A Theory of School-Choice Lotteries

Onur Kesten & M. Utku Ünver
Carnegie Mellon University *Boston College*

School Choice

- U.S. public schools
  Examples: Boston, Chicago, Florida, Minnesota, Seattle (since 1987)
- Centralized mechanisms were adopted (e.g. Boston and Seattle)
- Two kinds of mechanisms: both use lotteries for ETE (Abdulkadiroğlu & Sönmez *AER 2003*)
- Boston replaced its mechanism (2005) and NYC introduced a new mechanism (2004) based on Gale & Shapley's (*AMM 1962*) two-sided matching approach (Abdulkadiroğlu & Pathak & Roth & Sönmez *AERP&P 2005* and Abdulkadiroğlu & Pathak & Roth *AERP&P 2005, AER 2008*)
- New mechanism has superior *fairness* and *incentive* properties.
- However, school choice is different from two-sided matching.

School Choice vs. Two-sided Matching

Problem components

- students with preferences over schools
- schools with specific priority orders over students

A school is an "object":

- Efficiency
- Incentives

Priority orders are typically weak (i.e., large indifference classes exist)

- e.g. in Boston four priority groups (walk zone & sibling)
- random tie breaking is commonly used to sustain *fairness* among equal priority students.

Our Difference from Previous Approaches

- All previous literature is based on an *ex-post* idea assuming
  'priority orders are strict'
  OR
  'priority orders are made strict via a random draw'
- Our approach: *ex ante*
  Extends the study to random mechanisms as well
  Evidence from the random assignment problem (Bogomolnaia & Moulin *JET 2001* - BM hereafter)
- In the presence of indifference classes in priorities:
  School-choice problem $\approx$ Assignment (house allocation) problem

Our Contribution

- A new framework to study school-choice problems combining *random assignment problem* with the *deterministic school-choice problem*
- Two notions of "ex-ante" fairness instead of the existing "ex-post" fairness notions
- Two mechanisms that find special random matchings satisfying these ex-ante fairness notions

The Model

A **school-choice problem** $(I, C, q, P, \succsim)$:

- Finite set of students $I = \{1, 2, 3, \ldots |I|\}$
- Finite set of schools $C = \{a, b, c, \ldots |C|\}$
- Quotas of schools $q = (q_c)_{c \in C}$
- Strict preference profile of students $P = (P_i)_{i \in I}$
- Weak priority structure of schools $\succsim = (\succsim_c)_{c \in C}$

Outcome

A **random matching** is a (bi-stochastic) matrix $\rho = (\rho_{i,c})_{i \in I, c \in C}$ such that

1. for all i, c, $\rho_{i,c} \in [0, 1]$.
2. for all i, $\sum_c \rho_{i,c} = 1$.
3. for all c, $\sum_i \rho_{i,c} = q_c$.

A random matching row [or column] gives the marginal probability measure of the assignment of a student [or a school] to all schools [or all students].
A **matching** is a deterministic "random" matching (consisting of 0 or 1's only).

A **lottery** is a probability distribution over matchings.

Lotteries vs Random Matchings

What is the connection between *lotteries* and *random matchings*?
Each lottery induces a random matching. What about the converse?

Theorem

Birkhoff (1946) - von Neumann (1953): *Given any random matching there exists a lottery that induces it.*

Solution

We focus on random matchings rather then lotteries (by B-vN Theorem and our Propositon 1 below):

A **school-choice mechanism** selects a random matching for every problem.

A random matching ρ satisfies **ETE** if for all students i and j with identical preferences and equal priorities at all schools, $\rho_{i,c} = \rho_{j,c}$ for all $c \in C$.

A matching μ is **stable** if there is **no** *justified envy*

justified envy: There are i and c such that

$$cP_i\mu(i), \text{ and}$$
$$i \succ_c j \text{ for some } j \in \mu(c).$$

A random matching ρ is **ex-post stable** if there exists a lottery λ inducing ρ such that $\lambda_\mu > 0 \implies \mu$ is a stable matching.

Properties (continued)

2. New Fairness Properties: Strong Ex-ante Stability and Ex-ante Stability

A random matching $\rho = [\rho_{i,c}]_{i \in I, c \in C}$ is **ex-ante stable** if it does not induce **no** *ex-ante justified envy (toward a lower priority student)*

ex-ante justified envy: There are i and c such that for some a and j

$$cP_i a \ \text{ with } \rho_{i,a} > 0 \ , \ \text{ and}$$
$$i \succ_c j \text{ with } \ \rho_{j,c} > 0 \ .$$

An random matching ρ is **strongly ex-ante stable** if it is ex-ante stable and does not induce **no** *ex-ante discrimination (between equal priority students)*

ex-ante discrimination: There are i and c such that for some a and j

$$cP_i a \ \text{ with } \ \rho_{i,a} > 0 \ , \text{ and}$$
$$i \sim_c j \text{ with } \ \rho_{j,c} > \rho_{i,c} \ .$$

Related paper: Roth, Rothblum & vande Vate (*MOR 1994*)

Properties (continued)

Fairness (Continued)

Proposition

- *Ex-ante stability* $\Longrightarrow$ *ex-post stability,*
- *the converse is not correct,*
- *all lotteries that induce an ex-ante stable random matching are only over stable matchings, and*
- *the current NYC/Boston mechanism is not ex-ante stable.*

Why is Ex-ante Stability Important?

- The new mechanism was adopted for its superior fairness and incentive properties with respect to the previous mechanism.
- Students can potentially take legal action against school districts based on ex-post stability violations.
- However, they can similarly take legal action based on ex-ante stability violations.

Properties (Continued)

3. Efficiency: Ordinal vs Ex-post Pareto Efficiency

Ex-post: A random matching is **ex-post efficient** if there exists an equivakent lottery over Pareto-efficient matchings.
Interim: A random matching ρ **ordinally dominates** π if for all i, c
Prob {student i is assigned to c or a better school under ρ} $\geq$
Prob {student i is assigned to c or a better school under π}
(strict for at least one pair of i and c)
A random matching is **ordinally efficient** if it is not *ordinally dominated*.

Properties (Continued)

3. Computational Simplicity

A mechanism is **computationally simple** if there exists a deterministic algorithm that computes its outcome in a number of elementary steps bounded by a polynomial of the number of the inputs of the problem such as the number of students, schools, or total quota.

Some Known Results

Ex-post (ex-ante) stability and *ex-post efficiency* are incompatible (Roth *Ecma 1982*).

Ordinal efficiency implies *ex-post efficiency* but not vice versa (BM).

Ex-ante stability, *constrained efficiency*, *ETE*, and *strategy-proofness* are incompatible (BM).

Summary: Unified Ordinal School-Choice Framework

Through different models and mechanisms, previous studies examined:

- ex-ante efficiency gains (e.g. Featherstone & Niederle, *2008*; Abdulkadiroglu, Che, & Yasuda, *AER forth., 2008*): however school choice framework is *ordinal*
- ex-post efficiency gains (e.g. Erdil & Ergin, *AER 2008*): however school choice framework is *probabilistic* by the nature of fairness criteria.
- ex-post fairness through ex-ante tie breaking (e.g. Abdulkadiroğlu & Sönmez, *AER 2003*) however due to the problem's probabilistic nature, ex-post fair school-choice mechanisms may lead to ex-ante unfairness.

We unify these frameworks through an ordinal probabilistic framework.

What do we gain? (1) Constrained Ordinal Efficiency

Example: (BM)

P_1	P_2	P_3	P_4
a	a	b	b
b	b	a	a
c	c	d	d
d	d	c	c

with equal priority students

NYC/Boston Mechanism Outcome & Our Approach:

	a	b	c	d
1	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$
2	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$
3	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$
4	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{5}{12}$

	a	b	c	d
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$	0
3	0	$\frac{1}{2}$	0	$\frac{1}{2}$
4	0	$\frac{1}{2}$	0	$\frac{1}{2}$

What do we gain? (2) Elimination of ex-ante justified envy

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
5	4, 5	1, 3	$\vdots$
1	$\vdots$	$\vdots$	$\vdots$
2	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

P_1	P_2	P_3	P_4	P_5
c	a	c	b	b
a	d	d	d	a
d	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

NYC/Boston Mechanism Outcome:

$$\frac{1}{4}\begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 \\ \mathbf{d} & d & c & b & a \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & d & c & d & b \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & \mathbf{2} & 3 & 4 & 5 \\ c & \mathbf{a} & d & d & b \end{pmatrix}$$

1 has ex-ante justified envy toward student 2 for school a

What do we gain: (3) Elimination of ex-ante discrimination

Ex:

$\succ_a$	$\succ_b$	$\succ_c$
3	2	2
1, 2	1	1
	3	3

P_1	P_2	P_3
a	a	b
b	c	a
c	b	c

The NYC/Boston mechanism outcome

$$\frac{1}{2}\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}$$

Ex-ante discrimination between students 1 and 2 for school a

What do we gain? (4) Computational Simplicity

Our mechanisms can be executed in polynomial time to find the random matching outcome.

The new NYC/Boston mechanism cannot!

What do we lose? (1) Strategy-proofness

There is no mechanism that is constrained efficient, ETE, and strategy-proof (BM)

What do we lose? (2) Some ex-post efficiency

Our lotteries are not necessarily over student-optimal stable matchings, while NYC/Boston are.

NYC/Boston mechanism is not also ex-post constrained efficient.

Erdil-Ergin (*AER, 2008*) approach can be used here.

