REVIEWS

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Summing it Up: From One Plus One to Modern Number Theory. By Avner Ash and Robert Gross. Princeton Univ. Press, Princeton, NJ, 2016, ISBN: 9780691170190, $16.77. `http://press.princeton.edu/titles/10692.html`.

Reviewed by **Dominic Lanphier**

Number theory is unusual among branches of mathematics in that many of its fundamental concepts and problems can be understood by a lay audience. For example: prime numbers, writing an integer as a sum of squares, partitions of an integer, Diophantine equations, and even numerous open problems. The proper tools for studying these topics are another story, however, as they are often esoteric and inaccessible. This is illustrated by David Hilbert's well-known explanation from 1920 for why he never attempted to solve the easy-to-state Fermat's last theorem: "Before beginning, I should put in three years of intensive study, and I haven't that much time to squander on a probable failure." [**3**, p. 238] The distance between understanding the statement of a number theory problem and understanding the tools necessary for studying it properly can be vast and intimidating. Nevertheless, if someone with only some undergraduate mathematics decided to put in the time for intensive study of an arithmetic problem, or at least for really understanding the nature of arithmetic, then what should they study? A very good start would be books by Avner Ash and Robert Gross. They have been working to introduce deep, elegant, and extremely important tools of number theory to as large an audience as possible.

Summing it Up is their third book to cover deep number-theoretic topics in an engaging and accessible way, and for an audience that may only have some exposure to undergraduate mathematics. In the first two books, *Fearless Symmetry* [**1**] and *Elliptic Tales* [**2**], they introduced reciprocity laws, elliptic functions and curves, groups, and representations. In both books, however, the authors included only brief glimpses of one of the most important subjects of number theory: modular forms.

Fearless Symmetry discussed Andrew Wiles' proof of Fermat's last theorem, making a mention of modular forms unavoidable. *Elliptic Tales* discussed both the Birch–Swinnerton-Dyer conjecture and L-functions, both of which are related to modular forms. In neither book were these objects precisely defined or studied in any depth, but their appearances were crucial, with hints that much more lay beneath the surface. In *Summing it Up*, Ash and Gross provide a leisurely introduction to the subject, with proper attention given to the prerequisites from group theory and complex analysis.

Modular forms have long played a role in understanding arithmetic objects such as L-functions, representation numbers, and elliptic curves. It is often the case that in the background of a theory is a modular form controlling key properties. Classical examples include the Riemann zeta function (see Riemann's memoir on the zeta function [**13**]) and the partition function (in particular, see Hardy and Ramanujan's study of the partition function [**7**], [**12**]). In both cases, a type of modular form controls much of the behavior of the object of study.

More precisely, setting

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$$

for z in the complex upper half-plane, Euler [**6**], [**12**] essentially showed that

$$\sum_{n=0}^{\infty} p(n) e^{2\pi i n z} = e^{\pi i z/12} \eta(z)^{-1},$$

where $p(n)$ is the number of partitions of n with $p(0) = 1$. The function $\eta(z)$ (called the Dedekind eta function) possesses an interesting property. Namely, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the 2×2 special linear group with integer entries $SL_2(\mathbb{Z})$,

$$\eta\left(\frac{az+b}{cz+d}\right) = \nu_\eta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) (cz+d)^{1/2} \eta(z),$$

where ν_η is a 24th root of unity.

As another example, Riemann [**13**] showed that for $s \in \mathbb{C}$,

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} \int_0^{\infty} \left(\theta\left(\frac{ix}{2}\right) - 1\right) x^{s/2-1}\, dx,$$

where $\Gamma(s)$ is the gamma function. This expresses the zeta function $\zeta(s)$ as an integral of

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z},$$

for z in the upper half-plane. The function $\theta(z)$ (this is a Jacobi theta function) possesses a property very similar to the eta function,

$$\theta\left(\frac{az+b}{cz+d}\right) = \varepsilon_d \left(\frac{c}{d}\right) (cz+d)^{1/2} \theta(z),$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in a subgroup of $SL_2(\mathbb{Z})$ of finite index (called $\Gamma_0(4)$) and where ε_d and $\left(\frac{c}{d}\right)$ are explicit terms whose fourth powers are 1 [**9**]. The functions $\theta(z)$ and $\eta(z)$ are types of modular forms. The exponent of $cz + d$ in the above functional equations is called the weight of the modular form. The weights of both of these functions is $1/2$ which means that their functional equations are a bit more complicated than modular forms with integer weights.

