmez (1995) and Svensson (1999) for the result of Ma (1994). Our proof relies on building a contradiction with the claim that another mechanism with the four properties in the theorem's hypothesis can exist. Suppose such a mechanism exists and finds a matching different from that of 2S-TTC for some market. The 2S-TTC algorithm runs in rounds in which trading cycles are constructed and re-

[31] Theorem 4 is in stark contrast with similar results in the literature for stable mechanisms. The student- and college-optimal stable mechanisms are prone to admission quota manipulation by the colleges even under responsive preferences (see Sönmez 1997; Konishi and Ünver 2006). Thus, 2S-TTC presents a robust remedy for a common problem seen in centralized admissions that use the student-optimal stable mechanism and also in tuition exchange in a decentralized market (see theorems 10 and 11 in app. A).

[32] Our proposal also prevents some other manipulation possibilities. For example, right now if a college really likes one of its own students, then it may decrease its export quota preventing this student from being eligible, and the student, in the end, attends her home college through tuition remission. However, in our proposal a college's export quota also determines the set of its own students who are eligible for tuition remission. Thus, no ineligible student can attend her home college through tuition remission. We think that tuition exchange and tuition remission programs should be run together (see Sec. IV.A).

moved. Suppose $S(k)$ is the set of students removed in round k, while running 2S-TTC in such a way that in each round only one arbitrarily chosen cycle is removed and all other cycles are kept intact. We find a round k and construct an auxiliary market with the following three properties: (1) Eligibility quotas of home colleges of students in $S(k)$ are set such that these are the last certified students in their respective home institutions; (2) all preferences are kept intact except those of students in $S(k)$, whose preferences are truncated after their 2S-TTC assignments; and (3) all students in $S(k)$ are assigned $c_{\varnothing}$ under the alternative mechanism, while all students removed in the 2S-TTC algorithm before round k have the same assignment under 2S-TTC and the alternative mechanism. This contradicts the balanced efficiency of the alternative mechanism: we could give the students in $S(k)$ their 2S-TTC assignments while keeping all other assignments intact and obtain a Pareto-dominating balanced matching.

Among all the axioms, only the respect for internal priorities is based on exogenous rules. One might suspect that more students will benefit from the tuition exchange program if we allow the violation of respect for internal priorities. However, such mechanisms turn out to be manipulable by students.

Theorem 6. Any balanced and individually rational mechanism that does not assign fewer students than 2S-TTC and selects a matching in which more students are assigned whenever such a balanced and individually rational outcome exists is not strategy-proof for students, even under assumption 1.

A. *Market Implementation: Tuition Remission and Exchange*

Incorporating tuition remission programs by all participating colleges in tuition exchange is the best way to implement a centralized clearinghouse. If parallel remission and exchange programs are run, as in current practice, a student may receive multiple scholarship offers, one from her home college and one from the tuition exchange program. If the student accepts the home college's offer, the net balance of the college may deteriorate.

Although the current system is inflexible in accommodating this important detail, a clearinghouse utilizing 2S-TTC can easily combine tuition exchange with remission. Indeed, in assumption 1, we allowed a college to deem its own sponsored students to be acceptable. Hence, all our results in this section are robust to integration.

More specifically, we propose to run an indirect version of 2S-TTC in sequential stages in a semi-decentralized fashion: first, colleges announce their tuition exchange scholarship quotas and which of their students are eligible to be sponsored for both exchange and remission;

then eligible students apply for scholarships to the colleges they find acceptable; then colleges send out scholarship admission letters. At this stage, as students have also learned their opportunities in the parallel-running regular college admissions market, they can form better opinions about the relative ranking of the null college, that is, their options outside the tuition exchange market. Students submit rankings over the colleges that admitted them with a tuition exchange scholarship and the relative ranking of their outside option. Finally, 2S-TTC is run centrally to determine the final allocation.

B. *Allowing Tolerable Imbalances*

Some programs care about approximate balance over a moving time window. Here, we relax the zero-balance constraint and allow each $c \in C$ to maintain a balance within an interval $[\ell_c, u_c]$, where $\ell_c \leq 0 \leq u_c$.[33] When either ℓ_c or u_c equals zero for all $c \in C$, the market turns into the case studied in Section IV. Let $(\ell_c, u_c)_{c \in C}$ be the tolerance profile.

When the colleges hold a nonzero balance, then there may exist some colleges exporting (importing) more than they import (export). Then we cannot represent all allocations by cycles. Therefore, we need to consider chains in addition to the cycles. A *chain* is an ordered list $(c_1, s_1, c_2, s_2, \dots, c_k)$ such that c_1 points to s_1, s_1 points to c_2, ..., c_{k-1} points to s_{k-1}, and s_{k-1} points to c_k. We refer to c_1 as the tail and c_k as the head of the chain.

We use a mechanism similar to 2S-TTC referred as the *two-sided tolerable top trading cycles*. For any market and tolerance profile, its outcome is found as follows:

2S-TTTC Algorithm.