Fractional Deferred Acceptance (FDA) Approach:

- **Step 1:**
  - Each student applies to his favorite school.
  - Each school c considers its applicants. If the total number of applicants is greater than q_c, then applicants are tentatively assigned to school c one by one starting from the highest priority ones such that equal priority students, if assigned a fraction of a seat at this school, are assigned an equal fraction. Unassigned applicants (possibly, some being a fraction of a student) are rejected.

- **General Step k>0:**
  - Each student, who has a rejected fraction from the previous step, applies to the next best school that has not yet rejected any fraction of his.
  - Each school c considers its tentatively assigned applicants together with the new applicants. Applicant fractions are tentatively assigned to school c, starting from the highest priority ones such that equal priority students if assigned a fraction of a seat at this school, are assigned an equal fraction. Unassigned applicants (possibly, some being a fraction of a student) are rejected.
  - As an example, suppose a fraction of $1/3$ of students $1, 2, 3, 4$ apply to school a with quota 1 at some step k. Suppose all students have the highest priority at a. School a admits $1/4$ of each student and rejects a fraction of $1/12$ of each.
- We continue until no fraction of a student is unassigned. At this point, we terminate the algorithm by making all tentative assignments permanent.

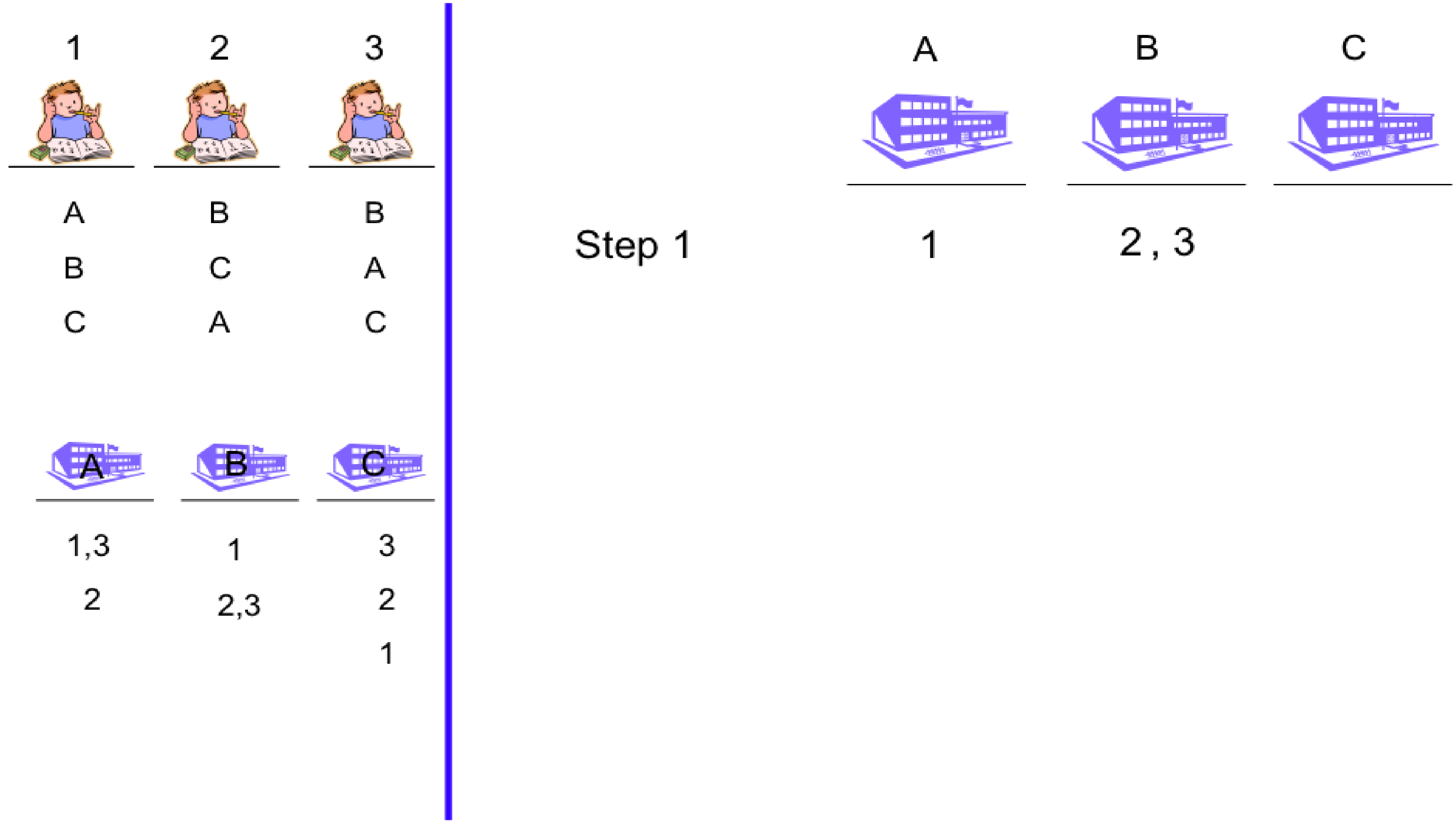
Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