In *Summing it Up*, a slightly simplified definition of "modular form" is given. A modular form is a holomorphic function $f(z)$ on the complex upper half-plane that is bounded as the imaginary part of z grows. We also have that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} f(z),$$

where the weight κ is a positive integer. Since the transformation property is assumed to hold for every element in $SL_2(\mathbb{Z})$, the modular form is of level 1. Higher levels imply that the property holds for certain subgroups of $SL_2(\mathbb{Z})$. Because of this structure, modular forms freely inhabit the worlds of analysis and arithmetic. Analytic methods can then be employed to study modular forms, and the subsequent discoveries regarding the properties of modular forms often yield arithmetic results. As a consequence, the subject of modular forms has played an increasingly central role in number theory.

A modular form has the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n,$$

where $q = e^{2\pi i z}$. This expression is called a q-expansion of $f(z)$, and the coefficients $a(n)$ often possess arithmetic data. As an example,

$$\eta^{24}(z) = \Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n,$$

where $\Delta(z)$ is a modular form of weight 12 and $\tau(n)$ is called the Ramanujan tau function. The function $\tau(n)$ satisfies numerous formulas and properties, such as this one:

$$\tau(n) = n^3\sigma_3(n) + 120 \sum_{m=1}^{[n/2]} (2n - 3m)\sigma_3(m)\sigma_3(n - m),$$

where $\sigma_k(n)$ is the sum of the kth powers of the positive divisors of n [**9**, p. 22].

We now follow the elegant and brief history of modular forms by Edixhoven, van der Geer, and Moonen [**5**]. For a long time, the objects now known as modular forms appeared sporadically in the works of several mathematicians. Nontrivial modular forms, as complex-valued functions with interesting properties, appeared already in work by Euler, but they appeared more extensively as theta functions in the work of Jacobi. They arise naturally in the study of elliptic functions, and for that reason were studied peripherally by mathematicians such as Eisenstein. Seeds of the subject also appeared in the works of Gauss, Dirichlet, and Riemann. The term "modular forms" (or "Modulformen" in German) first appeared in Klein and Fricke [**10**, p. 144]. They played increasingly significant roles in the work of several mathematicians of the late 19th and early 20th centuries, notably Ramanujan.

The elevation of modular forms to the center of a cohesive theory is most reasonably attributed to Hecke [**8**]. He revolutionized the subject, introducing new objects of study such as Hecke operators, L-functions of modular forms, and a converse theorem giving conditions under which a Dirichlet series is the series of a modular form. The general theory of modular forms was further developed and generalized by several mathematicians, notably Carl Ludwig Siegel. The theory's fundamental relevance to number theory became apparent in the 1950s and 1960s. The subject received contributions from too many people to give proper acknowledgment to all, but a small (and necessarily incomplete) list would include Eichler, Rankin, Gelfand, Piatetski-Shapiro, Langlands, Jacquet, Weil, Deligne, Serre, and Shimura. The introduction in the 1960s of the far-reaching conjectures that comprise the Langlands program was momentous. This program establishes, at least conjecturally, deep connections between numerous objects in number theory such as modular forms, L-functions, and representations. Such connections were demonstrated in dramatic fashion with Wiles' proof of Fermat's last theorem [**14**], and the later proof of the modularity theorem by Breuil, Conrad, Diamond, and Taylor [**4**], showing that elliptic curves over the rationals can be obtained from modular forms.

