On Determination of Optimal Reserve Price in Auctions with Common Knowledge about Ranking of Valuations*

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Abstract

This paper investigates a case study in designing auctions with optimal reserve prices when an analytical solution for the equilibrium bid functions does not exist. The Runge-Kutta integration method nested in a backward-shooting algorithm is used to solve the equilibrium bid functions for a first-price auction with bidders whose value ranking is common knowledge. The optimal reserve price is determined after a numerical search for the seller's maximum average revenue using simulated auction markets at equilibrium. We observe that setting the optimal reserve price can continue to raise, on average, more revenue for the seller in the first-price auction than in the second-price auction.

Keywords: Asymmetric auctions, dependent private value auctions, differential equations, first-price auctions, numerical methods, reserve price.
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1 Introduction

Traditional models of asymmetric auctions[1] assume that agents' values are believed to be drawn from heterogeneous distributions that are common knowledge. This is a strong assumption for real-life applications, where an agent might not be aware of the exact distribution of his opponents' values (even if it is believed that this value is distributed differently from his own). On the other hand, symmetric value auctions assume that agents believe that opposing bidders have values that are drawn from the same distribution as their own. This is a weak assumption in asymmetric settings, since agents can usually infer more than this in a real-life situation.

For instance, in the recent privatization of the PCS spectrum in the United States, it was known that the Pacific Bell Telephone Company had a distinct comparative advantage over other firms in the Los Angeles area and had a greater valuation for the frequency bands than the other participating firms: it was already in the wireless business in that area, had a reputable brand name, and had a database of the local customers. Moreover, the auction format helped them to win the auction at a low price. It was a hybrid of the second-price auction (SPA).

In the recent Glaxo and Wellcome merger, Glaxo publicly made clear that Wellcome was a very important investment before Wellcome was sold in the auction. All but a few potential bidders were deterred from entering the auction. Because of the auction format (i.e., open ascending auction), Glaxo won the auction easily at a low price (Klemperer (2002)).

We consider a model in which bidders draw their values from the same distribution, but in which the ranking of these values is common knowledge. So, while the belief structure in the auction is exante symmetric, the revelation of the ranking makes it expost asymmetric.

In most of the real-life auctions (including the examples above), the auctioneer sets a reserve price. It is well-known that the reserve price increases the expected revenue of the auctioneer by preventing the object from being sold at a low price (Myerson (1981)). However, the assumptions of the revenue equivalence theorem[2] (RET) do not hold in environments like ours involving the first-price auction (FPA) and SPA (Maskin and Riley

[1]An auction is referred to as 'asymmetric' if the bidders' belief functions (i.e., conditional densities) concerning their opponents' values are not the same as the bidders' own (examples are given in Maskin and Riley (2000a); Marshall et al. (1994)). The terminology is somewhat different from that of the private information game literature, which uses 'asymmetry' when referring to the private information structure.

[2]The RET was independently derived by Myerson (1981) and Riley and Samuelson (1981) using as principal assumptions risk neutrality, independence of bidders' reservation prices, lack of collusion among bidders, and symmetry of buyers' beliefs.

(2000a) and Lebrun (1998)). Landsberger et al. (2002) develop a model similar to what we consider below. For this model, the FPA dominates the SPA when the seller's reserve price is zero.[3]

Our contribution is to extend the Landsberger et al. (2002) model by introducing reserve prices and provide a well-outlined algorithm for the computation of the equilibrium bid functions for the FPA. We then determine the optimal reserve price.[4] The existence of a unique equilibrium in the FPA follows from convergence of the numerical algorithm. We provide numerical evidence for the proposition that the FPA can raise as much, or even more, revenue as the SPA, when optimal reserve prices are set. The FPA with common knowledge about the ranking of values introduces inefficiency into the auction. However, we find evidence that a positive reserve price reduces this inefficiency when the object is allocated. Beyond these numerical inspections, we analytically derive conditions for which the FPA would be more desirable than the SPA for small reserve prices. We pin down the behavior of the bid functions of the FPA in detail. Our findings suggest why the SPA was not very successful for sellers in the above examples with reserve prices.[5]

[3]Landsberger et al. (2002) point out that this particular relationship between bidders' valuations will give rise to an asymmetric auction with affiliated private values where the FPA dominates the SPA. However, Milgrom and Weber (1982) arrive at a different conclusion (i.e., SPA dominates FPA) under the affiliation assumption for a symmetric model.

[4]It is not a trivial exercise to introduce a reserve price in a FPA model with asymmetric dependent beliefs about private valuations. The behavior of the ordinary differential equations that are used to analyze the equilibria change substantially in presence of a reserve price.

[5]The model considered here is different from Maskin and Riley's (2000a), Marshall and Schulenberg's (1998) and Marshall et al.'s (1994) models in the following aspects. First, Maskin and Riley consider several models with asymmetry of beliefs due to buyers' private valuations (technically, bidders' valuations are believed to be drawn from *independent heterogeneous* distributions). These differences may be due to different opportunity costs of completing a project in the case of contract bidding or pre-existing different budget constraints in the case of major art auctions. Second, Marshall et al. and Marshall and Schulenberg consider models with asymmetries arising from bidders' coalitions (this is equivalent to considering a model with two bidders with *asymmetric* beliefs about private valuations, which are *independent*.) Here, we consider a model with exante independent valuations from the same densities, but with expost *asymmetric* beliefs after the ranking is revealed for the *dependent* private valuations. The characterization of equilibria substantially changes in existence of asymmetric but dependent beliefs about valuations over the cases where the beliefs are asymmetric and independent. Therefore, our case is not a subclass of problems handled by Maskin and Riley, Marshall et al., Marshall and Schulenberg or any other paper in the literature. Lizzeri and Persico (2000) and Maskin and Riley (2000b) also consider some asymmetric auction settings which differ from our model at certain points.