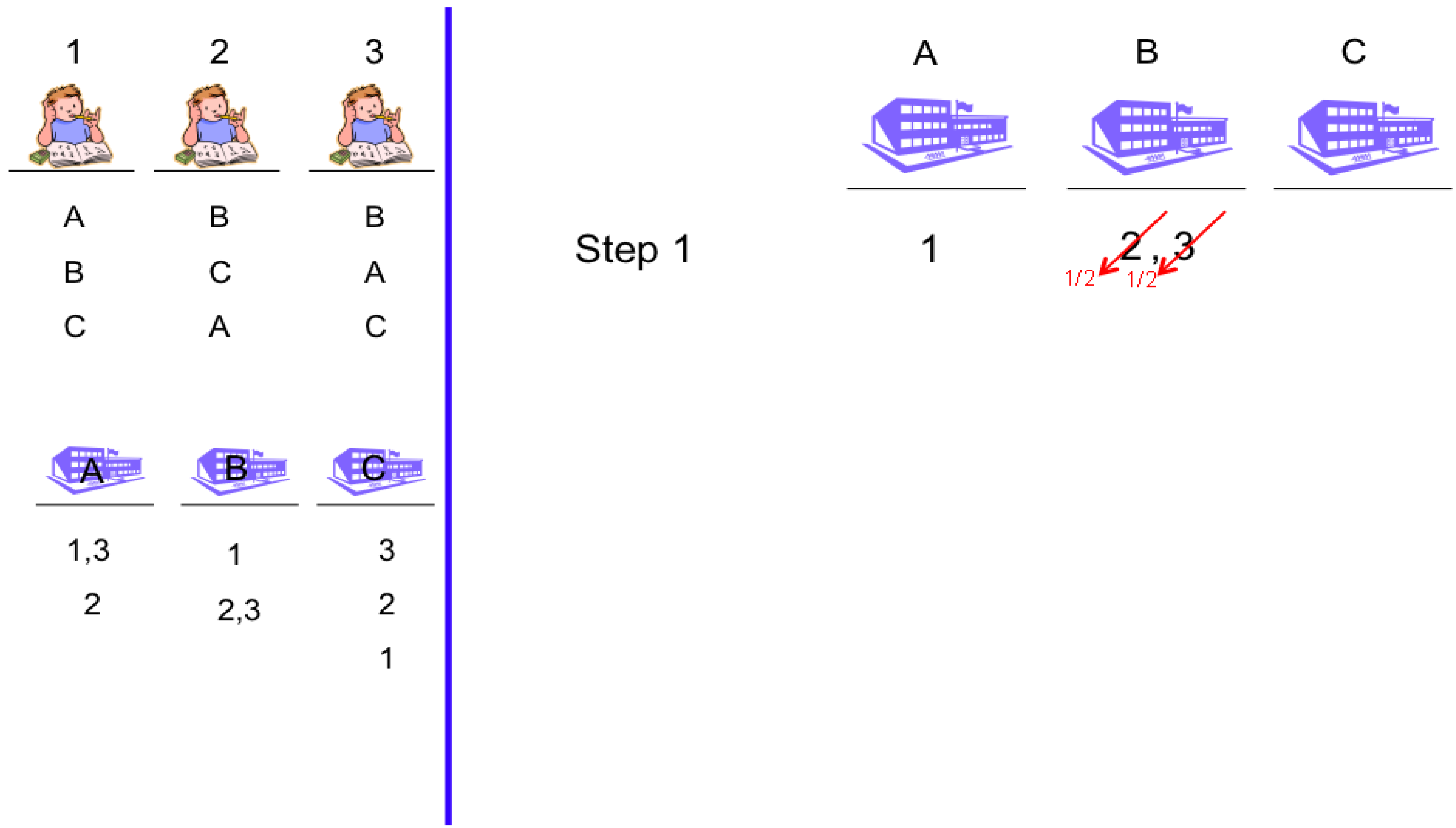
A	B	C
1,3	1	3
2	2,3	2
		1

A	B	C

Fractional Deferred Acceptance: An example



Fractional Deferred Acceptance: An example



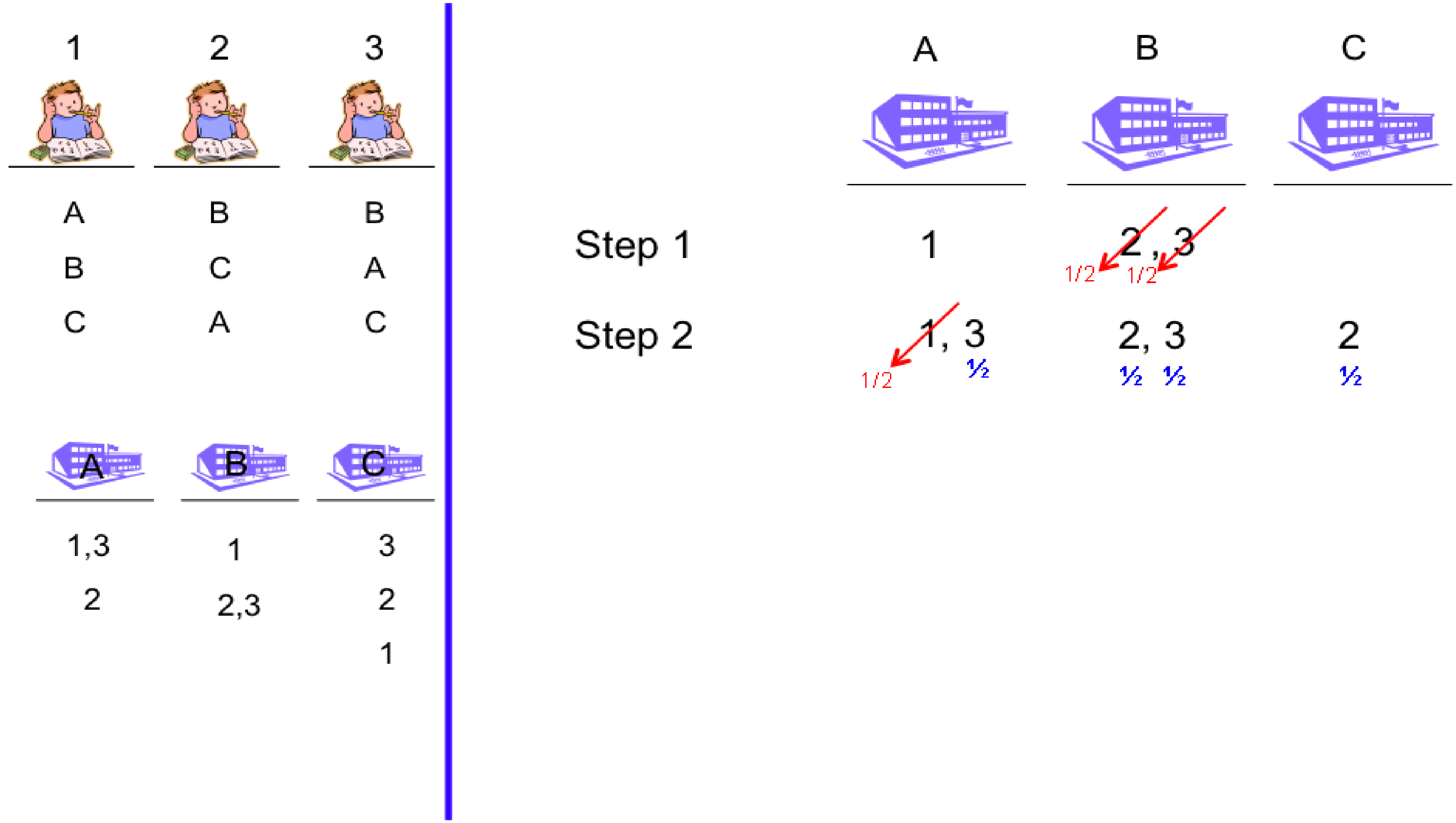
Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	2 , 3 1/2 1/2	
Step 2	1, 3 ½	2, 3 ½ ½	2 ½

Fractional Deferred Acceptance: An example



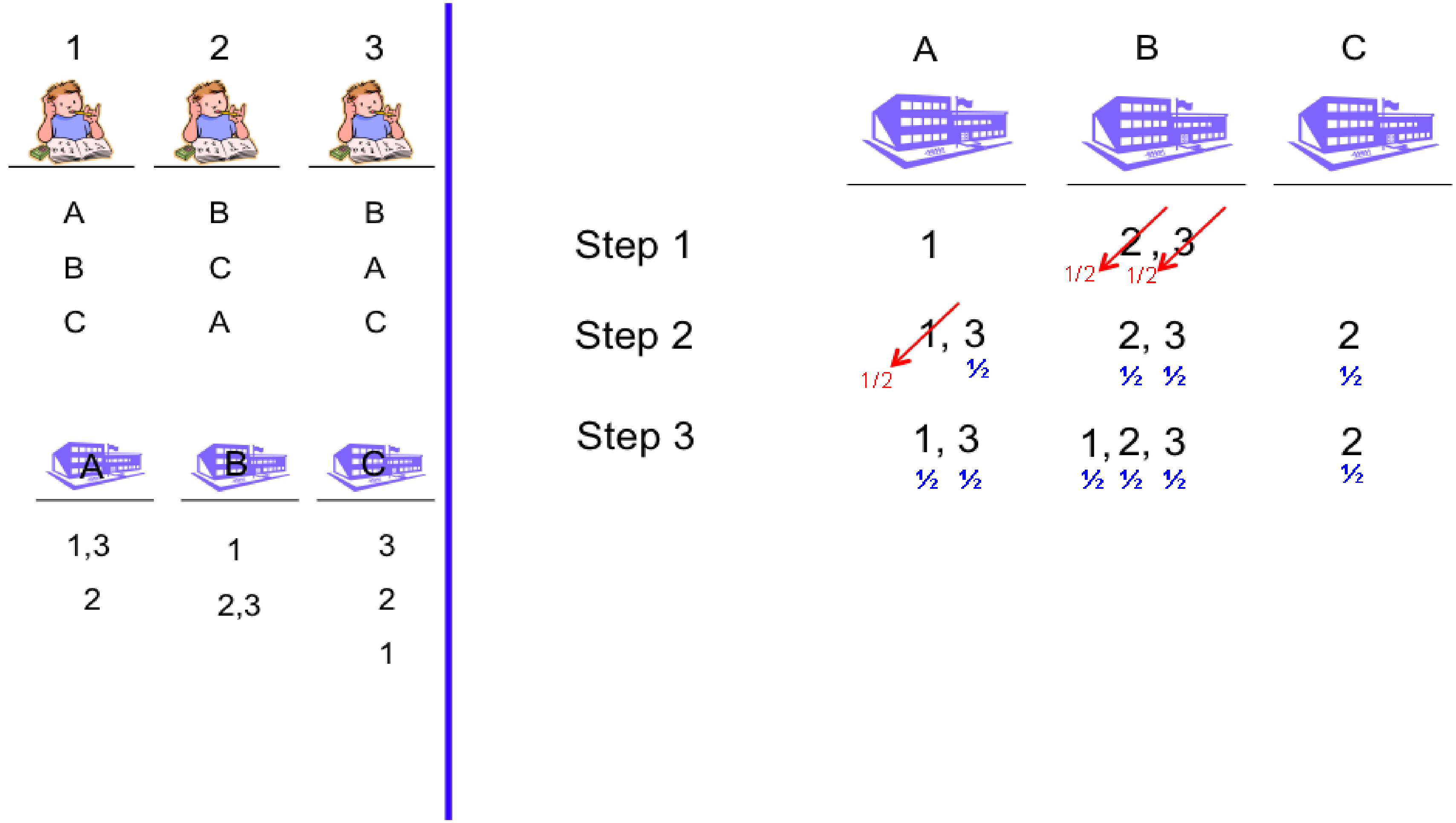
Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	~~1~~, 3 1/2 ½	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	2, 3 ½ ½	

Fractional Deferred Acceptance: An example



Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ (1/2 1/2)	
Step 2	~~1~~, 3 (1/2) ½	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	1, ~~2~~, ~~3~~ ½ ½ ½ (1/4 1/4)	2 ½

