Design of Kidney Exchange Mechanisms

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Follows mostly

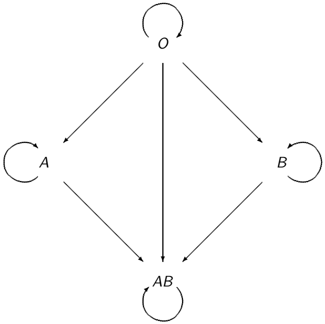
- Alvin E. Roth, Tayfun Sönmez and M. Utku Ünver, "Kidney Exchange," *Quarterly Journal of Economics* (2004)
- Alvin E. Roth, Tayfun Sönmez and M. Utku Ünver, "Pairwise Kidney Exchange," *Journal of Economic Theory* (2005)
- Alvin E. Roth, Tayfun Sönmez and M. Utku Ünver, "Efficient Kidney Exchange: Coincidence of Wants in Markets with Compatibility-Based Preferences," *American Economic Review* (2007)
- Alvin E. Roth, Tayfun Sönmez, and M. Utku Ünver, "Transplant Center Incentives in Kidney Exchange,"working paper. (2005)
- M. Utku Ünver, "Dynamic Kidney Exchange,"*Review of Economic Studies* (2010)

Introduction

- Transplantation is the preferred treatment for the most serious forms of kidney disease.
- 96,614 patients were waiting for a deceased donor kidney transplant in the United States as of 06/16/2013.
  in 2012:
  - 34,817 new patients joint the queue
  - 10,851 received deceased donor transplants
  - 5,088 received living donor transplants
  - 4,434 died while waiting
  - 8,306 were removed from the queue for other reasons
- Buying and selling a body part is illegal in many countries in the world including the U.S. (National Organ Transplant Act 1984/2007)
  Donation is the only source of kidneys in many countries.

Medicine of Donation: Blood–type Compatibility

A donor needs to be pass two **compatibility** tests before transplantation. There are four blood types O, A, B, AB

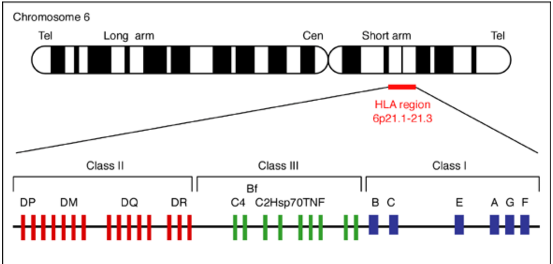


Blood–type donation compatibility is partial order $\triangleright$: $O \triangleright A, B \triangleright AB$.

Medicine of Donation: Tissue–type Compatibility

Tissue type or Human Leukocyte Antigen (HLA) type: Combination of several pairs of antigens on Chromosome 6.

HLA proteins A, B, and DR are especially important.



Gene map of the human leukocyte antigen (HLA) region

Expert Reviews in Molecular Medicine©2003 Cambridge University Press

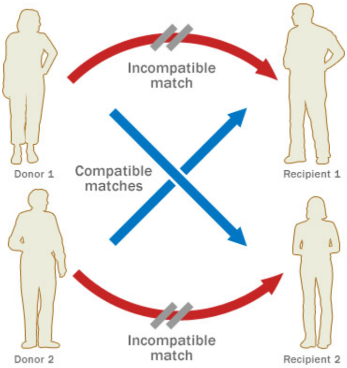
Prior to transplantation, the potential recipient is tested for the presence of preformed antibodies against donor HLA.

If the level of antibodies is above a **threshold** then the transplant cannot be carried out.

Donation Technologies

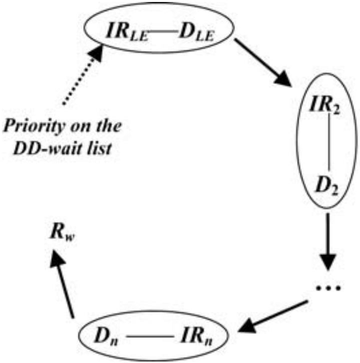
- **Deceased Donation:** Centralized priority allocation. Waiting time is always prioritized, however to different degrees for different organs. For kidneys for example like a **first-in–first-out** (FIFO) queue based on geography.
- **Live Donation:** Mostly loved ones of the patient come forward and if one is compatible with the patient, donation is conducted.
- **Live-Donor Organ Exchange:** If the live donor who came forward for his patient is not compatible, his organ is swapped with the organ from similar patient-donor pairs to find a compatible match for his patient.

Donation Technologies: Live-Donor Organ Exchange



Institutional Constraint: All transplants in one closed exchange has to be done simultaneously to prevent reneging of a donor whose patient already received a transplant.

List Exchange



<u>**An LE-chain exchange with n pairs**</u>

R_w: Recipient on the wait-list

IR - D: Pair including an intended recipient and incompatible donor.

$IR_1 - D_1$: Pair willing to participate in list exchange

... Indicates that additional pairs can participate in the chain exchange

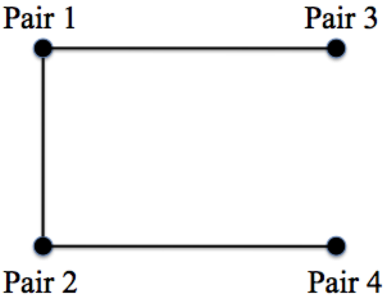
Kidney Exchange as a Market Design Problem

- The emerging field of Market Design applies insights and tools from economic theory to solve real-life resource allocation problems.
- In early 2000s, market designers observed that the two main types of kidney exchanges conducted in the U.S. correspond to the most basic forms of exchanges in house allocation models in matching literature.
- Building on the existing practices in kidney transplantation, Roth, Sönmez, & Ünver (2004, 2005, 2007) analyzed how an efficient and incentive-compatible system of exchanges might be organized, and what its welfare implications might be.
- The methodology and techniques advocated in this research program provided the backbone of several kidney exchange programs in the U.S. and the rest of the world. [NEPKE, APD, National Program, etc]

Progress of Live-Donor Exchange within the Last Decade

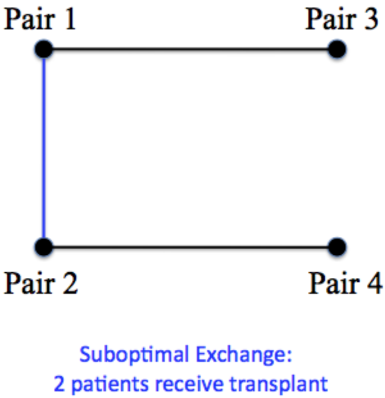
1. Organization and Optimization of Kidney Exchange
2. Utilizing Gains from Larger Exchanges
3. Integration of Altruistic Donors via Kidney Chains
4. Inclusion of Compatible Pairs for Increased Efficiency
5. Higher Efficiency via Larger Kidney Exchange Programs
6. Dynamic Optimization in Kidney Exchange

1. Organized Exchange & Optimization is Important



- Even in the absence of more elaborate exchanges, merely organizing the paired-exchanges may result in increased efficiency.