2 The FPA model with a positive reserve price

2.1 The basic model

A single object is to be auctioned to two bidders. Both bidders have risk neutral utility functions and independently drawn private values. These values V_1 and V_2 are exante identically distributed according to a differentiable probability density function $g : [0,1] \to R^{++}$ in the support set $[0,1]$. Assume that the seller has set a reserve price, $r \in [0,1]$. Let $G : [0,1] \to [0,1]$[6] be the cumulative probability distribution of a value v in $[0,1]$. Both G and r are common knowledge to the bidders. Each bidder j also knows whether she has the higher or the lower value, although she does not know her opponent's exact value. The ranking of values is also observable by the seller, who sets the reserve price.

Let H be the bidder with the higher value, $v_H = \max\{v_1, v_2\}$, and L be the bidder with the lower value, $v_L = \min\{v_1, v_2\}$. For the case when $r = 0$, Landsberger et al. (2002) prove that there exists a unique equilibrium in pure strategies for the FPA that is strictly increasing with respect to values. Moreover, Landsberger et al. (2002) show that if $r = 0$ and G is the uniform distribution, then both bid functions, b_L and b_H, are greater than the symmetric independent private value bid function, b_S (i.e., $b_j(v) > b_S(v)$, $j = L, H$).

The expost asymmetric FPA under uniform density therefore dominates both the expost asymmetric SPA (which has the same equilibrium bid functions as the symmetric SPA) and the symmetric FPA in terms of the expected revenue generated by the seller. Landsberger et al. (2002) prove these results only for the case of reserve price equal zero (i.e., $r = 0$). We shall show that the introduction of a relatively high positive reserve price, $r > 0$, changes the properties of the system substantially at the initial boundary point.

For each r, the low-value bidder's equilibrium bid function, b_L, and the high-value bidder's function, b_H, satisfy $b_L(v) > b_H(v)$, $\forall v \in (r, 1)$. This result implies inefficiency in the FPA (i.e., the low-value bidder getting the object with positive probability).

In this model, V_L and V_H each has a strictly positive marginal density in the interval $[0,1]$ such that the joint density (with a triangular support) is $f(v_H, v_L) = 2g(v_H)g(v_L)$ for $v_H \geq v_L$ and $(v_H, v_L) \in [0,1] \times [0,1]$.[7] Landsberger et al. (2002) point out that since values are stochastically dependent, V_H and V_L can be viewed as *affiliated à la* Milgrom and Weber (1982). That is, a higher value of the item for one bidder does not in general imply lower

[6] Landsberger et al. (2002) also assume, as a technical requirement, that G has a Taylor expansion around zero (i.e., $G(v) = \alpha v + \beta v^2 + ...$, with $\alpha > 0$).

[7] Note that marginal densities are $f_H(v) = 2G(v)g(v)$ and $f_L(v) = 2(1 - G(v))g(v)$ for all $v \in [0,1]$.

values for the other bidder. While the affiliation is between symmetric distributions of signals for Milgrom and Weber, this model can be seen as a special affiliated private-value model that considers a specific asymmetric (triangular) distribution of signals.

2.2 Analysis of pure strategy equilibrium for the FPA given a positive reserve price

We start by conjecturing the existence and uniqueness of equilibrium. This will help to illustrate the analytical difficulties with the model.

We seek a strictly increasing equilibrium (b_H, b_L) where $b_H : [r, 1] \to [r, t_r]$ is the differentiable bid function of the high-value bidder, and $b_L : [r, 1] \to [r, t_r]$ is the differentiable bid function of the low-value bidder, both having as an argument the bidder's own value, v_i. Note that these functions also depend on r. Assume that such an equilibrium exists. Let $l : [r, t_r] \to [r, 1]$ be the inverse bid function of the bidder L with respect to her value, v, at the reserve price, r (i.e. $l(b_L(v)) = v \ \forall \ v$). Similarly let $h : [r, t_r] \to [r, 1]$ be the inverse bid function of the bidder H with respect to her value, v, at the reserve price, r (i.e. $h(b_H(v)) = v \ \forall \ v$).