Step 0. Fix an exogenous priority order among colleges. Assign two counters for each $c \in C$, o_c^q, and o_c^e, and set them equal to q_c and e_c, respectively. Let b_c track the current net balance of c in the fixed portion of the matching. Initially set $b_c = 0$ for each $c \in C$. All colleges are marked as importing and exporting.

Step 1a.

- If $o_c^e = 0$ and either $o_c^q = 0$ or $b_c = u_c$, then remove c. If $o_c^e = 0$, $o_c^q > 0$, and $b_c < u_c$, then c becomes *nonexporting*.[34]
- If $o_c^q = 0$ and $b_c = \ell_c$, then remove c. If $o_c^q = 0$, $o_c^e > 0$, and $b_c > \ell_c$, then c becomes *nonimporting*.

[33] Here, ℓ_c and u_c are integers.

[34] That is, a college is nonexporting if it has available quota to import but all its sponsored students are removed. Therefore, a nonexporting college cannot point to a student.

Step 1b. Each available student points to her favorite among *available importing* colleges, which consider her acceptable, and $c_\varnothing$. Each *available exporting* college c points to the highest-priority available student in S_c according to $\triangleright_c$. Null college $c_\varnothing$ points to all students pointing to it.

Proceed to step 2 if there is no cycle. Otherwise, in each cycle assign each student to the option she is pointing to and remove her. For each cycle and college c:

- Reduce eligibility counter o_c^e by one if it has an eligible student in that cycle.
- Reduce import counter o_c^q by one if it is in that cycle.
- Return to step 1a.

Step 2. If there are no *exporting* colleges left, then the algorithm terminates.[35] If not, then we consider chains that end with *nonexporting* colleges.[36] If $b_c = \ell_c$ for each *available exporting* college c, then remove all *nonexporting* colleges and go to step 1a.[37] Otherwise, find among the considered chains the one whose tail has the highest priority among the *available exporting* colleges c with $b_c > \ell_c$. Assign each student in that chain to the college that she points to and remove her. Denote the tail and head of the chain by c_t and c_h, respectively. Observe that c_h is a nonexporting college. Other colleges in the chain are represented by $\tilde{c}$:

- Reduce eligibility counter $o_{\tilde{c}}^e$ and import counter $o_{\tilde{c}}^q$ of all $\tilde{c}$ by one.
- Reduce eligibility counter $o_{c_t}^e$ and current net balance b_{c_t} by one.
- Reduce import counter $o_{c_h}^q$ by one and increase current net balance b_{c_h} by one.
- Return to step 1a.

When the algorithm terminates, all remaining students are assigned to $c_\varnothing$. We call each repetition of these two steps a round.

The 2S-TTTC mechanism inherits the most desired features of 2S-TTC. We state two theorems to this end.

THEOREM 7. 2S-TTTC is strategy-proof for students, and for any market $[q, e, \succsim]$ and tolerance profile $(\ell_c, u_c)_{c\in C}$, there does not exist an acceptable matching ν that Pareto dominates the outcome of 2S-TTTC and $\ell_c \leq b_c^\nu \leq u_c$ for all $c \in C$.

[35] Note that this condition also captures the case "if no eligible students are left."
[36] If no student points to an available nonexporting college, then we would have a cycle.
[37] That is, no more chains respecting the tolerance profile can form after this point in the algorithm.

In the 2S-TTTC mechanism, a student is pointed to by the colleges in C after all the other students with higher internal priority are assigned to a college or $c_{\emptyset}$. Moreover, a student points only to the acceptable colleges that also consider her acceptable. As a consequence, the 2S-TTTC mechanism satisfies acceptability and respect for internal priorities.

Theorem 8. 2S-TTTC is acceptable, and it respects internal priorities.

Theorems 7 and 8 hold without any assumptions on preferences. Under a mild assumption on college preferences, we can show that 2S-TTTC is individually rational and it induces a dominant-strategy equilibrium for colleges' quota reporting game to certify all their students and report their true admission quota.

Although 2S-TTTC is defined in a static problem, we can easily extend it to the dynamic environment in which the aggregate balance over years matters. In particular, for each period t and $c \in C$, we can set counter b_c equal to c's aggregate balance in period $t - 1$, where the aggregate balance in period $t - 1$ is equal to the sum of balances between periods 1 and $t - 1$. Moreover, the exogenous priority rule used in period t can be determined on the basis of the aggregate balance colleges carry at the end of period $t - 1$ such that the highest priority can be given to the college with the highest aggregate balance, and so on.

V. Temporary Worker Exchanges

Many organizations have temporary worker exchange programs that can be modeled through our balanced two-sided matching framework. The first difference between such programs and tuition exchange is that these exchanges are usually temporary. Firms usually require a set of specific skills, for example, a mathematics teacher to replace their own mathematics teacher. Compatibility and ability to perform the task are the main preference criteria rather than a strict preference ranking. For example, finding a good teacher with a specific degree is the first-order requirement rather than finer details about the rankings of all good teachers.