1. Optimization is Important



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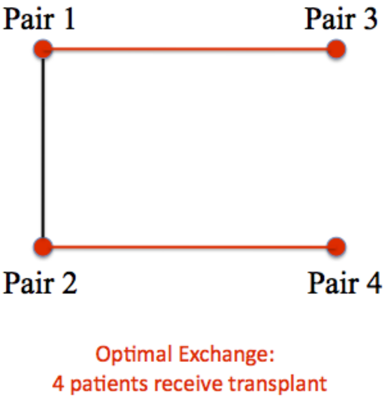
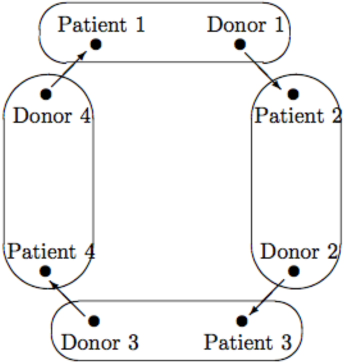


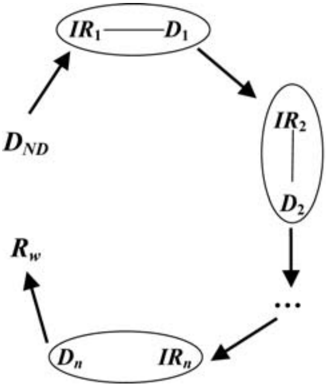
- Even in the absence of more elaborate exchanges, merely organizing the paired-exchanges may result in increased efficiency.

2. Gains from Larger Exchanges are Considerable



- Additional live-donor transplants may be possible through three-way, four-way, . . . , exchanges.
- Three-way exchange is especially important!

3. Gifts of Altruistic Donors Can Be Multiplied via Chains



An ND-chain exchange with *n* pairs

D_{ND}: Non-directed donor

R_w: Recipient on the wait-list

IR-D: Pair including an intended recipient and incompatible donor.

... Indicates that additional pairs can participate in the chain exchange

3. Gifts of Altruistic Donors Can Be Multiplied via Chains

- Simultaneity is not critical when a kidney-chain starts with a donation from an altruistic donor. Hence large kidney-chains can be utilized!

4. Inclusion of Compatible Pairs in Exchange is Important

- Typically a blood-type compatible pair participates in kidney exchange only when the donor is tissue-type incompatible with the intended recipient.
- In contrast, a blood-type incompatible pair is automatically referred to a kidney exchange program.
- Hence there are many more blood-type incompatible pairs in kidney exchange programs than blood-type compatible pairs.

$$\# \ O \ \textbf{Patients} >> \# \ O \ \textbf{Donors}$$

- This disparity can be minimized if compatible pairs can also be included in kidney exchange.

5. Higher Efficiency via Larger Kidney Exchange Programs

- Larger kidney exchange programs (such as national programs) provide a more efficient system than several smaller regional programs.
- Large national programs are especially beneficial for difficult-to-match patients such as those who are tissue-type incompatible with a large fraction of donor population (aka highly sensitized patients).

5. Higher Efficiency via Larger Kidney Exchange Programs

- In the US, due to vagueness of original NOTA regarding legality of exchanges NOTA had to be amended and the national kidney exchange program started late.
- Currently in the US, most activity is organized locally in regional programs, only leftover difficult-to-match pairs participate in the national program.
- Problem is even more severe than just a first-mover advantage: RSÜ '05c showed that there is **no incentive compatible** exchange system that would make all regional programs reveal their all pairs to the centralized national program (also see Ashlagi and Roth, 2014).

Progression of Kidney Exchanges

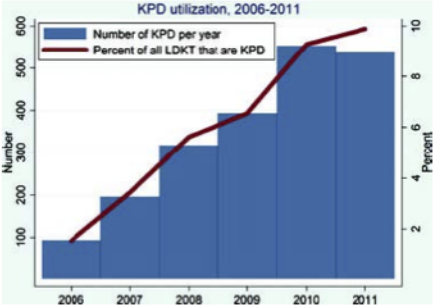


Figure from Massie et al AJT 2013

- A handful of kidney exchanges in the U.S. prior to 2004, increased to 93 in 2006 and to 553 in 2010.
- Currently kidney exchanges in the U.S. account for about 10% of all live donor kidney transplants.

Outline

In this presentation:

1. **Organization and Optimization of Kidney Exchange**
2. **Utilizing Gains from Larger Exchanges**

1. Organization and Optimization of Kidney Exchange

The Model

- A kidney exchange problem consists of
  1. a set of donor-recipient couples $\{(k_1, t_1), \ldots (k_n, t_n)\}$,
  2. a set of compatible kidneys
     $K_i \subset K = \{k_1, \ldots, k_n\}$ for each patient t_i, and
  3. a preference relation $\mathbf{R}_i$ over $K_i \cup \{k_i, w\}$
     w: priority in the waitlist in exchange for kidney k_i
- An outcome: **Matching** of kidneys/waitlist option to patients such that multiple patients can be matched with the w option (lotteries over matchings are possible).
- A kidney exchange **mechanism** is a systematic procedure to select a matching for each kidney exchange problem (lottery mechanisms are possible).