Nowadays, the importance of the subject toward an understanding of the issues at the core of many arithmetic problems can hardly be overstated. Results in number theory are often shown to be implied by statements about modular forms or related L-functions. Dirichlet's theorem about primes in arithmetic progressions, properties of L-functions, Fermat's last theorem, and the Hardy–Ramanujan–Rademacher formula are immediate examples. Over the last few decades we have seen generalizations of modular forms to adelic groups, automorphic representations, relations to differential forms on modular curves, a formulation of a non-abelian class field theory, and the Langlands program. These are just a few examples illustrating the rapid growth and attendant applications of the theory.

If you are interested in prime numbers, elliptic curves, partitions, sums of squares, Diophantine equations, arithmetic functions, or, really, any deep topic in number theory, then you should be interested in modular forms. There are obstacles, however, in introducing them to a general audience. To have some understanding of modular forms, one must first have a grasp of complex analysis, group theory, and Fourier series. Even then, the definitions are likely to come across as unmotivated and unnatural. For these reasons, the topic can be inaccessible to all but the most determined individuals.

In *Summing it Up*, Ash and Gross provide something novel: an accessible and fun introduction to modular forms. The book is divided into three parts: Finite Sums, Infinite Sums, and Modular Forms. The book's title references summations of course, which are among the main objects of study in the book. However, the book also brings a sense of completion to the introduction of number-theoretic methods, the study of which began with their first book. The first two parts of this book consist mostly of undergraduate mathematics, while the third part introduces modular forms.

In Part I, the authors discuss the arithmetic of finite sums. Since this topic requires an understanding of the integers, the authors give a brief course on classic number theory. Their discussion includes modular arithmetic and quadratic residues. They discuss the problem of expressing integers as sums of two squares, then three or four squares, and this leads to a discussion of Waring's problem. There follows a discussion of sums of arithmetic progressions and the fun methods (such as telescoping sums) that are used to evaluate certain finite sums. Figurative numbers and higher powers of integers are covered. This leads naturally to a discussion of Bernoulli numbers B_k, Bernoulli polynomials $B_k(x)$, and the nice expression (essentially Faulhaber's formula)

$$1^k + 2^k + \cdots + n^k = \frac{B_{k+1}(n+1) - B_{k+1}(1)}{k+1}.$$

An interesting application to estimating integrals is given at the end of this part.

Infinite sums are studied in Part II, starting with geometric series and the generalized binomial theorem. There follows an introduction to complex functions and the use of geometric series as a means to motivate the notion of analytic continuation. The exponential function is used as an example of expressing an analytic function using a power series. Then the authors give a quick introduction to generating functions, the gamma function, and Dirichlet series. As examples, the authors look at generating functions of certain partitions and sums of squares. These are classic examples that appear in the theory of modular forms. In particular, defining $r_k(n)$ to be the number of ways of writing n as a sum of k squares, where order, squares of 0, and squares of negative integers all matter, we can form the generating function

$$r_k(0) + r_k(1)q + r_k(2)q^2 + r_k(3)q^3 + \cdots.$$

This function is a modular form. In fact, it is $\theta^k(z)$, and we can use this connection and other similar results to study the numbers $r_k(n)$.

The authors make a quick foray into periodic functions and q-expansions. The authors outline a proof (from [**11**]) of the special values of the Riemann zeta function at positive even integers. They give Dirichlet series as an example of a class of generating functions, with a connection to prime numbers which delves a bit deeper than the usual Euler product of the Riemann zeta function. A theme here is that it is natural to define functions as infinite sums, and in this way some preparation is made for the book's final part.

In Part III, the authors finally introduce modular forms and provide applications. This requires some setup. The authors first introduce the geometry of the complex upper half-plane, non-Euclidean geometry, and the group $SL_2(\mathbb{Z})$ and its action on the upper half-plane. They give generators for the group and a discussion about the fundamental domain of the group action. After introducing modular forms in as simple a way as possible (in a section called "Modular Forms at Last") and with a q-expansion expression, the example of the weight 4 Eisenstein series is given. More generally, if κ is an even integer greater than or equal to 4, the Eisenstein series of weight κ and level 1 is

$$E_\kappa(z) = 1 - \frac{2\kappa}{B_\kappa}\sum_{n=1}^{\infty}\sigma_{\kappa-1}(n)q^n = \frac{1}{2\zeta(\kappa)}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(mz+n)^\kappa}.$$

It is a straightforward exercise to show that the double summation on the right has the appropriate functional equation of a modular form of weight κ.