At the equilibrium with probability $\frac{G(l(b))}{G(v_H)}$ bidder H gets the object if she bids b and has a value of v_H. Hence, the problem for the high-value bidder is to maximize the following function:

$$\max_b G(l(b))(v_H - b) \tag{1}$$

Bidder L wins the auction with probability $\frac{(G(h(b))-G(v_L))}{(1-G(v_L))}$ with value v_L if she bids b at the equilibrium. Hence the maximization problem of the bidder L is the following:

$$\max_b \left(G\left(h\left(b\right)\right) - G\left(v_L\right)\right)\left(v_L - b\right) \tag{2}$$

Notice that in equilibrium $v_L = l(b)$ when L bids b and $v_H = h(b)$ when H bids b; therefore, both equations (1) and (2) have a unique maximum. By the first-order necessary conditions, both maximization problems can be reduced to the following system of differential equations with two boundary conditions. Consider the following notation: $G_i = G \circ i$ for $i = l, h$.

$$G_l'(b) = \frac{G_l(b)}{G^{-1}(G_h(b)) - b} \tag{3}$$

$$G_h'(b) = \frac{G_h(b) - G_l(b)}{G^{-1}(G_l(b)) - b} \tag{4}$$

$$G_l(r) = G_h(r) = G(r) \tag{5}$$

$$\exists\ t_r \in [r, 1] \text{ such that } G_l(t_r) = G_h(t_r) = 1 \tag{6}$$

where G^{-1} is the inverse cumulative distribution function of G.

The first boundary condition, (5), follows from the fact that a bidder does not have incentives to decrease her bid below r at equilibrium when values are equal to r. In either case, she has zero utility. Since bidding above her value is weakly dominated by bidding exactly her value, condition (5) characterizes an equilibrium boundary condition. Note that at equilibrium, bidders do not submit bids when their values are below r. This is also true for the SPA. Note that bidding the value is a weakly dominant strategy and constitutes an equilibrium in the SPA.

The second boundary condition, (6), can be explained as follows: if the bids are not equal to each other at equilibrium, when the values are close to 1, the owner of the higher bid can lower her bid slightly and still can win the auction. Then, there exists some bid t_r such that at $v = 1$ both bid functions are equal to this bid.

A solution to equations (3), (4), (5), and (6) characterizes the equilibria for the FPA. Our conjectures are (i) that there exists a unique t_r for each r satisfying (3), (4), (5), and (6); (ii) that all pure strategy equilibrium bid functions acquire the same values in the interval $[r, 1]$; and (iii) that they are strictly increasing such that $b_L(v) \geq b_H(v)$ for all values of r and v. If the numerical algorithm me provide converges, this will be the numerical proof of the conjecture for the used probability density function g.

We use numerical methods to solve the system for each $r > 0$. The special feature of our model comes from information conditional on what each bidder knows: that is, whether they have the lower or higher value. The following conjecture justifies our effort to construct equilibrium bid functions in the FPA. Here we consider the *first-order ordinary differential equation system* (FOODES) described by (3), (4), and (6) and show that only one t_r of these solutions satisfies (5).

Conjecture 1 *There exist equilibria of the FPA game in pure strategies, and all the pure strategy equilibria are increasing and the same in the value interval* $[r, 1]$ *for each* $r \in [0, 1)$.

3 Expected revenue comparison at the equilibrium for small reserve prices

In this section, we try to derive general sufficient conditions regarding the expected revenue comparisons under both the FPA and the SPA. To make a general assessment, we need to do the following analysis:

Consider the following model: V_1 and V_2 are drawn identically and independently from G. The rankings are not revealed. In this model, let $b_S : [r, 1] \to [r, t_r^s]$ be the unique equilibrium increasing bid function, where t_r^s is the highest possible bid. Suppose that $\sigma : [r, t_r^s] \to [r, 1]$ denotes the equilibrium inverse bid function when the FPA is played by symmetric bidders with values drawn from G. Then, each bidder solves the following problem, since she wins the object with probability $G(\sigma(b))$:

$$\max_b G(\sigma(b))(v - b) \tag{7}$$

We need to introduce some further notation. Suppose that $G_\sigma = G \circ \sigma$, then

$$G'_\sigma(b) = \frac{G_\sigma(b)}{G^{-1}(G_\sigma(b)) - b} \tag{8}$$

$$G_\sigma(r) = G(r) \tag{9}$$

$$\exists\ t_r^s \in [r, 1] \text{ such that } G_\sigma(t_r^s) = 1 \tag{10}$$

by the first-order conditions.

We now consider the case when $r = 0$. Let's introduce the following notation: $\delta^* = \frac{G_\sigma}{G_l}$, and $\delta^{**} = \frac{G_l}{G_h}$. By the first-order conditions, the following lemmata apply:

Lemma 1 *If $r = 0$, then (i) $\exists\ \varepsilon > 0$ such that $b_S(v) < b_L(v)\ \ \forall\ v \in (0, \varepsilon)$. Moreover, (ii) $\exists\ \varepsilon > 0$ such that $b_S(v) < b_H(v)\ \ \forall\ v \in (0, \varepsilon)$ if and only if $g'(0)\ \le 0$.*

Proof. See Appendix. ■

Lemma 2 *If $r = 0$, then $b_S(v) < b_L(v)\ \ \forall\ v \in (0, 1)$.*

Proof. See Appendix. ■

Lemma 3 *If $r = 0$, then $\exists\ \varepsilon > 0$ such that. $b_S(v) < b_H(v)\ \ \forall\ v \in (1 - \varepsilon, 1)$.*

Proof. See Appendix. ■

Lemma 4 *If $r = 0$ and δ^{**} is increasing at each point, then $b_S(v) < b_H(v) \quad \forall\, v \in (0, 1]$.*