Roth, Sönmez and Ünver (Quart J Econ, 2004)

Assumptions:

- **Any number of** patient-donor pairs can participate in one exchange (medical constraint)
- **Heterogeneous** preferences over compatible kidneys: Opelz (1997) shows in his data set that among compatible donors, the increase in the number of HLA protein mismatches decreases the likelihood of kidney survival. Body size, age of donor etc. also affect kidney survival (medical constraint)
- List exchanges **are** allowed (institutional constraint)

Related Literature

Housing Markets, House Allocation with Existing Tenants: Shapley and Scarf (1974), Roth and Postlewaite (1977), Roth (1982), Abdulkadiroğlu and Sönmez (1999)

TTCC Mechanism

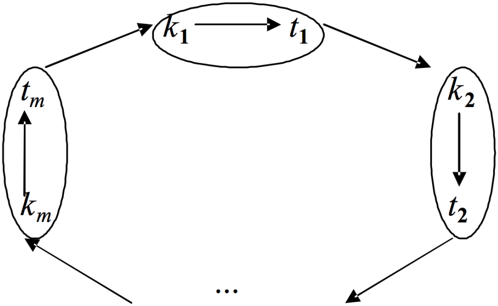
Cycles and w-Chains:

The mechanism relies on an algorithm consisting of several rounds. In each round

- each patient t_i points either towards a kidney in $K_i \cup \{k_i\}$ or towards w, and
- each kidney k_i points to its paired recipient t_i.

TTCC Mechanism

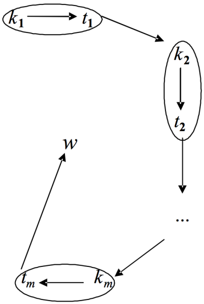
- **Cycle:** an ordered list of kidneys and patients $(k_1, t_1, k_2, t_2, ..., k_m, t_m)$ such that



Each cycle is of even size and no two cycles can intersect.

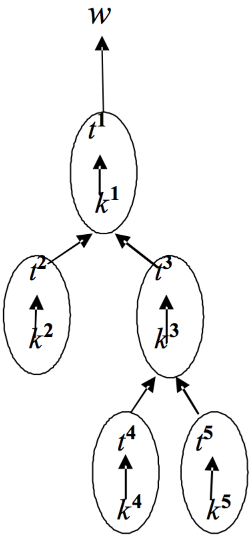
TTCC Mechanism

- **W-chain:** an ordered list of kidneys and patients $(k_1, t_1, k_2, t_2, \ldots, k_m, t_m)$ such that



Last pair (k_m, t_m): **Head**
First pair (k_1, t_1): **Tail**

TTCC Mechanism



TTCC Mechanism

- A **w-chain** is also of even size but, unlike in a cycle, a kidney or a patient can be part of several w-chains. One practical possibility is choosing among w-chains with a well-defined chain selection rule.
- Choice of chain selection rule has efficiency and incentive-compatibility implications.

TTCC Mechanism

[1.]Consider a graph in which both the patient and the kidney of each pair are distinct nodes as is the waitlist option w. Suppose each patient points to his first available choice, either towards a kidney or w, and each kidney points to its paired recipient. Then either there exists a cycle or each pair initiates a w-chain.

[2.a] If there is a cycle, clear all cycles and go to step 1, for the remaining patients and kidneys.
[2.b] if there is no cycle, choose and fix a w-chain based on a chain selection rule and go to step 1, for the remaining patients and kidneys.

Efficiency and Incentives

Theorem

Consider the following chain selection rules:
1. *Choose minimal w-chains and remove them.*
2. *Prioritize patient-donor pairs in a single list. Choose the w-chain starting with the highest priority pair and remove it.*
3. *Prioritize patient-donor pairs in a single list. Choose the w-chain starting with the highest priority pair and keep it.*

The TTCC mechanism, implemented with any of these chain selection rules makes it a ***dominant strategy*** *for patients:*

1. *reveal her preferences over other available kidneys truthfully, and*
2. *declare the whole set of her donors.*

Efficiency and Incentives

Theorem

Any w-chain selection rule that leaves the tail kidney in the problem is Pareto-efficient.

Logistical and Ethical Constraints

Following the first paper our discussions with the medical community disclosed that

- Larger than 2-way exchanges may not be possible to conduct, since all transplants in an exchange should be conducted simultaneously. Different designs may be needed to accommodate restricted exchanges.
- List exchanges may be ruled out due to ethical concerns about the harm on O blood-type patients.
- Compatible patient-donor couples may not participate in an exchange.

Design 2

Roth, Sönmez and Ünver (JET, 2005)

Assumptions:

- Only **two** patient-donor pairs can participate in one exchange. (medical constraint)
- **Indifference** over compatible kidneys: Delmonico (2004) and Gjertson and Cecka (2000) show in their data sets that among compatible donors, all live donor kidneys have the same likelihood of survival (medical constraint)
- List exchanges **are not** allowed (institutional constraint)

Related Literature

Cardinality Matching Problem, Matroids, Egalitarian Mechanism: *Operations Research:* Gallai (1963,1964), Edmonds (1965), *Economics:* Bogomolnaia and Moulin (2004)

The Pairwise Exchange Model

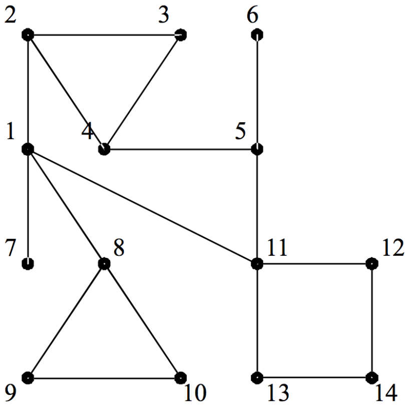
- N : set of patients each of whom has only incompatible donors.
- R : Patients $i, j \in N$ are **mutually compatible** if j has a compatible donor for i and i has a compatible donor for j. **Mutual compatibility matrix** $R = [r_{i,j}]_{i \in N, j \in N}$ defined by

$$r_{i,j} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are mutually compatible} \\ 0 & \text{otherwise} \end{cases}$$