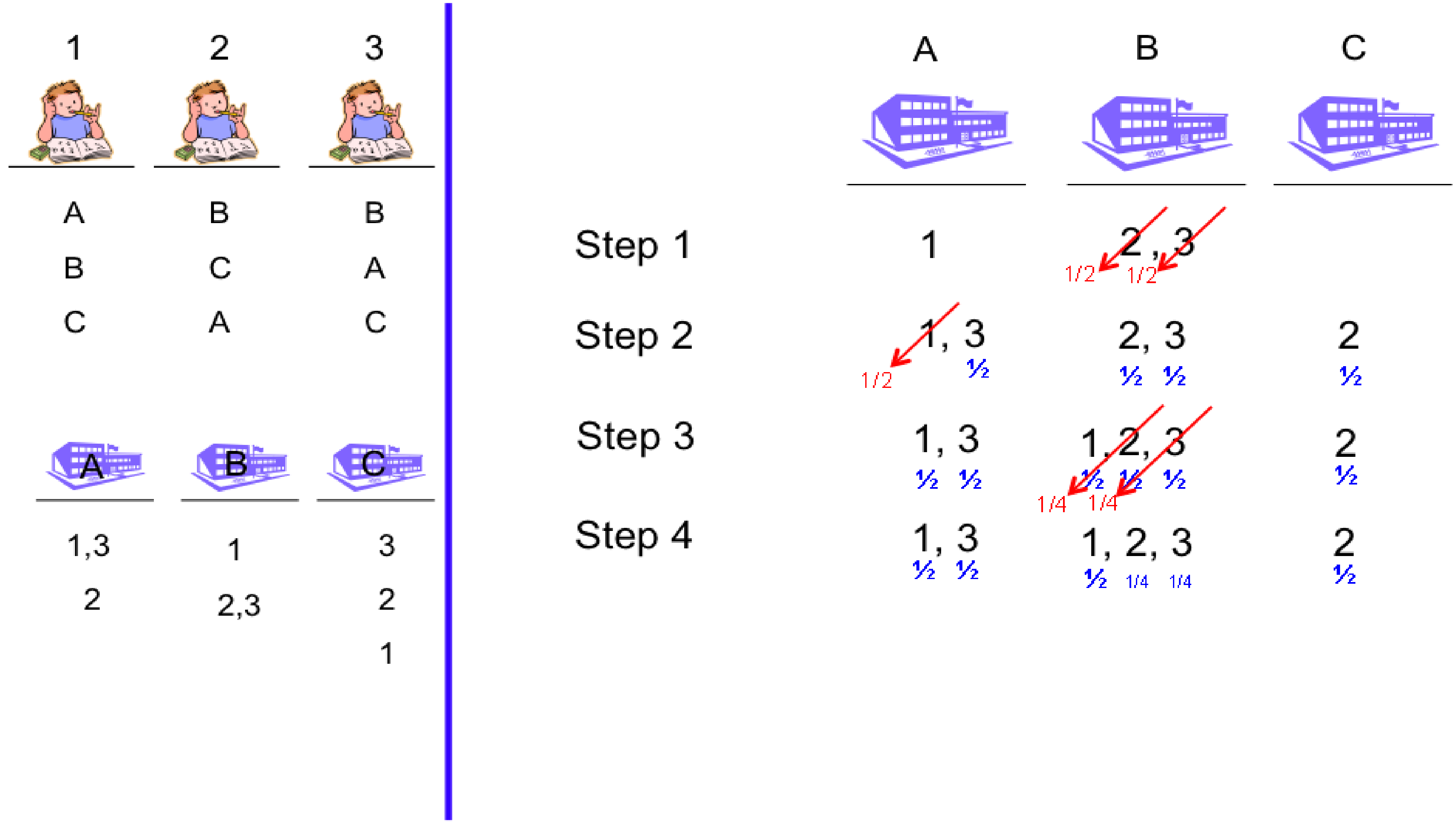
Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

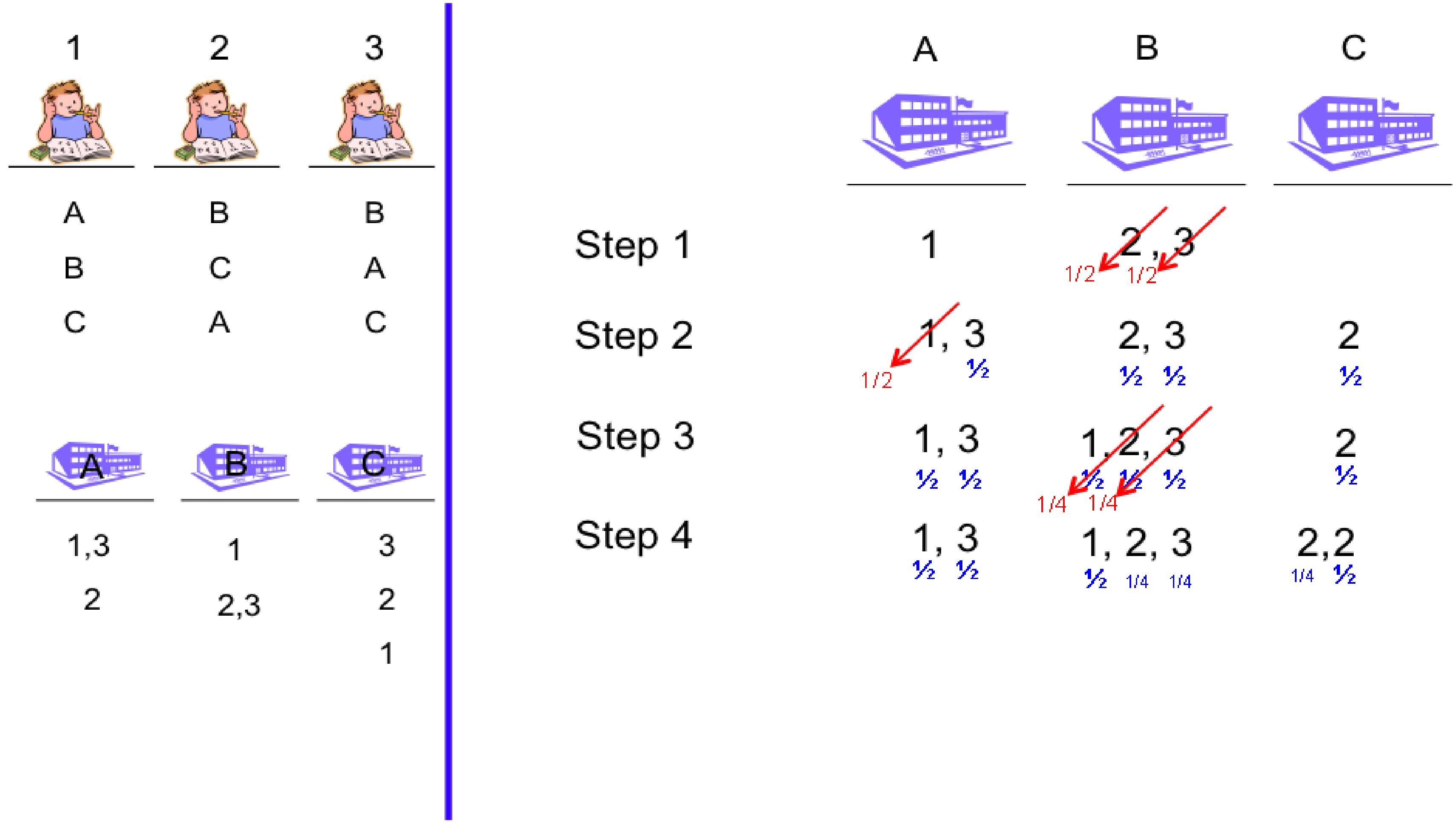
A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	~~1~~, 3 1/2 ½	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	1, ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½

Fractional Deferred Acceptance: An example



Fractional Deferred Acceptance: An example



Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	2 , 3 1/2 1/2	
Step 2	1, 3 1/2 ½	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	1, 2, 3 ½ ½ ½ 1/4 1/4	2 ½
Step 4	1, 3 ½ ½	1, 2, 3 ½ 1/4 1/4	2 3/4

Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	2 , 3 1/2 1/2	
Step 2	1, 3 1/2 ½	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	1, 2, 3 ½ ½ ½ 1/4 1/4	2 ½
Step 4	3, 1, 3 1/4 ½ ½	1, 2, 3 ½ 1/4 1/4	2 3/4

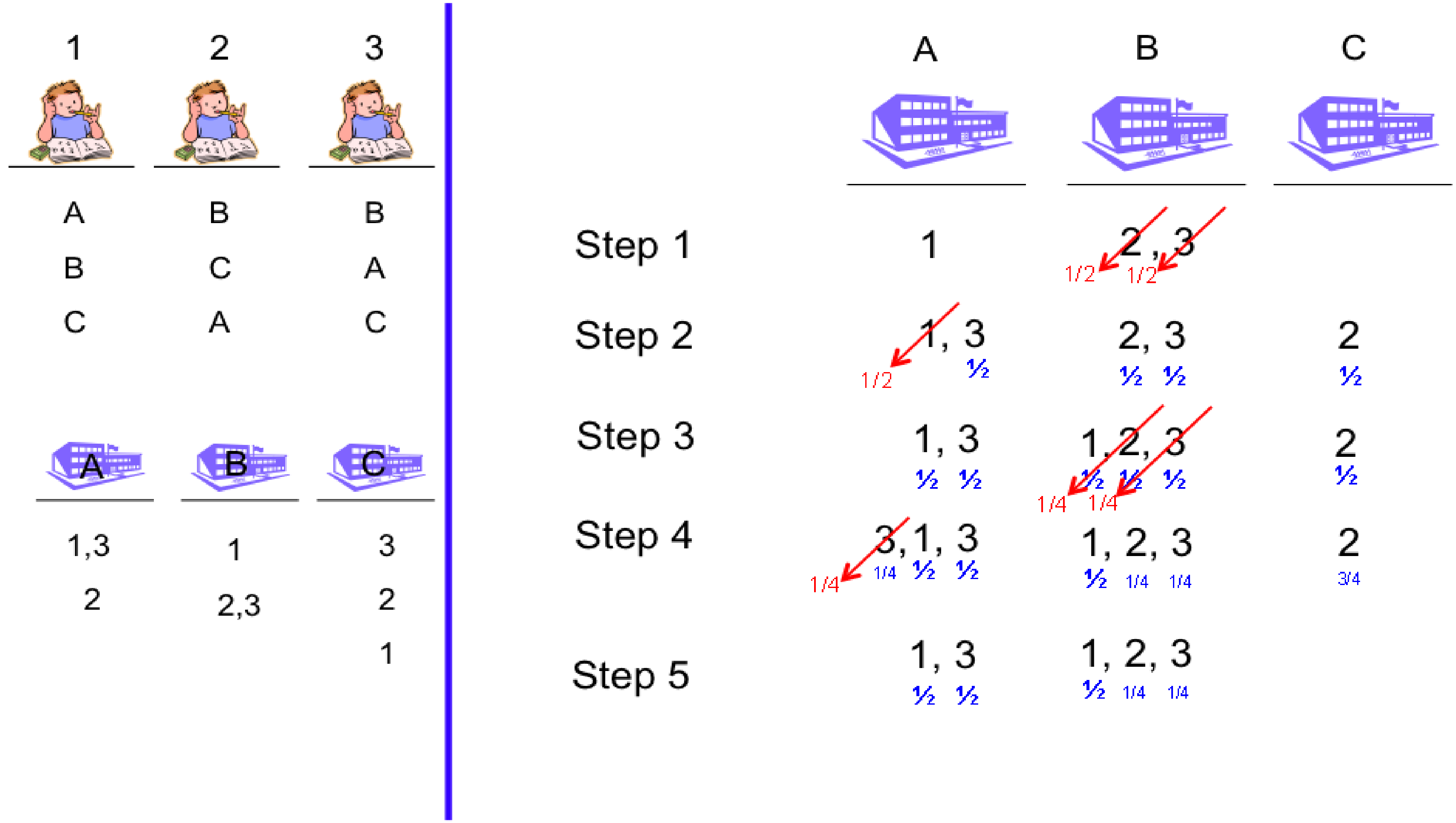
Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