Proof. See Appendix. ■

Corollary 1 *If $r = 0$ and δ^{**} is increasing at each point, then the FPA at the unique pure strategy equilibrium generates higher expected revenue to the seller than the SPA at the weakly dominant pure strategy equilibrium (i.e., $\delta^{**\prime}(b) > 0 \;\forall\; b \in (0, t_0] \Rightarrow ER_{FPA}(r = 0) > ER_{SPA}(r = 0)$)*

Proof. By Lemma 4, the FPA under the symmetric setting generates less expected revenue than the FPA of our model. By the RET of Myerson (1981), the SPA under symmetric setting generates the same expected revenue as the FPA under the symmetric setting. Note that the SPA under the symmetric setting raises the same expected revenue as the SPA in our model because the winning bid, second price and equilibria are identical. Therefore, the FPA in our model generates more expected revenue than the SPA. ■

When G satisfies certain conditions, the bid function under the symmetric case is lower than the bid functions under the asymmetric case for $r = 0$. There exist distributions that do not satisfy the condition in the hypothesis of the lemma above. For example, if $g'(0) > 0$ then $\delta^{**\prime}(0^+) < 0$: the condition is not satisfied.

The corollary translates one-to-one for small positive reserve prices by continuity of the expected revenue functions in r. Continuity of bid functions (and expected revenue) in r follows from a result for existence of nearby solutions for different initial conditions for the FPA.

For $r >> 0$, the analysis is more complicated. It can be shown that $b_H(v) < b_S(v)$ for any g for any v sufficiently close to r. We could not find an analytical solution. We give a numerical algorithm below in order to consider this situation.

4 Numerical analysis to determine the optimal reserve price

The following conjecture allows us to implement a search technique outlined in this section for determination of the unique expected revenue-maximizing reserve price.

Conjecture 2 *The seller's expected revenue is strictly quasi-concave in the reserve price. Therefore, there exists a unique $r^*_{FPA} \in [0, 1]$ that maximizes locally the seller's expected revenue at equilibria for the FPA.*

It should be noted that the grid search results in a strictly quasi-concave expected revenue as a function of reserve price. We find evidence for our conjecture in our numerical grid search. The analysis made in the previous section suggests that as r increases, t_r also increases.

For a given r and t, the equations (3) and (4) are numerically integrated, starting from the second boundary equation (6). For this purpose, the "backward-shooting algorithm," developed by Marshall et al. (1994), is nested within the fourth-order Runge-Kutta method. This procedure is similar to the one described in Marshall and Schulenberg (1998). However, some modifications are introduced in order to achieve a higher level of precision. First, the Runge-Kutta method is implemented using an adaptive-step-searching grid (called the Runge-Kutta-Fehlberg method). Second, a control variate is introduced in order to generate an accurate Monte Carlo estimate for the seller's expected revenue.

For each r considered, the Euclidean distance to the true initial boundary condition is minimized in the determination of t_r. Once the optimal t_r is found, the next step is the construction of the bid functions from the inverse bid functions. These lead to an approximation of the seller's expected revenue (denoted by $\widetilde{ER}_{FPA}(r)$) when each bidder uses the approximate equilibrium bid functions. A statistical Monte Carlo estimate (denoted by $\widehat{ER}_{FPA}(r)$) is used to estimate this approximation. The Monte Carlo sample size is denoted as N.

The determination of the optimal reserve price, r^*_{FPA}, is done by maximizing the Monte Carlo estimate of the seller's expected revenue with respect to r. Since the objective function in question is only a statistical estimate rather than an analytical function, a sensitive and detailed algorithm with control variates is used. The following is the outline of the steps of the search algorithm:

1. Search to determine the optimal reserve price for the FPA, r^*_{FPA}, in the interval $[0, 1]$ such that the estimate of the seller's expected revenue is maximized. To do this, the search interval for r, (r_{1_n}, r_{2_n}), is continuously narrowed in the direction where the average revenue increases. This search continues until $|r_{2_n} - r_{1_n}| < tol_r$ holds for the maximum tolerance tol_r. Notice that this would be possible if Conjecture 2 is correct. (This is called a golden section search.)

2. Evaluate numerically the inverse bid functions for each attempted value of r in order to find the highest bid t_r that satisfies the two boundary conditions. For each trial of t, the inverse bid function is calculated in the interval $[r, t]$. The criterion is to minimize

$(l^t(r) - r)^2 + (h^t(r) - r)^2$. The search interval for t_r, (t_{1_n}, t_{2_n}), is continuously narrowed in the direction where this square of the Euclidean distance decreases. The search stops when $|t_{2_n} - t_{1_n}| < tol_t$ holds for the maximum tolerance tol_t.[8]

3. Iterate the Runge-Kutta method 'backwards' until the bid b_{n_f} at the n_f^{th} step is within the tol_b neighborhood of r. The system can be re-written as follows for each t considered:

$$l^{t\prime}(b) = p_l(b, l^t(b), h^t(b)) \tag{11}$$

$$h^{t\prime}(b) = p_h(b, l^t(b), h^t(b)) \tag{12}$$

$$l^t(t) = 1 \text{ and } h^t(t) = 1 \tag{13}$$

   By simply numerically integrating this system of equations, we obtain the Euler method. After refining our steps four times, we yield a single step of the fourth-order Runge-Kutta method.

