Supplementary Material for

Efficient Kidney Exchange: Coincidence of Wants in Markets with Compatibility-Based Preferences

by

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(Online) Appendix B: Proofs of Proposition 4 and Theorem 2

Proof of Proposition 4: Inequality system 1 follows from the following five lemmas. Moreover for any incompatible pair i of type X-Y on the short side of the market, $v_i \geq 0$ and by Lemma 2, $p_{\text{X-Y}} > 0$. Therefore, $v_i + p_{\text{X-Y}} > 0$ implies pair i is matched and balanced budget condition together with Lemma 2 implies it is matched with a pair on the long side of the market.

Lemma 1: For any two mutually compatible types X-Y and W-Z,

$$p_{\text{X-Y}} + p_{\text{W-Z}} \geq 0.$$

Proof of Lemma 1: Let i, j be two mutually compatible pairs such that $\mathcal{T}(i) = $ X-Y, $\mathcal{T}(j) = $ W-Z, $v_i > 0$, and $v_j > 0$.[22] First suppose i is unmatched at μ. Then

$$0 \geq v_i - p_{\mathcal{T}(j)} \quad \Rightarrow \quad p_{\text{W-Z}} \geq v_i > 0 \quad \Rightarrow \quad v_j + p_{\text{W-Z}} > 0$$

and hence j is matched at μ. Therefore since pairs j and i are mutually compatible,

$$v_j + p_{\mathcal{T}(j)} = v_j - p_{\mathcal{T}(\mu(j))} \geq v_j - p_{\mathcal{T}(i)} \quad \Rightarrow \quad p_{\mathcal{T}(j)} + p_{\mathcal{T}(i)} = p_{\text{X-Y}} + p_{\text{W-Z}} \geq 0.$$

Next suppose i is matched at μ. Then since pairs j and i are mutually compatible,

$$v_i + p_{\mathcal{T}(i)} = v_i - p_{\mathcal{T}(\mu(i))} \geq v_i - p_{\mathcal{T}(j)} \quad \Rightarrow \quad p_{\mathcal{T}(i)} + p_{\mathcal{T}(j)} = p_{\text{X-Y}} + p_{\text{W-Z}} \geq 0$$

completing the proof. ◊

Lemma 2:

$$p_{\text{X-Y}} < 0 \quad \text{for any X-Y} \in \mathcal{L},$$
$$p_{\text{X-Y}} > 0 \quad \text{for any X-Y} \in \mathcal{S}.$$

Proof of Lemma 2: Let X-Y $\in \mathcal{L} \setminus \{$A-B$\}$ and suppose $p_{\text{X-Y}} \geq 0$. Then for any pair i with type X-Y and valuation $v_i > 0$,

$$v_i + p_{\text{X-Y}} > 0$$

and hence any pair i of type X-Y (with the possible exception of those with 0 valuation) are matched at μ by utility maximization. This contradicts Assumption 2. Hence $p_{\text{X-Y}} < 0$ for any X-Y $\in \mathcal{L} \setminus \{$A-B$\}$ and therefore $p_{\text{X-Y}} > 0$ for any X-Y $\in \mathcal{S} \setminus \{$B-A$\}$ by Lemma 1, proving the lemma for all types but A-B and B-A.

Next suppose $p_{\text{A-B}} \geq 0$ and let i be a type A-B pair with $v_i > 0$. Since $v_i + p_{\text{A-B}} > 0$, $\mu(i) \neq i$. Moreover $C(\text{A-B}) \subset \mathcal{S}$ and we have already shown that $p_{\text{X-Y}} > 0$ for any X-Y $\in \mathcal{S} \setminus \{$B-A$\}$. Therefore pair $\mu(i)$ has to be of type B-A for otherwise $p_{\mathcal{T}(i)} + p_{\mathcal{T}(\mu(i))} > 0$ contradicting balanced budget at competitive equilibrium. But that means any pair i of type A-B with $v_i > 0$ is matched with a pair of type B-A contradicting Assumption 3. Hence $p_{\text{A-B}} < 0$ and therefore $p_{\text{B-A}} > 0$ by Lemma 1. ♢

Lemma 3: $p_{\text{X-X}} = 0$ for any X-X $\in \{$O-O, A-A, B-B, AB-AB$\}$.