Example: The Pairwise Exchange Model

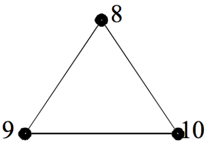
- Graph representation of a kidney exchange problem: Set of Vertices: N. Set of Edges: R

Example: $N = \{1, 2, ..., 14\}$, Problem (N, R) is given as



Example: The Pairwise Exchange Model

Subproblem (I, R_I) with $I = \{8, 9, 10, 11, 12, 13, 14\}$

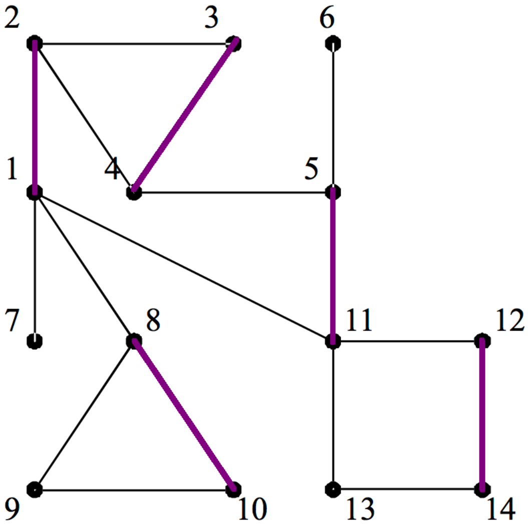


Odd Component $\{8, 9, 10\}$, Even Component $\{11, 12, 13, 14\}$

The Pairwise Exchange Model

- Deterministic outcome is a **matching** $\mu : N \longrightarrow N$ such that
  1. $\mu(i) = j \Leftrightarrow \mu(j) = i$ (only **pairwise exchanges** are possible)
  2. for $i \neq j$, $\mu(i) = j \Rightarrow r_{i,j} = 1$ (only **mutually compatible exchanges** are possible)
- Stochastic outcome is a **lottery** $\lambda = (\lambda_\mu)_{\mu \in \mathcal{M}}$ that is a probability distribution on all matchings.
- **Utility** of a patient i under a lottery λ is the probability the patient getting a transplant and is denoted by $u_i(\lambda)$.
- **Utility profile** of lottery λ is $u(\lambda) = (u_i(\lambda))_{i \in N}$.

Example: A matching



Efficient Pairwise Exchange

- A matching is **Pareto efficient** if there is no other matching that makes every patient weakly better off and some patient strictly better off.
- A lottery is **ex-post efficient** if it gives positive weight to only Pareto efficient matchings.
- A lottery is **ex-ante efficient** if there is no other lottery that makes every patient weakly better off and some patient strictly better off.

Efficient Pairwise Exchange

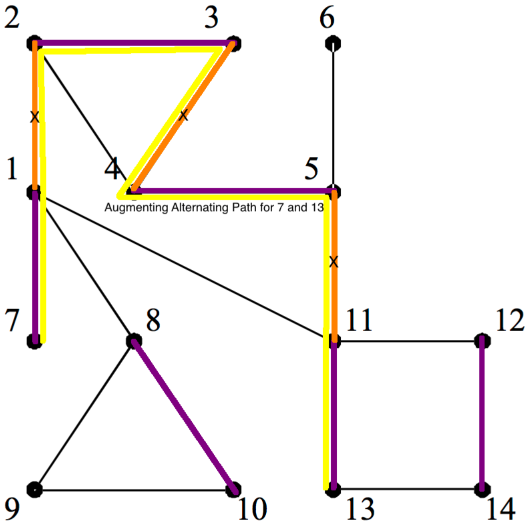
Lemma

The same number of patients are matched at each Pareto efficient matching.

- Lemma 4 would not hold if exchange was possible among three or more patients.
- *In this problem*, for any lottery,

$$\text{Ex-ante Efficiency} \iff \text{Ex-post Efficiency}$$

Example: An efficient matching



Efficient Matchings

Theorem

A matching is efficient if and only if there is no augmenting alternating path.

Priority Mechanism

Given a priority ordering of patients, a **priority mechanism** matches **Priority 1** patient if she is mutually compatible with a patient; skips her otherwise.

⋮

matches **Priority k** patient in addition to all the previously matched patients if possible, skips her otherwise.

Theorem

The ***priority mechanism*** *is not only Pareto efficient but also it makes it a dominant strategy for a patient to reveal both*

a. *her full set of compatible kidneys, and*

b. *her full set of available donors.*

Structure of a Problem

Underdemanded Patient: A patient for whom there exists a Pareto efficient matching that leaves her unmatched. (Set N^U).

Overdemanded Patient: A patient who is not underdemanded and is a neighbor of an underdemanded patient (Set N^O).

Perfectly-matched Patient: A patient who is not underdemanded and is not a neighbor of an underdemanded patient (Set N^P).

Structure of a Problem

Lemma (Gallai-Edmonds Decomposition (GED))

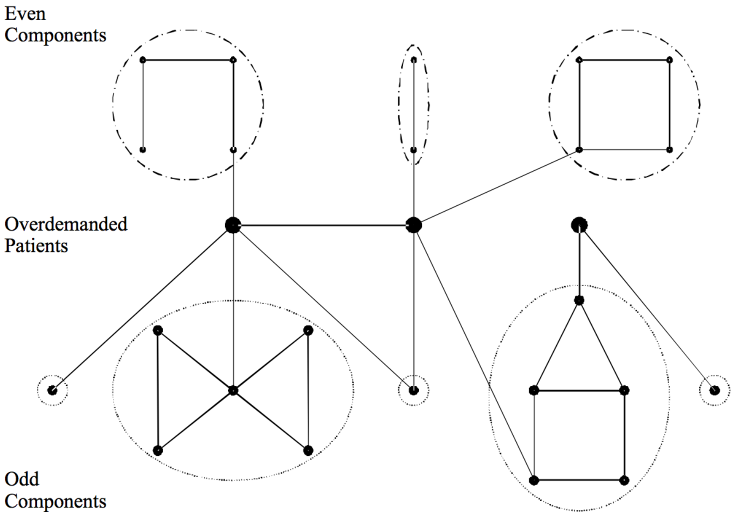
Let μ be any Pareto efficient matching for the original problem (N, R) and (I, R_I) be the ***subproblem*** *for $I = N \setminus N^O$:*