4. Calculate the values of the functions l^t_{n+1} and h^t_{n+1} from their previous values, l^t_n and h^t_n, and from the step-size for b_n, db_n, which is adjusted by the increments dh^t_n and dl^t_n within a certain ratio of reference values. This maximum tolerance ratio is called ε and the reference values are set "close" to the previous values of the functions. Moreover, the adaptive step-size becomes larger at the flat portions of the functions l^t and h^t.

5. Calculate the inverse bid functions in - at most - K step points backward beginning from $b_0 = t$. This calculation is done once in every s iteration of the Runge-Kutta method. The maximum iterations of the Runge-Kutta algorithm is therefore set equal to $K \times s$. The step size is initially set to db_0. The iteration stops when $K \times s$ values are calculated, $n = K \times s$, or b_n is in tol_b - neighborhood of r for $n \leq K \times s$. The squared distance to the smallest bid r is calculated at the last values found, $l^t(b_{n_f})$ and $h^t(b_{n_f})$, where $n_f \leq K \times s$ is the final calculated point index.

6. Finally, calculate a Monte Carlo estimate of the average revenue by simulating the FPA at reserve price r with bidders playing the equilibrium strategies. In order to reduce the variance of this estimate, the following control variate is used (Davidson and MacKinnon, 1993):

$$c = R_{SPA}(v_L, v_H, r) - ER_{SPA}(r) \tag{14}$$

[8] Marshall et al. (1994) suggest that the nonexistence of Nash equilibrium would typically manifest itself in the form of cycles in the numerical search for t_r.

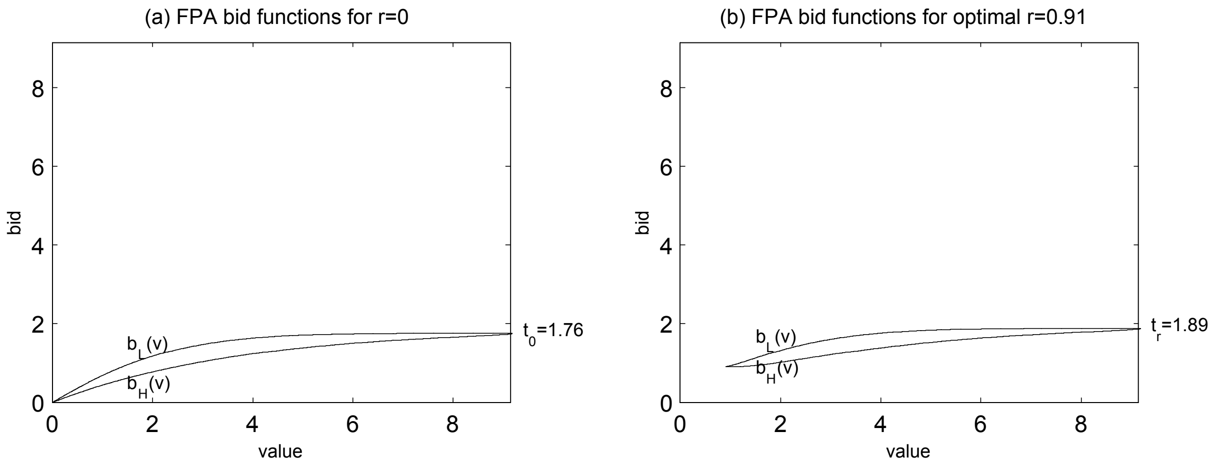

Figure 1: The equilibrium bid functions of the bidders in the FPA when $r = 0$ (Graph a) and $r = r^*_{FPA}$ (Graph b). The example uses exponential density.

This control variate represents the difference between the revenue under the SPA, R_{SPA}, and the expected revenue under the SPA, ER_{SPA}, at the weakly dominant strategy equilibrium. It is straightforward to verify that

$$R_{SPA}(v_L, v_H, r) = \begin{cases} v_L \text{ if } v_L, v_H \geq r \\ r \text{ if } v_L < r \text{ and } v_H \geq r \\ 0 \text{ otherwise} \end{cases} \tag{15}$$

We set for our purposes the following parameters:

ε	tol_r	tol_t	tol_b	dx_0	s	N	K
10^{-18}	10^{-7}	10^{-7}	10^{-7}	-10^{-4}	5	10^6	10^4

The numerical analyses of the FPA derived for the optimal reserve price in the FPA, $r = r^*_{FPA}$, for the optimal reserve price in the SPA, $r = r^*_{SPA}$, and for zero reserve price, $r = 0$, are shown in Table 1 for exponential density. Figure 1 shows the bid functions of the FPA derived for $r = 0$ (Graph a) and $r = r^*_{FPA}$ (Graph b).[9]

[9]For the numerical analysis, Pascal implementation of Press et al. (1996) and Borland Delphi compiler are used on an IBM PC compatible Pentium microprocessor-based machine. The program is available from the authors upon request.