The concepts of vector spaces and dimension, and q-expansions of Eisenstein series are then introduced and applied to modular forms. The ideas behind the basic results are well motivated, though not proved. Identities for sum-of-divisor functions are established by using the finite-dimensionality of the vector space of modular forms of fixed weight and level, as well as the q-expansion of Eisenstein series. As an example, it is shown that

$$E_{10}(z) = E_4(z)E_6(z).$$

By comparing coefficients in the respective q-expansions we get an interesting formula involving sum-of-divisors functions,

$$264\sigma_9(n) = 504\sigma_5(n) - 240\sigma_3(n) + 504\cdot 240\sum_{m=1}^{n-1}\sigma_3(m)\sigma_5(n-m).$$

Ash and Gross then discuss modular forms of higher level, fundamental domains of congruence subgroups, and the notion of cuspforms. A cuspform (of level 1) is a modular form whose q-expansion has a constant term of 0. Cuspforms are of particular importance, not least because we can define a Dirichlet series of a cuspform and under reasonably nice conditions the series has an Euler product. This creates a whole new area of study for researchers in modular forms. It is an area of great importance and depth, as illustrated by [**4**] and [**14**]. The function $\Delta(z)$, mentioned earlier, is an example of a cuspform.

As classic applications of q-expansions, the authors briefly discuss partitions and the problem of representing integers as sums of squares. For example, they mention that the basic results about modular forms can be used to show that

$$r_4(n) = 4(d_1(n) - d_3(n)),$$

where $d_k(n)$ is the number of positive divisors of n congruent to k modulo 4. Further, they outline the argument to obtain the more complicated expression, originally due to Ramanujan:

$$r_{24}(n) = \frac{16}{691}\sigma_{11}^*(n) - \frac{128}{691}\big(512\tau\left(\frac{n}{2}\right) + (-1)^n 259\tau(n)\big),$$

where $\sigma_k^e(n)$ is the sum of the kth powers of even divisors of n, $\sigma_k^o(n)$ is the sum of the kth powers of the odd divisors of n. We also have that

$$\sigma_k^*(n) = \begin{cases} \sigma_k^e(n) - \sigma_k^o(n) & \text{if } n \text{ is even} \\ \sigma_k(n) & \text{if } n \text{ is odd} \end{cases}$$

and that $\tau(n/2)$ is 0 for n odd. Such equations illustrate deep and nonobvious connections between seemingly unrelated modular forms. The authors also give a foray into the further theory of modular forms, including Hecke operators, L-functions of cuspforms, and newforms. They end with a survey of related topics such as Galois representations, elliptic curves, monstrous moonshine, and the Sato–Tate conjecture (now a theorem).

The book is engaging and conversational, without losing accuracy or essential rigor. It is somewhat less technical than their previous books. It should be noted that the book puts forth challenging goals for itself. There are steep hills to climb in moving a reader with only a little mathematical background toward an understanding of modular forms and how they are studied. As an example, comprehending the basic theorem that the vector space of the set of holomorphic modular forms of fixed weight and level is finite-dimensional requires knowledge of complex functions, vector spaces and their dimensions, the weight and level of modular forms, and the notion of cuspforms. For a student armed only with calculus, this is material from at least a few courses yet to be taken. Only after these ideas are established can nontrivial applications be shown. The authors do this as gently as possible, and their elegant prose amply justifies the writing of this book. However, some of the examples, such as partition functions, are treated very briefly with regard to their intersection with modular forms. The book only mentions these applications, without going into detail. This makes the examples seem somewhat less motivated than they might have been. The last part of the book can seem rushed, like a laundry list of further topics. However, this does give a sense that only the surface of the subject has been scratched, and that much more is waiting to be discovered by the determined student.

Overall the book is successful and highly recommended for a reader with some mathematical background who wants a basic understanding and feel for the subject. The book will be helpful in developing intuition. The reader should come away with an understanding of modular forms, why they are important, and a good sense of their behavior.

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