Proof of Lemma 3: Fix X-X $\in \{$O-O, A-A, B-B, AB-AB$\}$. Type X-X is mutually compatible with itself and therefore $2p_{\text{X-X}} \geq 0$ by Lemma 1 and hence $p_{\text{X-X}} \geq 0$. By Lemma 2, $p_{\text{X-Y}} > 0$ for any X-Y$\in \mathcal{S}$ and therefore by the balanced budget condition a pair of type X-X is not matched with a pair of a type on the short side at competitive equilibrium.

Pick any pair i of type X-X with $v_i > 0$. Since $v_i + p_{\text{X-X}} > 0$, $\mu(i) \neq i$. Moreover $C(\text{X-X}) \subset \mathcal{S} \cup \{$X-X$\}$ and we have already shown that $\mathcal{T}(\mu(i)) \notin \mathcal{S}$. Hence $\mu(i)$ is of type X-X. Therefore $2p_{\text{X-X}} = 0$ by the balanced budget condition and thus $p_{\text{X-X}} = 0$. ♢

Lemma 4: Let X-Y, W-Z $\in \mathcal{S}$ be such that $C(\text{X-Y}) \subseteq C(\text{W-Z})$. Then $p_{\text{W-Z}} \geq p_{\text{X-Y}}$.

Proof of Lemma 4: Let X-Y, W-Z $\in \mathcal{S}$ with $C(\text{X-Y}) \subseteq C(\text{W-Z})$ and suppose $p_{\text{W-Z}} < p_{\text{X-Y}}$. By Lemma 2, $p_{\text{X-Y}} > 0$ and $p_{\text{W-Z}} > 0$. Let i be a type X-Y pair such that $v_i > 0$. Since $v_i + p_{\text{X-Y}} > 0$, $\mu(i) \neq i$ and by balanced budget at competitive equilibrium

$$p_{\mathcal{T}(\mu(i))} = -p_{\text{X-Y}}.$$

Consider any pair j of type W-Z such that $v_j > 0$. Since $v_j + p_{\text{W-Z}} > 0$, $\mu(j) \neq j$. Since $C(\text{X-Y}) \subseteq C(\text{W-Z})$, pair j is mutually compatible with pair $\mu(i)$. But then

$$v_j + p_{\mathcal{T}(j)} = v_j + p_{\text{W-Z}} < v_j + p_{\text{X-Y}} = v_j - p_{\mathcal{T}(\mu(i))}$$

contradicting pair j maximizes utility at competitive equilibrium. Hence $p_{\text{W-Z}} \geq p_{\text{X-Y}}$. ♢

Lemma 5: For any type X-Y $\in \mathcal{T}$, $p_{\text{X-Y}} = -p_{\text{Y-X}}$.

Proof of Lemma 5: The result follows from Lemma 3 for types O-O, A-A, B-B and AB-AB. Consider any type X-Y $\in \mathcal{S}$. First observe that $p_{\text{X-Y}} < \bar{v}$ since all incompatible pairs of type X-Y are matched at competitive equilibrium, and if $p_{\text{X-Y}} \geq \bar{v}$, only pairs of valuation $\bar{v}$ could possibly be willing to match with them. That would yield a contradiction since the mass of the set of pairs with valuation $\bar{v}$ is $0 < \#^I(\text{X-Y})$.