Table 1: Numerical analysis of the FPA for exponential density

FPA - Equilibrium	$r = r^*_{FPA}$	$r = r^*_{SPA}$	$r = 0$
Reserve Price (r)	0.9111	1	0
Highest Bid (t^*_r)	1.8853	1.9144	1.7605
Seller			
Average Approx. Revenue	0.6806	0.6788	0.5955
	(0.0002263)	(0.0002095)	(0.0004030)
High-Value Bidder			
Average Approx. Payoff	0.5524	0.5220	0.7575
	(0.0009864)	(0.0001004)	(0.0001012)
Approx. Prob. of Winning	0.5523	0.5221	0.7575
	(0.0004973)	(0.0004995)	(0.0004536)
Low-Value Bidder			
Average Approx. Payoff	0.03741	0.03308	0.07987
	(0.0004737)	(0.0004624)	(0.0005734)
Approx. Prob. of Winning	0.08972	0.07813	0.2425
	(0.0002858)	(0.0002684)	(0.0004286)
Approx. Allocative Inefficiency	0.1397	0.1302	0.2425

Table 2: Equilibrium analysis of the SPA for exponential density

When $b_j(v_j) = v_j$, the weakly dominant strategy			
SPA - Equilibrium	$r = r^*_{SPA}$	$r = r^*_{FPA}$	$r = 0$
Reserve Price (r)	1	0.9111	0
Seller			
Expected Revenue	0.6681	0.6662	0.5
High-Value Bidder			
Expected Payoff	0.6004	0.6425	1
Prob. of Winning	0.6004	0.6425	1

Table 3: Seller's expected revenue comparison

$\frac{ER_{SPA}(r^*_{SPA})}{\widehat{ER}_{FPA}(r^*_{FPA})}$	$\frac{ER_{SPA}(0)}{\widehat{ER}_{FPA}(0)}$
0.9816	0.8396

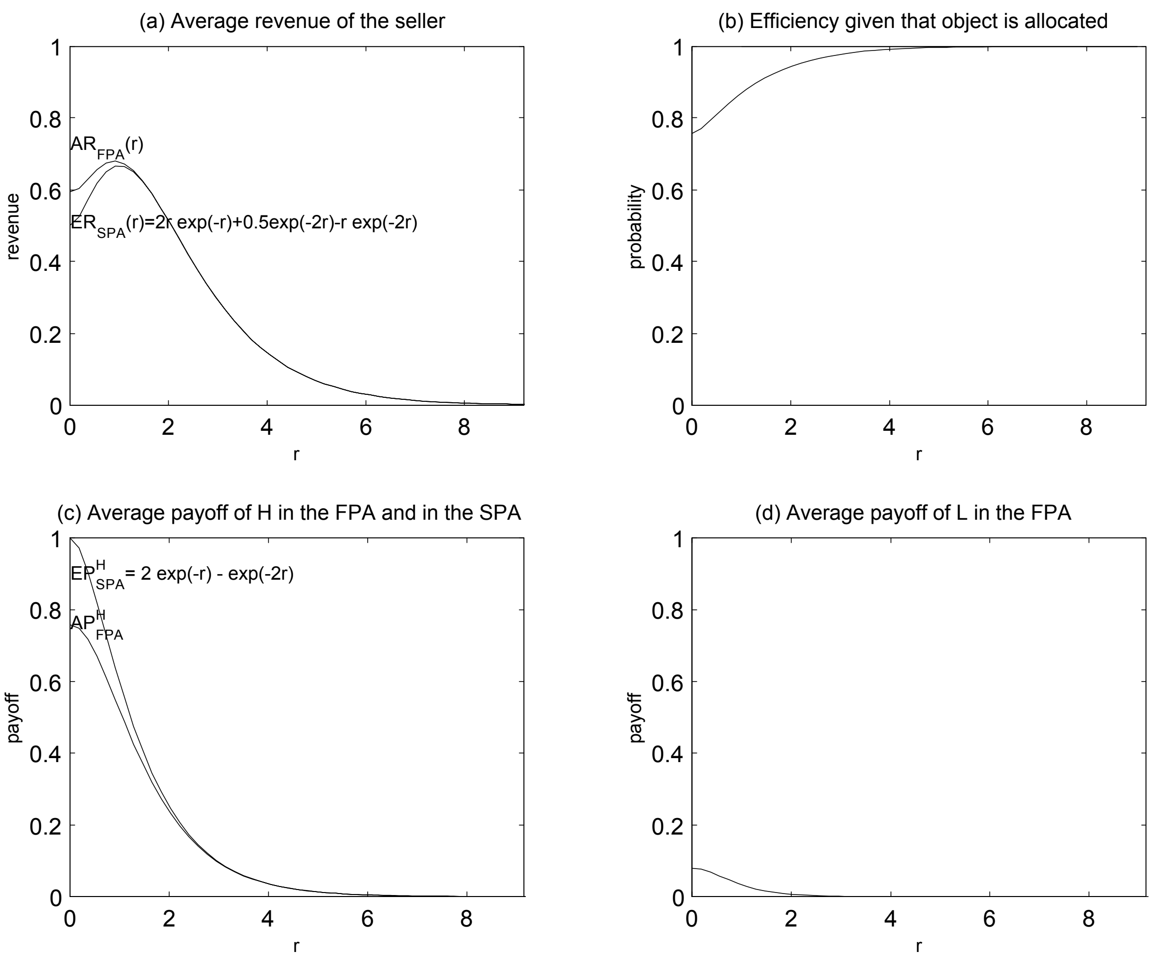

Figure 2: The average revenue in the FPA and the SPA (Graph a), the allocative efficiency of the FPA (Graph b), the average payoff of the high value bidder in the FPA and in the SPA (Graph c), and the average payoff the low value bidder in the FPA (Graph d). This numerical example uses exponential density.