1. *Any overdemanded patient is matched with an underdemanded patient under μ.*
2. *$J \subseteq N^P$ for any* ***even component*** *J of the subproblem (I, R_I) and all patients in J are matched with each other under μ.*
3. *$J \subseteq N^U$ for any* ***odd component*** *J of the subproblem (I, R_I) and for any patient $i \in J$, it is possible to match all remaining patients with each other under μ. Moreover under μ*
   a. *either one patient in J is matched with an overdemanded patient and all others are matched with each other,*
   b. *or one patient in J remains unmatched while the others are matched with each other.*

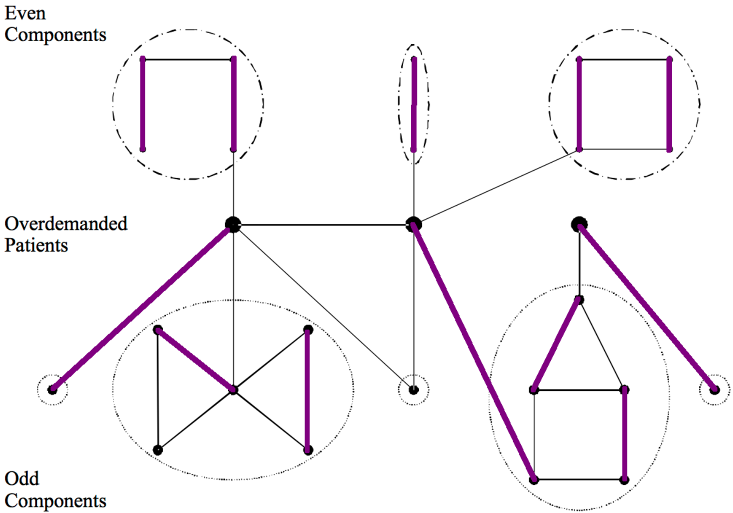
Competition Among Odd Components

- $\mathcal{D} = \{D_1, \ldots, D_p\}$: Set of odd components remaining in the problem when overdemanded patients are removed from the problem.
- By the GED Lemma, each Pareto efficient matching leaves unmatched $|\mathcal{D}| - |N^O|$ patients each of whom is in a distinct odd component D.

Example:



Example of a **Pareto-efficient** matching:



Skip Equity

Equity and The Egalitarian Mechanism

- Utility is the probability of receiving a transplant.
- Equalizing utilities as much as possible may be considered very plausible from an equity perspective.
- A feasible utility profile is **Lorenz-dominant** if
  - the least fortunate patient receives the highest utility among all feasible utility profiles, and

    ⋮
  - the sum of utilities of the **k** least fortunate patients is the highest among all feasible utility profiles.
- **Question:** Is there a feasible Lorenz-dominant utility profile? The answer is YES. We call it u^E.

Equity and The Egalitarian Mechanism

Let

- $\mathcal{J} \subseteq \mathcal{D}$ be an arbitrary set of odd components,
- $I \subseteq N^O$ be an arbitrary set of overdemanded patients, and
- $C(\mathcal{J}, I)$ denote the **neighbors** of $\mathcal{J}$ among I.

Equity and The Egalitarian Mechanism

Suppose only overdemanded patients in I are available to be matched with underdemanded patients in $\bigcup_{J \in \mathcal{J}} J$.

Question: What is the upper-bound of the utility that can be received by the *least fortunate* patient in $\bigcup_{J \in \mathcal{J}} J$?

Answer:

$$f\left(\mathcal{J}, I\right) = \frac{\left|\bigcup_{J \in \mathcal{J}} J\right| - \left(\left|\mathcal{J}\right| - \left|C\left(\mathcal{J}, I\right)\right|\right)}{\left|\bigcup_{J \in \mathcal{J}} J\right|}$$

Equity and The Egalitarian Mechanism

The above upper-bound can be received only if:

1. all underdemanded patients in $\bigcup_{J \in \mathcal{J}} J$ receive the same utility, and
2. all overdemanded patients in $C(\mathcal{J}, I)$ are committed for patients in $\bigcup_{J \in \mathcal{J}} J$.

Equity and The Egalitarian Mechanism

So partition $\mathcal{D}$ as $\mathcal{D}_1, \mathcal{D}_2, \ldots$ and N^O as $N^O_1, N^O_2, \ldots$ as follows:

- *Step 1.*

$$\begin{aligned} \mathcal{D}_1 &= \arg\min_{\mathcal{J} \subseteq \mathcal{D}} f\left(\mathcal{J}, N^O\right) \text{ and} \\ N^O_1 &= C\left(\mathcal{D}_1, N^O\right) \end{aligned}$$

- *Step k.*

$$\begin{aligned} \mathcal{D}_k &= \arg\min_{\mathcal{J} \subseteq \mathcal{D} \setminus \bigcup_{\ell=1}^{k-1} \mathcal{D}_\ell} f\left(\mathcal{J}, N^O \Big\backslash \bigcup_{\ell=1}^{k-1} N^O_\ell\right) \text{ and} \\ N^O_k &= C\left(\mathcal{D}_k, N^O \Big\backslash \bigcup_{\ell=1}^{k-1} N^O_\ell\right) \end{aligned}$$

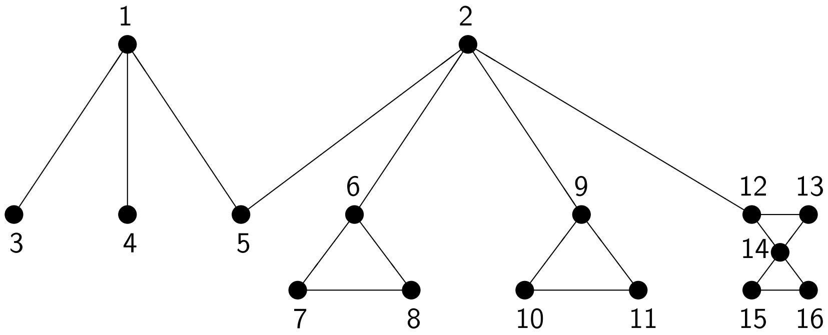
The Egalitarian Utility Profile

Construct the vector $u^E = (u_i^E)_{i \in N}$ as follows:

1. For any overdemanded patient and perfectly-matched patient $i \in N \setminus N^U$,
$$u_i^E = 1.$$
2. For any underdemanded patient i whose odd component left the above procedure at Step $k(i)$,
$$u_i^E = f(\mathcal{D}_{k(i)}, N^O_{k(i)}).$$

Example

Let $N = \{1, \ldots, 16\}$ be the set of patients and consider the reduced problem given by the graph in Figure 1.