Fix type Y-X$\in \mathcal{L}$. Pick any pair i of type Y-X with $v_i = \bar{v}$. Pair i is mutually compatible with any pair of its opposite type X-Y $\in \mathcal{S}$, and since $v_i - p_{\text{X-Y}} = \bar{v} - p_{\text{X-Y}} > 0$, $\mu(i) \neq i$. Let pair $\mu(i)$ be of type W-Z. By the balanced budget condition at competitive equilibrium

$$p_{\text{Y-X}} + p_{\text{W-Z}} = 0,$$

and therefore W-Z $\in \mathcal{S}$ by Lemma 2. Moreover since type W-Z is mutually compatible with type Y-X, the transitivity of the compatibility relation implies $C(\text{X-Y}) \subseteq C(\text{W-Z})$. Therefore $p_{\text{W-Z}} \geq p_{\text{X-Y}}$ by Lemma 4 and hence $p_{\text{Y-X}} + p_{\text{X-Y}} \leq 0$. But we also have $p_{\text{Y-X}} + p_{\text{X-Y}} \geq 0$ by Lemma 1 and thus $p_{\text{Y-X}} + p_{\text{X-Y}} = 0$. ♢

Proof of Theorem 2: We will establish the existence of a competitive equilibrium by providing a procedure that determines which types on the short side clear with their opposite types in segregated markets and which types on the short side are pooled. The procedure assures that the inequality system 1 holds.

For each type X-Y $\in \mathcal{S}$ on the short side, let $S_{\text{X-Y}}(p) = \#(\text{X-Y})(p)$ denote the supply function, and for each type Y-X $\in \mathcal{L}$ on the long side, let $D_{\text{Y-X}}(p) = \#(\text{Y-X})(-p)$ denote the demand function. As in Section IV B, each demand function $D_{\text{Y-X}}(p)$ is continuous and strictly decreasing in the interval $[0, \bar{v}]$ with $D_{\text{Y-X}}(0) = \#^I(\text{Y-X})$ and $D_{\text{Y-X}}(\bar{v}) = 0$ whereas each supply function $S_{\text{X-Y}}(p)$ is continuous and strictly increasing in the interval $[0, \bar{v}]$ with $S_{\text{X-Y}}(0) = \#^I(\text{X-Y}) < \#^I(\text{Y-X}) = D_{\text{Y-X}}(0)$.

Fix X-Y $\in \mathcal{S}$. Let $p^*_{\text{X-Y}} \in (0, \bar{v})$ be such that

$$D_{\text{Y-X}}(p^*_{\text{X-Y}}) = S_{\text{X-Y}}(p^*_{\text{X-Y}}).$$

Analogous to the case of the segregated market A-B&B-A, $p^*_{\text{X-Y}}$ is the unique market clearing price in the segregated market X-Y&Y-X. Repeat this for each type on the short side to determine the price vector p^*. If p^* satisfies the inequality system 1 then it is a competitive price and it supports the competitive equilibria determined by the six segregated markets. If p^* does not satisfy the inequality system 1 then either $p^*_{\text{B-A}}$ is too high or $p^*_{\text{AB-O}}$ is too low (or both). In this case some types on the short side will be pooled at competitive equilibria. We determine the types to be pooled and the resulting equilibrium prices with the following two step procedure. The first step assures that the price of type B-A is no more than the price of types B-O and AB-A whereas the second step assures that the price of type AB-O is no less than the price of any other type.

Step 1 (Pooling type B-A to reduce its price). If $p^*_{\text{B-A}} \leq \min\{p^*_{\text{B-O}}, p^*_{\text{AB-A}}\}$ then let $q^*_{\text{AB-A}} = p^*_{\text{AB-A}}$, $q^*_{\text{B-O}} = p^*_{\text{B-O}}$, $q^*_{\text{B-A}} = p^*_{\text{B-A}}$ and skip to Step 2. If $p^*_{\text{B-A}} > \min\{p^*_{\text{B-O}}, p^*_{\text{AB-A}}\}$, then the price of type B-A is too high. W.l.o.g. suppose $p^*_{\text{B-O}} \leq p^*_{\text{AB-A}}$ (the other case is analogous). Pool the short side types B-A and B-O and uniquely determine $\tilde{p} \in (p^*_{\text{B-O}}, p^*_{\text{B-A}})$ as follows:

$$D_{\text{O-B}}(\tilde{p}) + D_{\text{A-B}}(\tilde{p}) = S_{\text{B-O}}(\tilde{p}) + S_{\text{B-A}}(\tilde{p}).$$