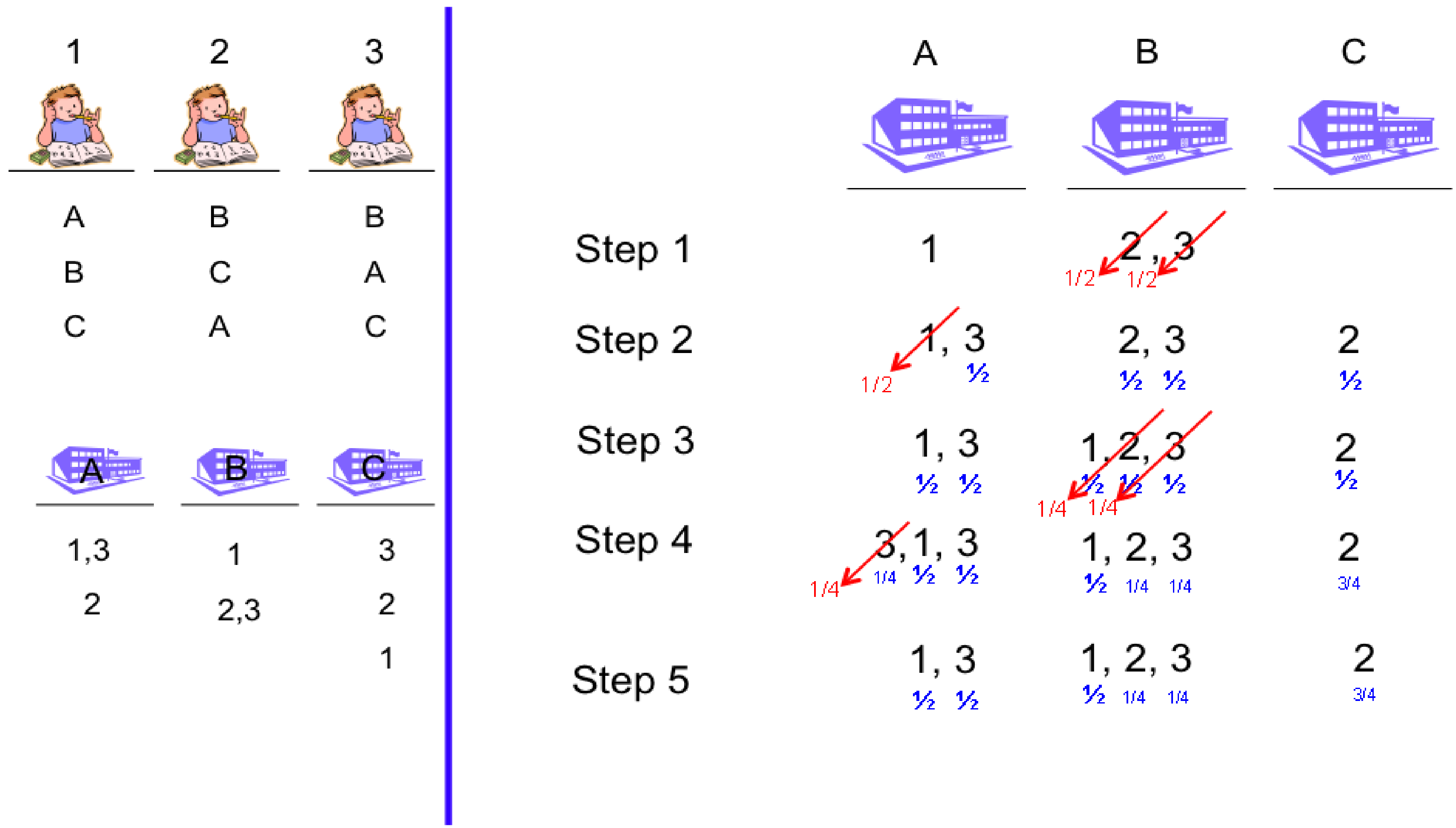
A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	~~1~~, 3 1/2 ½	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	1, ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½
Step 4	~~3~~, 1, 3 1/4 1/4 ½ ½	1, 2, 3 ½ 1/4 1/4	2 3/4

Fractional Deferred Acceptance: An example



Fractional Deferred Acceptance: An example



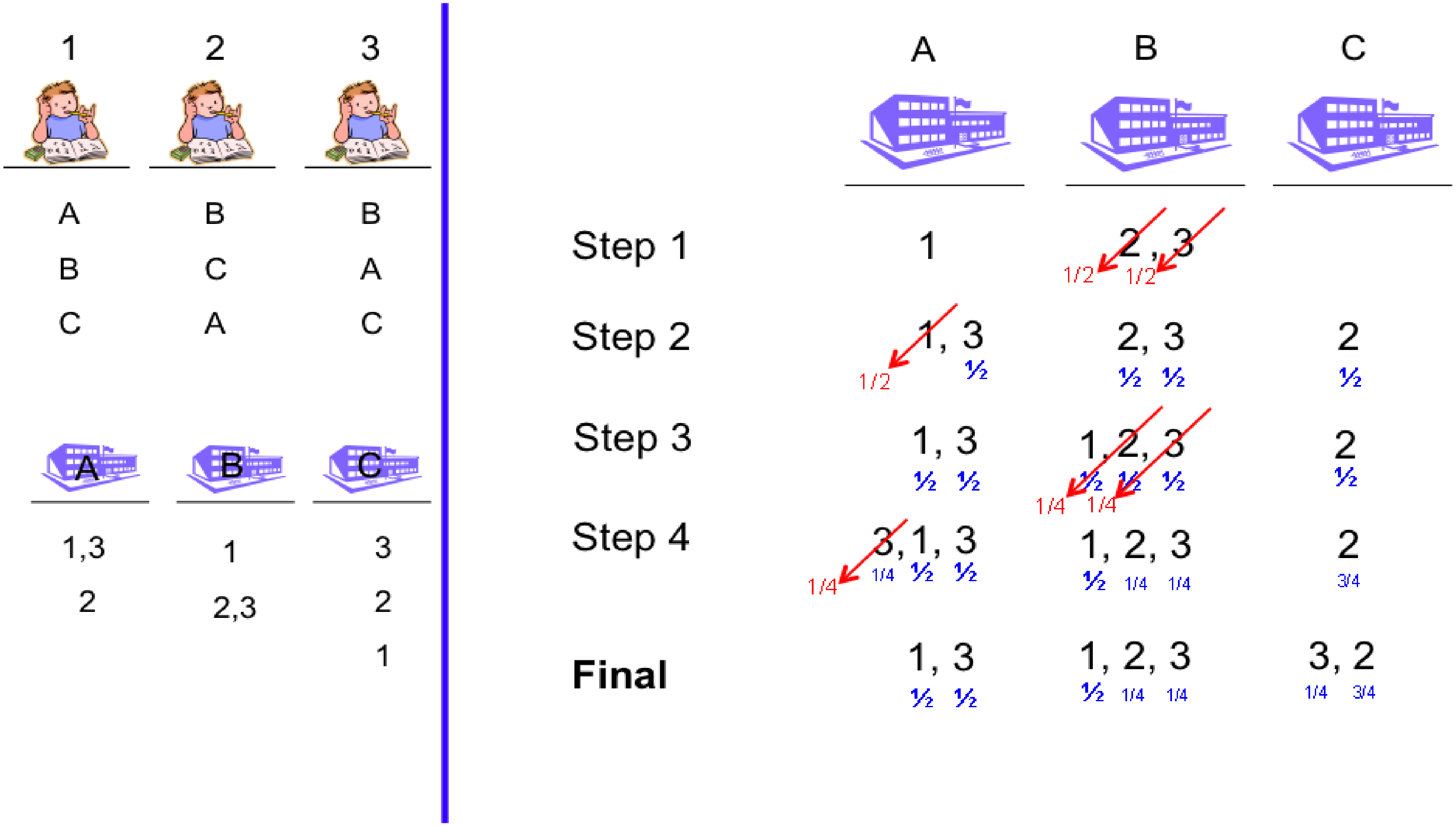
Fractional Deferred Acceptance: An example

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	~~1~~, 3 1/2 ½	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	1, ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½
Step 4	~~3~~, 1, 3 1/4 1/4 ½ ½	1, 2, 3 ½ 1/4 1/4	2 3/4
Step 5	1, 3 ½ ½	1, 2, 3 ½ 1/4 1/4	3, 2 1/4 3/4