Graphical Representation for Example 2.

Example

Each patient except 1 and 2 can be left unmatched at some Pareto efficient matching and hence $N^U = \{3, \ldots, 16\}$ is the set of underdemanded patients.

Since both patients 1 and 2 have links with patients in N^U, $N^O = \{1, 2\}$ is the set of overdemanded patients.

$$\mathcal{D} = \{D_1, \ldots, D_6\}$$

where

$$\begin{aligned} D_1 &= \{3\}, D_2 = \{4\}, D_3 = \{5\}, D_4 = \{6, 7, 8\} \\ D_5 &= \{9, 10, 11\}, D_6 = \{12, 13, 14, 15, 16\} \end{aligned}$$

Example

Consider $\mathcal{J}_1 = \{D_1, D_2\} = \{\{3\}, \{4\}\}$. Note that . Since $f(\mathcal{J}_1, N^O) = \frac{1}{2} < \frac{2}{3} < \frac{4}{5}$, none of the multi-patient odd components is an element of $\mathcal{D}_1$. Moreover patient 5 has two overdemanded neighbors and $f(\mathcal{J}, N^O) > f(\mathcal{J}_1, N^O)$ for any $\mathcal{J} \subseteq \{\{3\}, \{4\}, \{5\}\}$ with $\{5\} \in \mathcal{J}$. Therefore

$$\begin{aligned} \mathcal{D}_1 &= \mathcal{J}_1 = \{\{3\}, \{4\}\}, \quad N_1^O = \{1\}, \\ u_3^E &= u_4^E = \frac{1}{2}. \end{aligned}$$

Example

Next consider $\mathcal{J}_2 = \{D_3, D_4, D_5\} = \{\{5\}, \{6, 7, 8\}, \{9, 10, 11\}\}$. Note that $f(\mathcal{J}_2, N^O \setminus N_1^O) = \frac{7-(3-1)}{7} = \frac{5}{7}$. Since $f(\mathcal{J}_2, N^O \setminus N_1^O) = \frac{5}{7} < \frac{4}{5}$, the 5-patient odd component D_6 is not an element of $\mathcal{D}_2$. Moreover

$$\begin{aligned}
f(\{D_3\}, N^O \setminus N_1^O) &= f(\{D_4\}, N^O \setminus N_1^O) \\
&= f(\{D_5\}, N^O \setminus N_1^O) = 1, \\
f(\{D_3, D_4\}, N^O \setminus N_1^O) &= f(\{D_3, D_5\}, N^O \setminus N_1^O) = \frac{3}{4}, \\
f(\{D_4, D_5\}, N^O \setminus N_1^O) &= \frac{5}{6}.
\end{aligned}$$

Example

Therefore

$$\begin{aligned} \mathcal{D}_2 &= \mathcal{J}_2 = \{\{5\},\{6,7,8\},\{9,10,11\}\}, \\ N_2^O &= \{2\}, \\ \text{and} \quad u_5^E &= \cdots = u_{11}^E = \frac{5}{7}. \end{aligned}$$

Finally since $N^O \setminus (N_1^O \cup N_2^O) = \emptyset$,

$$\begin{aligned} \mathcal{D}_3 &= \{\{12,13,14,15,16\}\}, \\ N_3^O &= \emptyset, \\ \text{and} \quad u_{12}^E &= \cdots = u_{16}^E = \frac{4}{5}. \end{aligned}$$

Hence the egalitarian utility profile is

$$u^E = (1, 1, \frac{1}{2}, \frac{1}{2}, \frac{5}{7}, \frac{5}{7}, \frac{5}{7}, \frac{5}{7}, \frac{5}{7}, \frac{5}{7}, \frac{5}{7}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5})$$

Results

Theorem

The vector u^E is feasible.

Theorem

The utility profile u^E ***Lorenz dominates*** *any other feasible utility profile (efficient or not).*

Theorem

The egalitarian mechanism is not only Pareto efficient but also it makes it a dominant strategy for a patient to reveal both

a. *her full set of compatible kidneys, and*

b. *her full set of available donors.*

2. Utilizing Gains from Larger Exchanges
Roth, Sonmez, Unver, AER, 2007

What is an Upper-bound to the Number of Patients matched under Two-way Exchanges?

Assumption 1 (Upper bound assumption). No patient is tissue-type incompatible with another patient's donor

Assumption 2. (Large Population of Incompatible Patient-Donor Pairs) Regardless of the maximum number of pairs allowed in each exchange, pairs of types O-A, O-B, O-AB, A-AB, and B-AB are on the "long side" of the exchange in the sense that at least one pair of each type remains unmatched in each feasible set of exchanges

What is an Upper-bound to the Number of Patients matched under Two-way Exchanges?

Proposition

For any patient population obeying Assumptions 1 and 2, the maximum number of patients who can be matched with only two-way exchanges is:

$$
\begin{aligned}
&2\left(\#(A\text{-}O)+\#(B\text{-}O)+\#(AB\text{-}O)+\#(AB\text{-}A)+\#(AB\text{-}B)\right)\\
&+\left(\#(A\text{-}B)+\#(B\text{-}A)-|\#(A\text{-}B)-\#(B\text{-}A)|\right)\\
&+2\left(\left[\frac{\#(A\text{-}A)}{2}\right]+\left[\frac{\#(B\text{-}B)}{2}\right]+\left[\frac{\#(O\text{-}O)}{2}\right]+\left[\frac{\#(AB\text{-}AB)}{2}\right]\right)
\end{aligned}
$$

Gains from Three-Way Exchanges

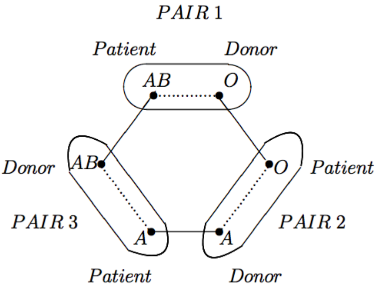


Figure 1: When three-way exchanges are feasible, each type AB-O pair can form a three-way exchange with two pairs on the long side.