The second difference is that each position and each worker should be matched, in contrast to the tuition exchange application. The workers are currently working for their home firms. Thus, the firms consider these workers necessarily acceptable. By contrast, in tuition exchange, colleges are not required to admit all the dependents of their employees. In temporary worker exchanges, a worker who does not want to go to a different firm necessarily stays employed in her home firm. We need to use a variant of the tuition exchange model to facilitate balanced-efficient trade in such circumstances.

We can use the model introduced in Section III with slight changes. Since each firm accommodates its current workers, $q_c = |S_c|$ for each $c \in C$. In Section III, in the definition of a matching, students who are not eligible are taken as assigned to $c_\varnothing$. However, for worker exchange programs, the workers who are not certified as eligible continue to work in their home firms in a matching. Formally, a *matching* is a correspondence $\mu : C \cup S \twoheadrightarrow C \cup S$ such that (1) $\mu(c) \subseteq S$, where $|\mu(c)| = q_c = |S_c|$ for all $c \in C$; (2) $\mu(s) \subseteq C$, where $|\mu(s)| = 1$ for all $s \in S$; (3) $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in C$ and $s \in S$; and (4) $\mu(s) = c$ for all $s \in S_c \setminus E$ and $c \in C$. Observe that each matching is balanced in this environment by definition. Thus, balanced efficiency and Pareto efficiency are equivalent.

To capture the features of worker exchange programs, we make certain assumptions about the preferences of workers and firms. Since worker $s \in S_c$ is already working at firm c, we assume that s finds c acceptable and c finds s acceptable, that is, $cP_s c_\varnothing$ and $sP_c\varnothing$ for all $s \in S_c$ and $c \in C$. As discussed above, acceptable workers do not have huge differences for the firms. The compatibility assumption and assumption 1 together imply that each firm weakly prefers an unconstrained matching with zero net balance to another unconstrained matching with nonpositive balance as long as it gets weakly more acceptable workers under the former one. We formally state these assumptions on preferences as follows.

Assumption 2. (1) Weakly size-monotonic firm preferences: for any $c \in C$ and $\mu, \nu \in \mathcal{M}^u$, if $b_c^\mu = 0$, $b_c^\nu \leq 0$, and $|\{s \in \mu(c) : sP_c \varnothing\}| \geq |\{s \in \nu(c) : sP_c \varnothing\}|$, then $\mu \succsim_c \nu$. (2) Acceptability of the current match: For any $c \in C$ and $s \in S_c$, we have $cP_s c_\varnothing$ and $sP_c\varnothing$.

On the basis of assumption 2, a (balanced) mechanism that allows employees to get better firms, which consider them acceptable, improves the total welfare without hurting anyone. Hence, 2S-TTC can be applied to temporary exchange programs with a minor change such that when a firm c is removed, all its remaining workers are assigned to it.[38] In this environment, 2S-TTC inherits its desired features. Moreover, the characterization result also holds. Additionally, acceptability of 2S-TTC implies that it is strategy-proof for firms.

Theorem 9. Under assumption 2, 2S-TTC is a (balanced) Pareto efficient, individually rational, acceptable, and strategy-proof mechanism that also respects internal priorities, and it is the unique Pareto efficient, acceptable, worker-strategy-proof mechanism that also respects internal priorities.[39]

[38] Since $q_c = |S_c|$ for all $c \in C$, a firm is removed when its eligibility counter reaches zero.

[39] Moreover, 2S-TTC is stable in this domain. This result is noteworthy, because the widely used worker-proposing deferred-acceptance (DA) mechanism with exogenous tie breaking favoring own workers over the others is not Pareto efficient, although it is stable and balanced in this special environment. If tie breaking does not favor own workers, the outcome of DA may not be a "matching" in this domain. In the proof of theorem 9 we also show that 2S-TTC is stable in this domain.

VI. Conclusions

This paper proposes a centralized market solution to overcome problems observed in decentralized exchange markets. We used tuition exchange and temporary exchange programs as our leading examples, in which more than 300,000 people participate annually.

Our paper, besides introducing a new applied problem and proposing a solution to it, has six main theoretical and conceptual contributions: We introduce a new two-sided matching model that builds on the two most commonly used matching models in the literature: discrete object allocation, including school choice, and standard many-to-one two-sided matching models; but it differs in many fronts from these. As far as we know, this is the first time object allocation and exchange algorithms inspire the mechanism design for a two-sided matching model. This is one of the few instances when axiomatic mechanism design is used in practical market design to come up with the correct mechanism. A natural axiomatic representation is given for a TTC-based mechanism. This is one of the rare occasions in which the stable matching theory of Gale and Shapley is extended to a setting with externalities with tractable existence, equilibrium, and comparative static results (see app. A). Finally, our paper is one of the few studies that propose a dynamic matching mechanism with good properties for a dynamic applied problem.

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