Fractional Deferred Acceptance: An example



Fractional Deferred Acceptance: An example

1	2	3
1/2 A	1/4 B	1/4 B
1/2 B	3/4 C	1/2 A
C	A	1/4 C

A	B	C
1,3	1	3
2	2,3	2
		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	~~1~~, 3 1/2 ½	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	1, ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½
Step 4	~~3~~, 1, 3 1/4 1/4 ½ ½	1, 2, 3 ½ 1/4 1/4	2 3/4
Final	1, 3 ½ ½	1, 2, 3 ½ 1/4 1/4	3, 2 1/4 3/4

Fractional Deferred Acceptance: “Cycle Issue”

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	2 , 3	

Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	2 , 3 1/2 1/2	
Step 2	1	2, 3 ½ ½	

Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	2, 3 1/2 1/2	
Step 2	1	2, 3 ½ ½	2 ½

Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	2 , 3 1/2 1/2	
Step 2	**3**,1 ½	2, 3 ½ ½	2 ½

Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	~~2~~ 1/2, ~~3~~ 1/2	
Step 2	3 ½, ~~1~~ 1/2	2, 3 ½ ½	2 ½

Fractional Deferred Acceptance: “Cycle Issue”

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	**3**, ~~1~~ ½ 1/2	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	2, 3 ½ ½	2 ½

Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	**3**, ~~1~~ ½ 1/2	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	**1**, 2, 3 ½ ½ ½	2 ½

Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	**3**, ~~1~~ ½ 1/2	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	**1**, ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½

Fractional Deferred Acceptance: “Cycle Issue”

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	~~2~~, ~~3~~ 1/2 1/2	
Step 2	**3**, ~~1~~ ½ 1/2	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	**1**, ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½
Step 4	1, 3 ½ ½	1, 2, 3 ½ 1/4 1/4	2 ½

Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	**3** , ~~1~~ ½ 1/2	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	**1** , ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½
Step 4	**3** , 1, 3 1/4 ½ ½	1, 2, 3 ½ 1/4 1/4	2 ½

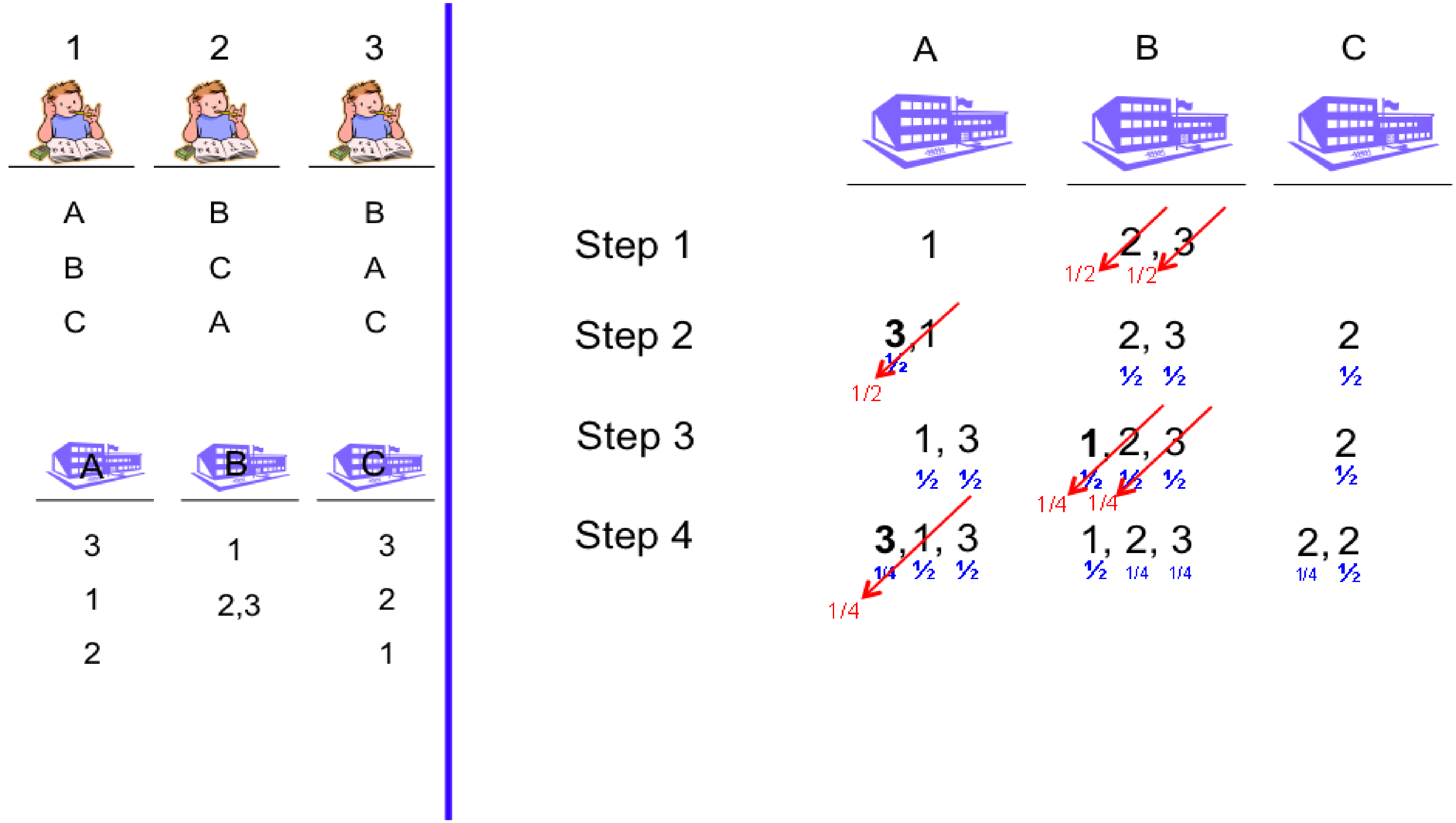
Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	**3**, ~~1~~ ½ 1/2	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	**1**, ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½
Step 4	**3**, ~~1~~, 3 1/4 ½ ½ 1/4	1, 2, 3 ½ 1/4 1/4	2 ½

Fractional Deferred Acceptance: “Cycle Issue”



Fractional Deferred Acceptance: “Cycle Issue”

1	2	3
A	B	B
B	C	A
C	A	C

A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	~~2~~ , ~~3~~ 1/2 1/2	
Step 2	**3**, ~~1~~ ½ 1/2	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	**1**, ~~2~~, ~~3~~ ½ ½ ½ 1/4 1/4	2 ½
Step 4	**3**, ~~1~~, 3 1/4 ½ ½ 1/4	1, 2, 3 ½ 1/4 1/4	2, 2 1/4 ½
Step 5	1, 3 1/4 3/4	1, 2, 3 ½ 1/4 1/4	2 3/4

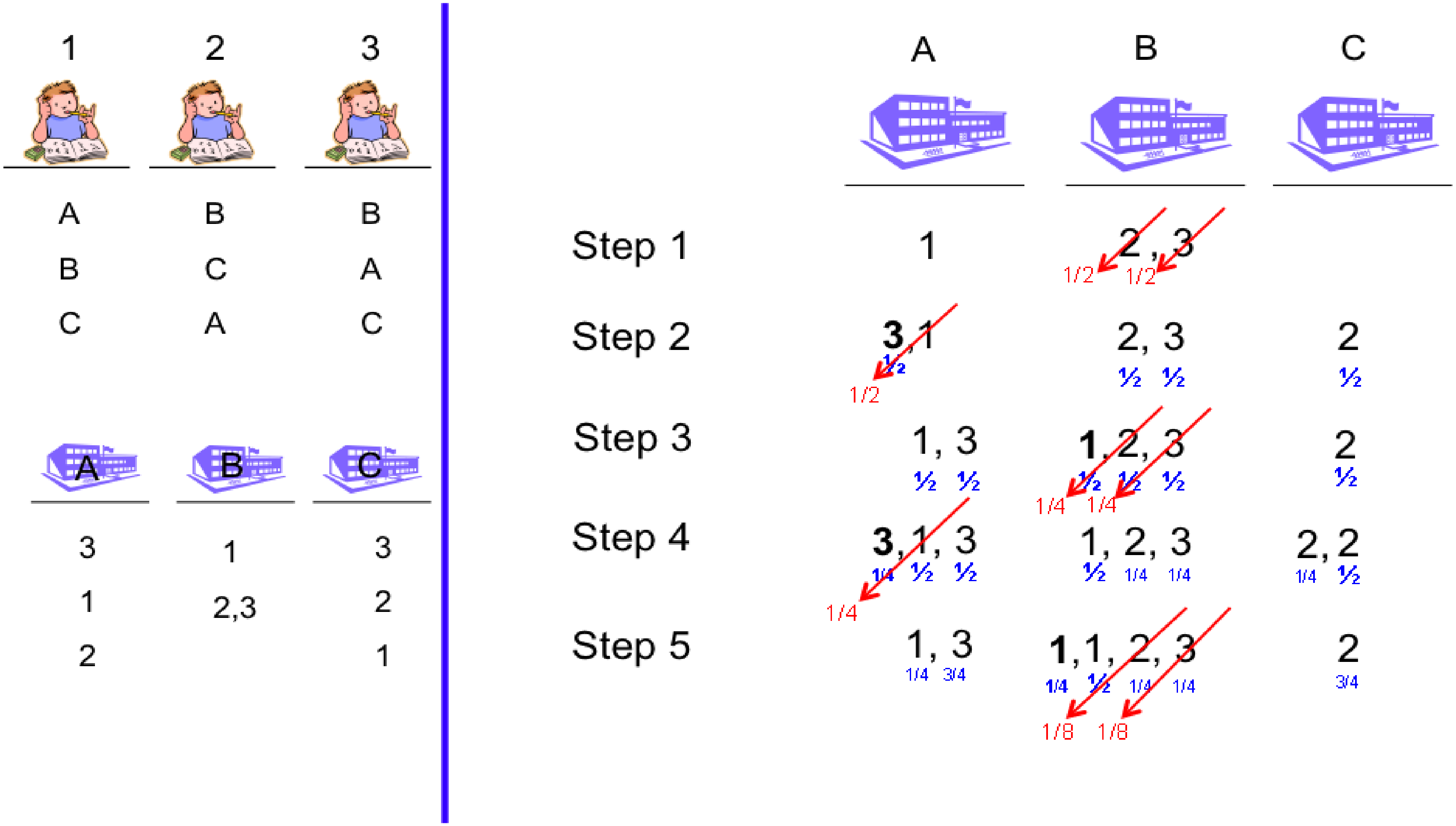
Fractional Deferred Acceptance: "Cycle Issue"