Gains from Three-Way Exchanges

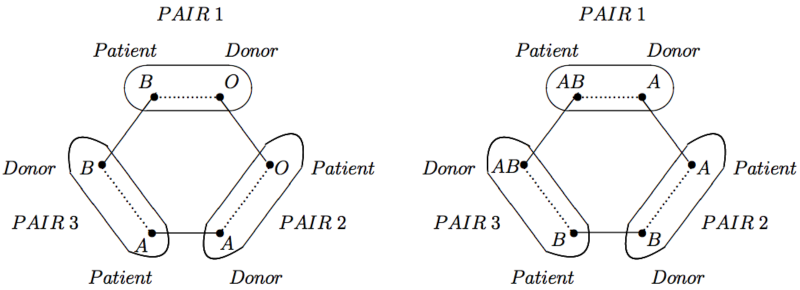


Figure 2: When three-way exchanges are feasible and #(A-B) > #(B-A), each type B-O pair can form a three-way exchange with two pairs on the long side. The same is also true for each type AB-A pair.

Gains from Three-way Exchanges

- An example helps illustrate why 3-way exchange is important. Consider a population of 9 incompatible patient-donor pairs. (A pair is denoted as type x-y if the patient and donor are ABO blood-types x and y respectively.)
- Example: There are five pairs of patients who are blood-type incompatible with their donors, of types O-A, O-B, A-B, A-B, and B-A; and four pairs who are incompatible because of positive crossmatch, of types A-A, A-A, A-A and B-O. For simplicity in this example there are no positive crossmatches between patients and other patients' donors.
- If 2-way exchanges are possible:
  (A-B,B-A); (A-A,A-A); (B-O,O-B)
- If 3-way exchanges are also feasible
  (A-B,B-A); (A-A,A-A,A-A); (B-O,O-A,A-B)

Gains from Three-way Exchanges

The 3-way exchanges allow

1. an odd number of A-A pairs to be transplanted (instead of only an even number with 2-way exchanges), and
2. an O donor to facilitate three transplants rather than only two.

Gains from Three-way Exchanges

Assumption 3. #(A-B) > #(B-A).

Assumption 4. There is either no type A-A pair or there are at least two of them. The same is also true for each of the types B-B, AB-AB, and O-O.

Proposition

For any patient population for which Assumptions 1-4 hold, the maximum number of patients who can be matched with two-way and three-way exchanges is:

$$
\begin{aligned}
&2\,(\#(A\text{-}O) + \#(B\text{-}O) + \#(AB\text{-}O) + \#(AB\text{-}A) + \#(AB\text{-}B)) \\
&+ (\#(A\text{-}B) + \#(B\text{-}A) - |\#(A\text{-}B) - \#(B\text{-}A)|) \\
&+ (\#(A\text{-}A) + \#(B\text{-}B) + \#(O\text{-}O) + \#(AB\text{-}AB)) \\
&+ \#(AB\text{-}O) \\
&+ \min\{(\#(A\text{-}B) - \#(B\text{-}A)), (\#(B\text{-}O) + \#(AB\text{-}A))\}
\end{aligned}
$$

Gains from Three-way Exchanges

To summarize, the marginal effect of three-way kidney exchanges is:

$$
\begin{aligned}
&\#(\text{A-A}) + \#(\text{B-B}) + \#(\text{O-O}) + \#(\text{AB-AB}) \\
&-2\left(\left[\frac{\#(\text{A-A})}{2}\right] + \left[\frac{\#(\text{B-B})}{2}\right] + \left[\frac{\#(\text{O-O})}{2}\right] + \left[\frac{\#(\text{AB-AB})}{2}\right]\right) \\
&+\#(\text{AB-O}) \\
&+\min\{(\#(\text{A-B}) - \#(\text{B-A})), (\#(\text{B-O}) + \#(\text{AB-A}))\}
\end{aligned}
$$

Gains from Four-Way Exchanges

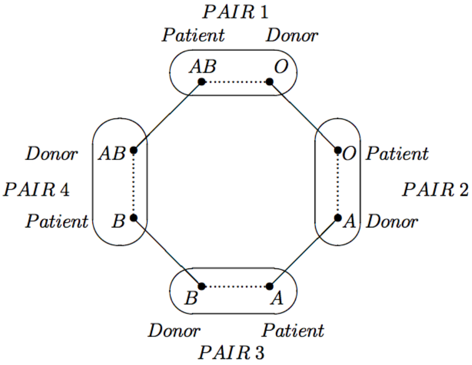


Figure 3: When four-way exchanges are feasible, each type AB-O pair can form a four-way exchange with three pairs on the long side.

Gains from Four-way Exchanges

Proposition

For any patient population in which Assumptions 1-4 hold, the maximum number of patients who can be matched with two-way, three-way and four-way exchanges is:

$$
\begin{aligned}
& 2\left(\#(A\text{-}O)+\#(B\text{-}O)+\#(AB\text{-}O)+\#(AB\text{-}A)+\#(AB\text{-}B)\right) \\
& +\left(\#(A\text{-}B)+\#(B\text{-}A)-|\#(A\text{-}B)-\#(B\text{-}A)|\right) \\
& +\left(\#(A\text{-}A)+\#(B\text{-}B)+\#(O\text{-}O)+\#(AB\text{-}AB)\right) \\
& +\#(AB\text{-}O) \\
& +\min\{(\#(A\text{-}B)-\#(B\text{-}A)), \\
& \qquad\qquad (\#(B\text{-}O)+\#(AB\text{-}A)+\#(AB\text{-}O))\}
\end{aligned}
$$

Therefore in the absence of tissue-type incompatibilities between patients and other patients' donors, the marginal effect of four-way kidney exchanges is bounded above by the rate of the very rare AB-O type.