Since $\tilde{p} > p^*_{\text{B-O}}$, quantity demanded by the less flexible O-B type demander decreases[23] (compared with the outcome in the segregated market) and quantity supplied by the more flexible B-O type supplier increases. Similarly since $\tilde{p} < p^*_{\text{B-A}}$, quantity demanded by the more flexible A-B type demander increases and quantity supplied by the less flexible B-A type supplier decreases. Hence it is feasible to clear a pooled B-O,B-A&O-B,A-B market by matching

- each one of $D_{\text{O-B}}(\tilde{p})$ type O-B demanders with a type B-O supplier,
- each one of $S_{\text{B-A}}(\tilde{p})$ type A-B demanders with a type B-A supplier, and
- each one of $D_{\text{A-B}}(\tilde{p}) - S_{\text{B-A}}(\tilde{p}) = S_{\text{B-O}}(\tilde{p}) - D_{\text{O-B}}(\tilde{p})$ type A-B demanders with a type B-O supplier.

Throughout the procedure whenever we pool a number of types, the quantity supplied by the more flexible suppliers as well as the quantity demanded by the more flexible demanders will be increased compared with the quantities in their segregated markets and hence it will be feasible to clear the pooled market.

If $\tilde{p} \leq p^*_{\text{AB-A}}$, then let $q^*_{\text{B-A}} = q^*_{\text{B-O}} = \tilde{p}$, $q^*_{\text{AB-A}} = p^*_{\text{AB-A}}$ and proceed to Step 2. At this point we know that short side types B-O and B-A shall be pooled (possibly together with other types to be determined in Step 2).

If $\tilde{p} > p^*_{\text{AB-A}}$, then the price of type B-A is still too high. In this case pool all three of short side types B-A, B-O, AB-A and uniquely determine $q^* \in (p^*_{\text{AB-A}}, \tilde{p})$ as follows:

$$D_{\text{A-AB}}(q^*) + D_{\text{O-B}}(q^*) + D_{\text{A-B}}(q^*) = S_{\text{AB-A}}(q^*) + S_{\text{B-O}}(q^*) + S_{\text{B-A}}(q^*).$$

Let $q^*_{\text{AB-A}} = q^*_{\text{B-A}} = q^*_{\text{B-O}} = q^*$ and proceed to Step 2. At this point we know that short side types AB-A, B-O and B-A shall be pooled (possibly together with other types to be determined in Step 2).

Step 2 (Pooling type AB-O to increase its price). If $p^*_{\text{AB-O}} \geq \max\{p^*_{\text{A-O}}, p^*_{\text{AB-B}}, q^*_{\text{AB-A}}, q^*_{\text{B-A}}, q^*_{\text{B-O}}\}$, then the only types to be pooled are those determined in Step 1.

If $p^*_{\text{AB-O}} < \max\{p^*_{\text{A-O}}, p^*_{\text{AB-B}}, q^*_{\text{AB-A}}, q^*_{\text{B-A}}, q^*_{\text{B-O}}\}$, then the price of type AB-O is too low and the short-side type with the highest price must be pooled with type AB-O. In case there are multiple short-side types with the highest price (e.g. one way that may happen is if the price in the pooled market from Step 1 is the highest price), then all these types shall be pooled with type AB-O. This increases the price of type AB-O. If the market clearing price in the pooling market that involves type AB-O is no less than the remaining prices then we are done and the pooled markets obtained in Step 1 and in Step 2 together with the segregated markets of remaining types yield a competitive price and competitive equilibria. Otherwise, the price of type AB-O is still not high enough and the type (or types) with highest price shall also be pooled with type AB-O further increasing the size of the pooled market and the procedure continues until type AB-O has one of the highest prices.

The above construction shows that the inequality system 1 can be satisfied once various types of pairs are adequately pooled.