1	2	3
A	B	B
B	C	A
C	A	C

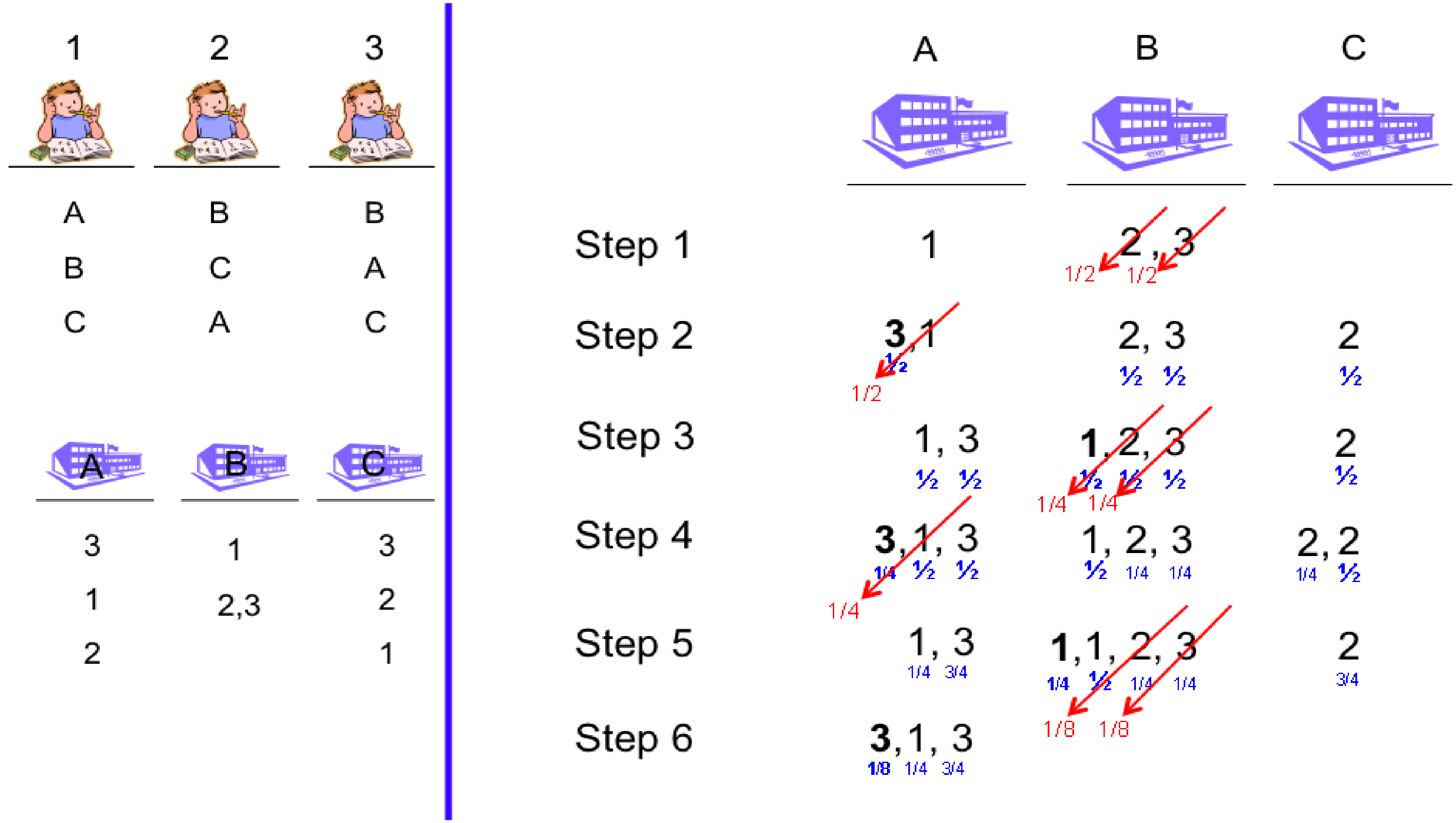
A	B	C
3	1	3
1	2,3	2
2		1

	A	B	C
Step 1	1	2 , 3 1/2 1/2	
Step 2	3 , 1 ½ 1/2	2, 3 ½ ½	2 ½
Step 3	1, 3 ½ ½	1 , 2, 3 ½ ½ ½ 1/4 1/4	2 ½
Step 4	3 , 1, 3 1/4 ½ ½ 1/4	1, 2, 3 ½ 1/4 1/4	2 , 2 1/4 ½
Step 5	1, 3 1/4 3/4	1 ,1, 2, 3 1/4 ½ 1/4 1/4	2 3/4

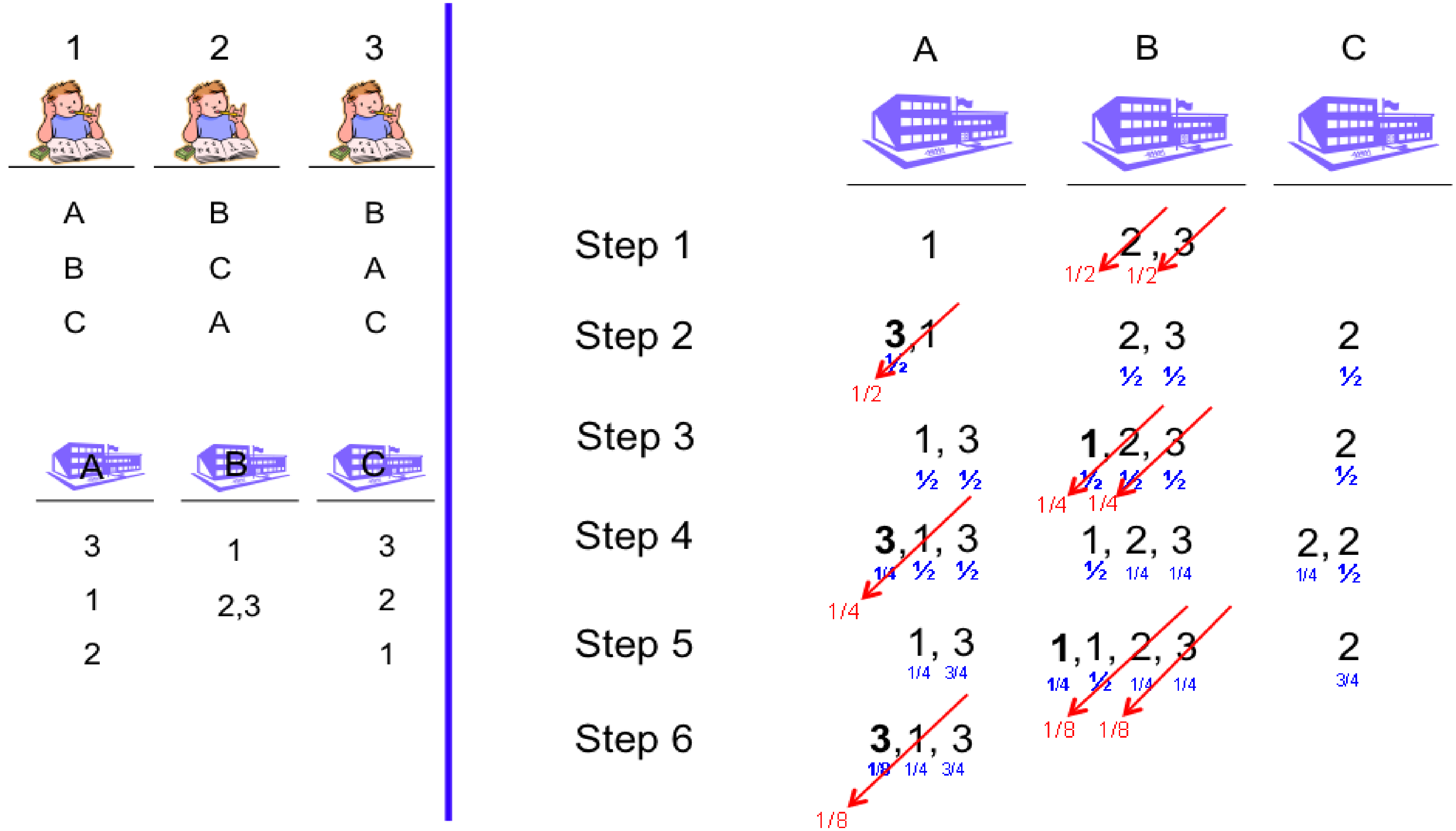
Fractional Deferred Acceptance: “Cycle Issue”