Sufficiency of Four-way Exchanges

Theorem (4-way exchange suffices)

Consider a patient population for which Assumptions 1, 2, 4 hold and let μ be any maximal matching (when there is no restriction on the size of the exchanges that can be included in a matching). Then there exists a maximal matching ν which consists only of two-way, three-way, and four-way exchanges, under which the same set of patients benefit from exchange as in matching μ.

Simulations

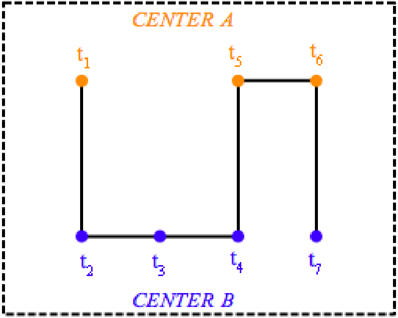
Pop. Size	Method	Type of Exchange: Two-way	Two-way, Three-way	Two-way, Three-way, Four-way	No Constraint
n=25	**Simulation**	8.86 (3.4866)	11.272 (4.0003)	11.824 (3.9886)	11.992 (3.9536)
	Upper bound 1	12.5 (3.6847)	14.634 (3.9552)	14.702 (3.9896)	
	Upper bound 2	9.812 (3.8599)	12.66 (4.3144)	12.892 (4.3417)	
n=50	**Simulation**	21.792 (5.0063)	27.266 (5.5133)	27.986 (5.4296)	28.09 (5.3658)
	Upper bound 1	27.1 (5.205)	30.47 (5.424)	30.574 (5.4073)	
	Upper bound 2	23.932 (5.5093)	29.136 (5.734)	29.458 (5.6724)	
n=100	**Simulation**	49.708 (7.3353)	59.714 (7.432)	60.354 (7.3078)	60.39 (7.29)
	Upper bound 1	56.816 (7.2972)	62.048 (7.3508)	62.194 (7.3127)	
	Upper bound 2	53.496 (7.6214)	61.418 (7.5523)	61.648 (7.4897)	

Table 2: Simulation results about average number of patients actually matched and predicted by the formulae to be matched. The standard errors of the population are reported in parantheses. The standard errors of the averages are obtained by dividing population standard errors by square root of the simulation number, 22.36.

Real Life

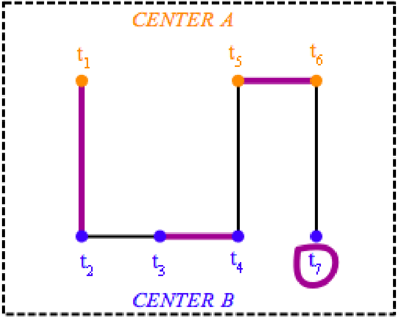
- NEPKE formed by RSU and doctors (2004) , Priority Mechanism with 3-way exchanges(integer programming techniques for 3-way and larger, maximize total exchanges)
- Alliance for Paired Donation formed by doctors and helped by RSU (2006)
- Chain exchanges are being conducted (RSU' 07; and Rees et al. '09)
- National Exchange Program in the US (2010)

Center Incentives



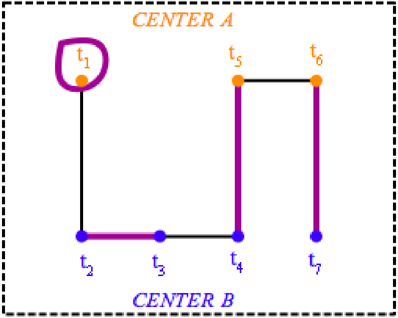
Two centers with 3 and 4 pairs each. Each center wants to maximize the number of its pairs that receive transplant. They can either internally match them or release to the National Program, which uses an efficient mechanism.

Center Incentives



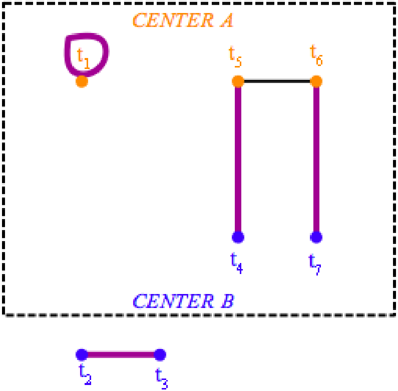
Efficient Matching 1

Center Incentives



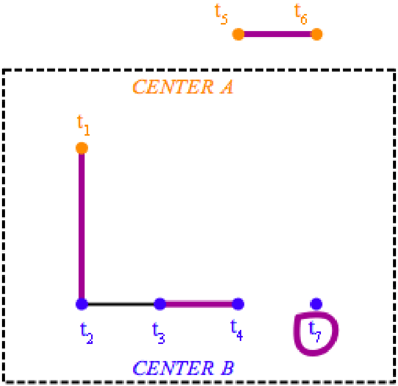
Efficient Matching 2

Center Incentives



If Efficient Matching 1 was chosen in the original problem, Center B withholds t_2 and t_3, matches them internally, and and helps its all pairs to be matched.

Center Incentives



If Efficient Matching 2 was chosen in the original problem, Center A withholds t_5 and t_6, matches them internally, and and helps its all pairs to be matched.

Summary: Progress of Live-Donor Exchange within the Last Decade:

1. Organization and Optimization of Kidney Exchange ✓
   (RSÜ '04, '05a, '07a and in practice)
2. Utilizing Gains from Larger Exchanges ✓
   (RSÜ '07, Saidman et al. '06 and in practice)
3. Integration of Altruistic Donors via Kidney Chains ✓
   (RSÜ et al. '07, and in practice)
4. Inclusion of Compatible Pairs for Increased Efficiency ✗
   (RSÜ '04, '05b, Nicolo Rodrigo-Alvarez '16, Sönmez and Ünver '14, Sönmez, Ünver, Yenmez '17)
5. Higher Efficiency via Larger Kidney Exchange Programs ✗
   (RSÜ '05c, Ashlagi and Roth, '14,)
6. Dynamic Optimization (Ünver '10) ✗

See the policy survey for better references