4.1 An example for which the FPA dominates the SPA at equilibrium

We derive expected revenue for the seller, the probability of H winning the object, and expected payoff of H in the SPA at equilibrium. We also determine the optimal reserve price, r^*_{SPA}.

Tables 1 and 2 show the analyses of expected revenue for exponential density with speed parameter 1 (i.e., $g(v) = \exp(-v)$ for each $v \in [0, +\infty)$ and zero otherwise) in the FPA and the SPA.[10] Our algorithm converges, therefore this is a numerical verification of Conjecture 1.

In our example, the optimal reserve price for the SPA, r^*_{SPA}, is greater than the optimal reserve price for the FPA, r^*_{FPA} (i.e., $r^*_{SPA} > r^*_{FPA}$). We find that the FPA generates more expected approximate revenue than the SPA, once optimal reserve prices are calculated for the auction with exponential density (i.e., $\widetilde{ER}_{FPA}(r^*_{FPA}) > ER_{SPA}(r^*_{SPA})$). The null hypothesis that $\widetilde{ER}_{FPA}(r^*_{FPA}) = ER_{SPA}(r^*_{SPA})$ is rejected ($t-$stat= 55.23, $p < 0.0001$). Table 3 shows the ratio of the revenue in the SPA to that in the FPA. Optimal reserve price increases the ratio, nevertheless it is still less than 1. Graph a in Figure 2 shows the seller's expected revenue for both auction formats with respect to r. For the FPA, it is found that the curve is strictly quasi-concave as was conjectured.

The usage of $t-$test can be justified by the fact that the average approximate revenue $\widehat{ER}_{FPA}(r^*_{FPA})$ is a Monte Carlo estimate and is asymptotically normally distributed with the mean expected approximate revenue $\widetilde{ER}_{FPA}(r^*_{FPA})$ (obtained at equilibrium with the approximated equilibrium bid functions). Its mean and its variance are estimated by the mean and variance in the Monte Carlo simulation. We are testing the null hypothesis that $\widetilde{ER}_{FPA}(r^*_{FPA}) = ER_{SPA}(r^*_{SPA})$. We are not testing whether the "exact" expected revenue in the FPA, $ER_{FPA}(r^*_{FPA})$, is equal to the expected revenue in the SPA, $ER_{SPA}(r^*_{SPA})$, as we do not have a statistical estimate of the precision of the approximation.

The degree of *allocative* inefficiency can be defined as $\Pr\{L \text{ wins the object} \mid \text{object is allocated}\}$. Under these conditions, it is shown that efficiency increases with the introduction of reserve prices. For exponential density, the probability of allocative inefficiency decreases from 0.24 to 0.14. So, the introduction of a reserve price can be used to increase allocative efficiency in the FPA. These results are described in Table 1. Graph b in Figure 2 also displays the efficiency rate of the FPA with respect to r in case the object is allocated.

[10] In numerical analysis, we assume that the support of valuations is limited to the interval $[0, u]$ for $u = 9.2103$ (where $G(u) = 0.9999$).

The high-value bidder prefers the SPA over the FPA, where she might lose the auction to the low-value bidder with positive probability. This can be seen in Figure 2 Graph c, which shows the high-value bidder's expected payoffs in the FPA and SPA with respect to r. On the other hand, it can be established that the low-value bidder prefers the FPA over the SPA, for which she has zero expected revenue. This can be seen in Figure 2 Graph d, which is the plot of average approximate payoff of the low-value bidder in the FPA with respect to r. Also the values of the payoffs are given in Tables 1 and 2 for the reserve prices considered.

In an extended working paper (Elbittar and Ünver (2001)), we analyze different underlying density distributions in the existence of more bidders. In each case, the findings are similar to the example above. The FPA is more desirable than the SPA for zero reserve price. For the optimal reserve price setting the FPA continues to be at least as desirable as the SPA for the seller.

5 Conclusion

In designing real-life auctions, the symmetry assumption is often unreasonable due to different demographies of bidders. In this study, given the existence of a particular asymmetry, we find evidence that the FPA is still at least as desirable as the SPA, once optimal reserve prices are set. Furthermore, we show that allocative inefficiency of the FPA is reduced after imposing a positive reserve price. This gives strong generality to Landsberger et al.'s (2000) results.

Finally, we explicitly give an algorithm that can be used for solving equilibria for models of auctions. Two important points should be noted: the application of an adaptive-step method, which shows more accuracy than fixed-step integration methods, and the introduction of a control variate as a variance reduction technique, which helps to increase the accuracy of the numerical results significantly. We believe that these improvements and our calculations will enhance our comprehension of different auction environments with positive reserve prices that seem analytically intractable. In our case, it has helped us to numerically probe and formulate conjectures for reserve prices greater than zero.

Appendix

Lemma 1 Proof. We will prove this lemma by comparing first a couple of terms of the expansions of the functions in consideration around 0. Now, by l'H$\widehat{o}$pital's rule $\sigma'(0) = 2$. Similarly, by l'H$\widehat{o}$pital's rule $h'(0) = 2$ and $l'(0) = \frac{4}{3}$. Now $\sigma(0) = h(0) = l(0) = 0$.

(i) Therefore, the following is trivial by continuity of σ and l in a neighborhood of zero (excluding zero):

$$l(b) = \frac{4}{3}b + ... < \sigma(b) = 2b + ... \tag{16}$$

by expansions around zero. So $b_S(v) < b_L(v)$ in a neighborhood of zero.