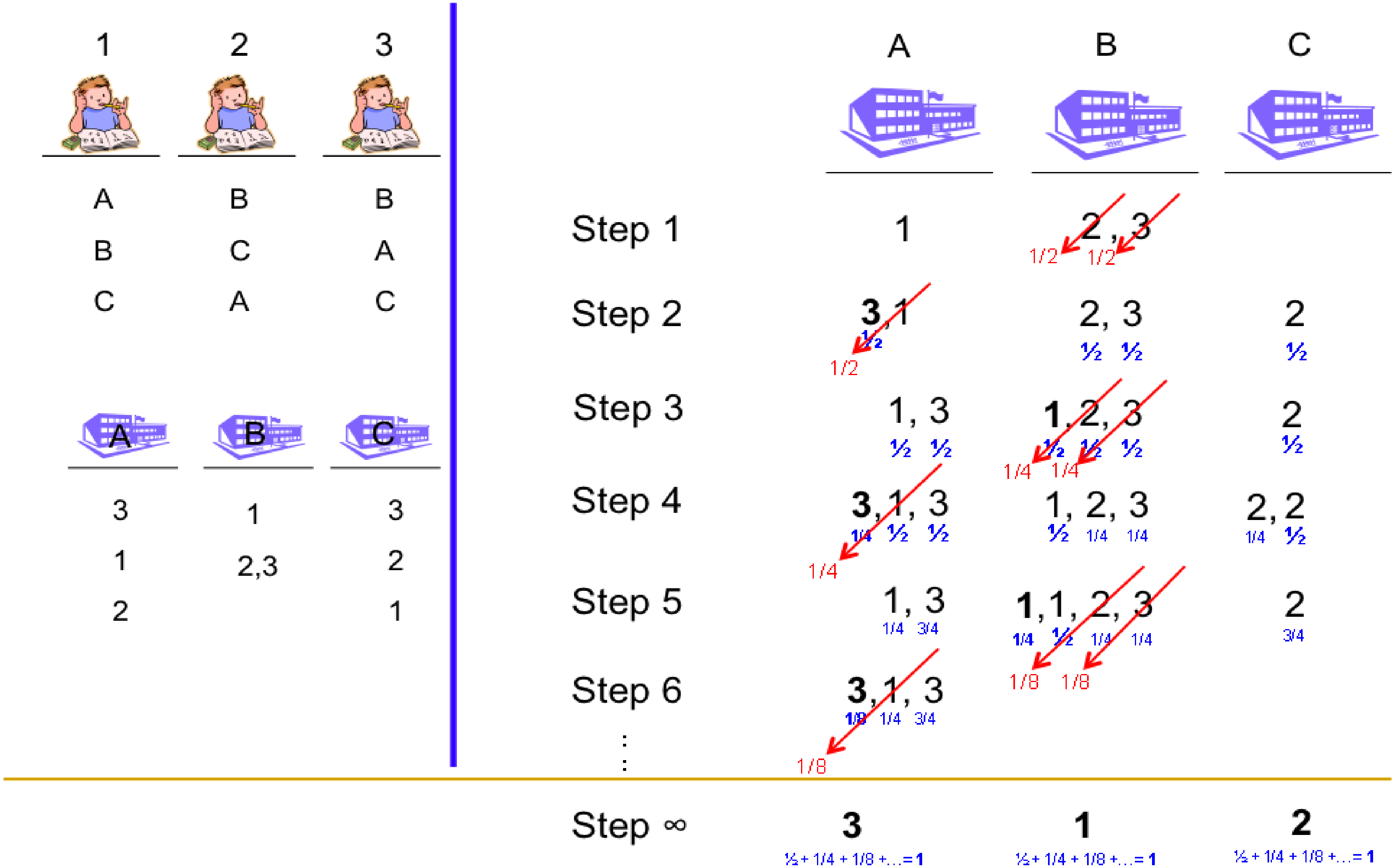
Fractional Deferred Acceptance: "Cycle Issue"



Fractional Deferred Acceptance: "Cycle Issue"



Fractional Deferred Acceptance: "Cycle Issue"



Difficulty and Solution

- This approach can cycle as shown above:

$$i_1, a_1 \rightsquigarrow i_2, a_2 \rightsquigarrow \dots \rightsquigarrow i_k, a_k \rightsquigarrow i_1, a_1$$

- **Solution:**
    - Randomly order students
    - Students make offers one at a time according to this order
    - Detect a cycle as soon as it happens: there exists a single current cycle

Proposition

We can resolve such a cycle in one step by determining the eventual fractional assignments resulting from this infinite cycle as a sum of infinite convergent series.

FDA Mechanism

- **FDA algorithm** is obtained by resolving cycles as they occur.
- **FDA mechanism** is the mechanism whose outcome is obtained through this algorithm.

Results on the FDA Mechanism

Proposition

The FDA mechanism has a polynomial algorithm.

Theorem

The FDA mechanism is strongly ex-ante stable.

Theorem

- *The FDA mechanism ordinally dominates any other strongly ex-ante stable mechanism.*
- *Thus, regardless of the order of students, it converges to the same outcome.*

Other related paper: Alkan & Gale (*JET 2005*)

Ex-ante Stability and Probability Trading

$(i, a) \blacktriangleright^{\rho} (j, b)$ (means i **top-priority envies** j **at school** b **through school** a) if

- i envies j at b through a, i.e. $\rho_{j,b} > 0$, $\rho_{i,a} > 0$, $bP_i a$,
- i is one of the highest priority students envying j at b.

An ex-ante stable improvement cycle is

$$(i_1, a_1) \blacktriangleright^{\rho} (i_2, a_2) \blacktriangleright^{\rho} \ldots \blacktriangleright^{\rho} (i_k, a_k) \blacktriangleright^{\rho} (i_1, a_1).$$

Generalization of deterministic stable improvement cyles of Erdil & Ergin (*AER 2008*).

What is Its Implication?

Theorem

An ex-ante stable random matching is

$$\text{constrained ordinally efficient} \iff \text{it does not include any ex-ante stable improvement cycle.}$$

Fractional Deferred Acceptance and Trading (FDAT) Approach

- **Step 0:** Run the FDA algorithm to find ρ^0.
- **General Step k>0:** Given $\rho^{\text{k-1}}$, in order to preserve ETE, find *all* ex-ante stable improvement cycles, and satisfy all of them simultaneously with an *equal maximum* possible fraction f, obtain ρ^{k}.
- Continue until no stable improvement cycle remains.

Difficulty and Solution

- Finding all ex-ante stable improvement cycles is in general computationally infeasible.
- Even if we found them, how do we satisfy them simultaneously?
- **Solution:** Use a network flow approach proposed for *housing markets* (Shapley & Scarf, *JME 1974*) by Yilmaz (*GEB 2009*) and Athanassoglou & Sethuraman (*2007* - AS hereafter)
  - Assume that the probabilities found at ρ^0 as the endwoments of students at schools.
  - Find all top-priority envy relationships: these are feasible school assignments.
  - Run the *constrained consumption algorithm* of AS to satisfy all ex-ante stable improvement cycles simultaneously (without explicitly finding them) for each step of the FDAT approach.
- **FDAT algorithm** combines constrained consumption algorithm with the FDAT approach.
- **FDAT mechanism** is the mechanism whose outcome is obtained through this algorithm.

Results on the FDAT Mechanism

Proposition

The FDAT mechanism has a polynomial algorithm.

Theorem

The FDAT mechanism satisfies ex-ante stability and ETE.

Theorem

The FDAT mechanism is constrained ordinally efficient within the ex-ante stable class.

FDAT fosd DA		DA fosd FDAT		FDAT=DA	not comp.		Overall	
62.9%		1.5%		30.7%	5.0%		100%	
FDAT	**DA**	**FDAT**	**DA**	**FDAT=DA**	**FDAT**	**DA**	**FDAT**	**DA**
0.760	0.456	0.407	0.482	1.000	0.498	0.414	0.815	0.621
0.168	0.331	0.565	0.496		0.256	0.311	0.127	0.231
0.055	0.131	0.026	0.020		0.179	0.177	0.044	0.091
0.013	0.049	0.002	0.001		0.055	0.066	0.011	0.034
0.003	0.020				0.010	0.020	0.002	0.013
0.001	0.008				0.002	0.007	0.000	0.005
	0.003				0.001	0.003		0.002
	0.001					0.001		0.001
						0.001		0.000
Justifiably ex-ante envious students in DA					5.6%			

Concluding Comments

- *Design of the lottery inducing the designed random matching(s)*
  - Straightforward using Proposition 1 and constructive proof of B-vN Theorem and Edmonds (1965) algorithm
  - with no more than $|I||C|$ stable matchings in the support
- *Incentives (future work)*
  - BM impossibility result in our domain: there is no constrained efficient, ex-ante stable, strategy-proof, and ETE mechanism.
  - How much is strategic manipulation a problem with FDAT in *higher* information settings?
- *Ex-post stable and constrained ordinally efficient lottery design (future work)*
  - The FDAT is not necessarily constrained ordinally efficient within the **ex-post** stable class.
  - Currently no such mechanism is known, since ex-post stability is not characterized using ex-ante random matching constraints.