(ii) Consider the second derivatives of the inverse bid functions:

$$\frac{l''(b)}{l'(b)} = \frac{2 - h'(b)}{h(b) - b} - \frac{g'(l(b))l'(b)}{g(l(b))} \tag{17}$$

$$\frac{h''(b)}{h'(b)} = \frac{2 - l'(b) - \frac{g(l(b))l'(b)}{g(h(b))h'(b)}}{l(b) - b} - \frac{g'(h(b))h'(b)}{g(h(b))} \tag{18}$$

$$\frac{\sigma''(b)}{\sigma(b)} = \frac{2 - \sigma'(b)}{\sigma(b) - b} - \frac{g'(\sigma(b))\sigma'(b)}{g(\sigma(b))} \tag{19}$$

So by l'H$\widehat{o}$pital's rule at $b = 0$, $h''(0) = -\frac{12}{13}\frac{g'(0)}{g(0)}$ and $\sigma''(0) = -\frac{4}{3}\frac{g'(0)}{g(0)}$. If $g'(0) < 0$ then in a neighborhood of 0 (excluding zero)

$$h(b) = 2b - \frac{12}{13}\frac{g'(0)}{g(0)}b^2 + ... < \sigma(b) = 2b - \frac{4}{3}\frac{g'(0)}{g(0)}b^2 + ... \tag{20}$$

by series expansions around 0. Then, $b_S(v) < b_H(v)$ in a neighborhood of zero - excluding zero - if $g'(0) < 0$. Now also note that when $g'(0) = 0$, Landsberger et al. (2002) prove that $b_S(v) < b_H(v)$ in a neighborhood of zero by excluding zero. Conversely, suppose that $b_S(v) < b_H(v)$ in a neighborhood of zero - excluding zero. Suppose that $g'(0) > 0$. Then the above inequality is reversed, a contradiction. ∎

Lemma 2 Proof. Suppose that $G_\sigma(b^*) = G_l(b^*)$ for some $b^* > 0$. Now recall that $\delta^* = \frac{G_\sigma}{G_l}$ and $\delta^{*\prime}(b^*)$ is positive: $\delta^{*\prime}(b^*) = \delta^*(b^*)\{\frac{G\sigma'}{G\sigma} - \frac{G_l'}{G_l}\} = \delta^*(b^*)\{\frac{1}{\sigma(b^*)-b^*} - \frac{1}{h(b^*)-b^*}\} > 0$ since $G_h(b^*) > G_l(b^*) = G_\sigma(b^*)$. Hence, there exists $\varepsilon > 0$ such that $l(b) > \sigma(b)\ \forall\ b \in (b^* - \varepsilon, b^*)$. Because of the previous lemma, suppose that $\delta^*(b^* - \varepsilon) = 1$. But with a similar argument, we approach arbitrarily close to 0 (i.e., $\delta^*(0^+) = 1$). This is a contradiction to Lemma 1. ∎

Lemma 3 Proof. Now the maximum bid when $r = 0$ with symmetric bidders is lower than the maximum bid within the asymmetric model. To see this, suppose that $t_0 \leq t_0^S$. Then $G_l(t_0) = 1 \geq G_\sigma(t_0)$. This is a contradiction to Lemma 2. So $t_0 > t_0^S$. Now $G_\sigma(t_0^S) = 1 > G_h(t_0^S)$. So in a right neighborhood of t_0^S $G_\sigma(b) > G_h(b)$ by continuity. ∎

Lemma 4 Proof. Rewrite high-value bidder's maximization problem when the opponent plays an equilibrium strategy and assuming that $b_H(s) = b$:

$$\max_{s} G(l(b_H(s)))(v - b_H(s)) \tag{21}$$

The first-order necessary conditions imply that, when $s = v$ at the equilibrium,

$$\frac{d}{dv}[G(l(b_H(v)))b_H(v)] = v\frac{dG(l(b_H(v)))}{dv} \tag{22}$$

Integrating both sides we obtain:

$$b_H(v) = \frac{\int_0^v sdG(l(b_H(s)))}{G(l(b_H(v)))} \tag{23}$$

Similarly, rewriting the symmetric model bidder maximization problem when the opponent plays an equilibrium strategy and assuming that $b_S(s) = b$:

$$b_S(v) = \frac{\int_0^v sdG(s)}{G(v)} \tag{24}$$

Now, $b_H(v) > b_S(v) \ \forall \ v \in (0, 1]$ if the distribution $\frac{G(l(b_H(s)))}{G(l(b_H(v)))}$ first-order stochastically dominates $\frac{G(s)}{G(v)}$: that is, if $\frac{G(s)}{G(v)} > \frac{G(l(b_H(s)))}{G(l(b_H(v)))} \ \ \forall \ s < v \Longleftrightarrow b_H(v) > b_S(v) \ \ \forall v \in (0, 1]$ if $\frac{G(l(b_H(v)))}{G(v)}$ is increasing at each $v \in (0, 1] \Longleftrightarrow$ by $h(b) = v$, $b_H(v) > b_S(v) \ \ \forall \ v \in (0, 1]$ if $\delta^{**}(b) = \frac{G(l(b))}{G(h(b))}$ is increasing at each $b \in (0, t_0]$. ■

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