Research Statement

1. Introduction

This note provides a quick guide to my research. As it is intended for experts, I do not provide much background. Rather, I will briefly overview the various areas I have worked on and my contributions to each. Since I have been asked to focus on recent work, I will present the topics and papers in reverse chronological order (reversing the natural development of ideas).

2. Research Area I: Curves on Fano varieties

2.1. **Background.** My recent work has focused on Geometric Manin's Conjecture, a conjectural description of the moduli spaces of curves on Fano varieties. This conjecture combines several strands of mathematical research:

- The explicit description of (rational) curves on Fano varieties, a key tool in the study of Fano threefolds and hypersurfaces.
- Manin's Conjecture; via the global field analogy, arithmetic results over number fields should correspond with geometric results for curves over $\mathbb{F}_q$.
- The study of homology groups of moduli spaces of curves (e.g. the Cohen-Jones-Segal conjecture).

More concretely, Geometric Manin's Conjecture can be divided into two parts. The first part is the explicit description of the irreducible components of the moduli space of sections of a Fano fibration. As in Manin's Conjecture, there are two types of components: "good" components parametrizing free curves, and "exceptional" components with pathological behavior. The exceptional components should be controlled by a finite set of data. The second part is the homological stability of the good components of the moduli space. Via standard counting arguments, such stability would lead to point-counting results over $\mathbb{F}_q$.

My work thus far has focused on the first step – classifying irreducible components of the moduli space. I am currently starting to study the second step – homological stability of irreducible components.

2.2. **Results.**

(1) Non-free curves on Fano varieties (w/ E. Riedl, S. Tanimoto)
to appear on Osaka J. Math., arXiv:2304.05438 [math.AG]

(2) Non-free sections of Fano fibrations (w/ E. Riedl, S. Tanimoto)
submitted, arXiv:2301.01695 [math.AG]

(3) Rational curves on del Pezzo surfaces in positive characteristic (w/ R. Beheshti, E. Riedl, S. Tanimoto)
Trans. Amer. Math. Soc. 10 (2023), 407451

(4) Classifying sections of del Pezzo fibrations, II (w/ S. Tanimoto)
Geom. & Top. 26 (2022), no. 6, 2565-2647

(5) Classifying sections of del Pezzo fibrations, I (w/ S. Tanimoto)
J. Eur. Math. Soc. 26 (2024), no. 1, 289–354

(6) Moduli spaces of rational curves on Fano threefolds (w/ R. Beheshti, E. Riedl, S. Tanimoto)
Adv. Math. 408 (2022), Paper No. 108557

(7) Rational curves on prime Fano threefolds of index 1 (w/ S. Tanimoto)
J. Alg. Geom. 30 (2021), no. 1, 151-188

(8) Geometric Manin's Conjecture and rational curves (w/ S. Tanimoto)
Compos. Math. 155 (2019), no. 5, 833-862

The most comprehensive result is (2). This paper verifies in the maximum possible generality (in characteristic 0) that exceptional families of sections of a Fano fibration come from a bounded family of maps known as "accumulating maps". This verifies the conjectural description of the exceptional set in our geometric setting. The paper includes several other powerful results, and in particular proves the first "bounds of Manin type" over global function fields if we reverse the order of limits and first allow the prime q go to infinity. The follow-up (1) recasts these results for Fano varieties (instead of Fano fibrations).

The earliest paper (8) was our initial attempt to describe the theoretical behavior of exceptional components and good components on Fano varieties.

The papers (4) and (5) address the moduli space of sections for del Pezzo surface fibrations. In addition to classifying exceptional components, they also provide a classification scheme for good components. In some cases we get a complete description of these moduli spaces.

Similarly, the papers (6) and (7) address the moduli space of rational curves for Fano varieties of dimension 3. They focus on applying the general theory in these concrete cases (and have laid the groundwork for many subsequent extensions).

(3) addresses Geometric Manin's Conjecture in characteristic p. The key question in characteristic p is whether new "pathological" phenomena match up: do inseparable families of curves come from inseparable accumulating maps? This paper answers this question (and several others) for weak del Pezzo surfaces in arbitrary characteristic.

3. Research Area II: Manin's Conjecture

3.1. **Background.** Let X be a smooth projective Fano variety over a number field k. Conjecturally rational points are potentially dense on X. Manin's Conjecture gives a quantitative version of this prediction: the "amount" of negative curvature of K_X should predict the "amount" of rational points on X.

An important concept in Manin's Conjecture is the idea of an "exceptional set". It is possible for rational points on X to accumulate along certain special subvarieties. In this case we cannot hope for the asymptotic behavior to be controlled by global geometric invariants and we must remove the exceptional set to obtain the correct predicted growth rate. While we expect the exceptional set to be "small", an example of Batyrev and Tschinkel shows that the exceptional set can be Zariski dense. Peyre has conjectured that the exceptional set is a thin set (in the sense of Serre).

Recently there have been several attempts to provide a (conjectural) description of the exceptional set. My work has focused on a particular approach via "accumulating maps". Suppose given a generically finite morphism $f : Y \to X$ such that the geometric invariants in Manin's Conjecture are larger on Y than on X. Then we should expect $f(Y(k))$ to be included in the exceptional set – indeed, this is necessary for the consistency of Manin's Conjecture. In all known examples, the exceptional set is entirely accounted for by such maps.

Thus to understand the exceptional set in Manin's Conjecture, we must understand the geometric invariants and their behavior over generically finite morphisms $f : Y \to X$. My work has accomplished this goal using tools from the Minimal Model Program. This perspective has found recent applications to Manin's Conjecture for stacks, and in particular, in novel formulations of Malle's Conjecture.

3.2. **Results.**

(9) On exceptional sets in Manin's Conjecture (w/ S. Tanimoto)
Res. Math. Sci. 6 (2019), no. 1, Paper No. 12, 41 pp.

(10) Geometric consistency of Manin's Conjecture (w/ A.K. Sengupta, S. Tanimoto)
Compos. Math. 58 (2022), no. 6, 1375-1427

(11) On the geometry of thin exceptional sets in Manin's Conjecture (w/ S. Tanimoto)
Duke Math. J. 166 (2017), no. 15, 2815-2869

(12) Balanced line bundles on Fano varieties (w/ S. Tanimoto, Y. Tschinkel)
J. Reine Angew. Math. 743 (2018), 91-131

The most comprehensive result is (10). This paper proves the "boundedness" of accumulating maps in the maximum possible generality (in characteristic 0). As a consequence, we see that the exceptional set proposed above is always contained in a thin set of rational points, verifying a prediction of Peyre.

The earlier papers (11) and (12) take the first steps toward the general result. They also develop the tools necessary for analyzing the exceptional set in specific examples.

The paper (9) is mostly a survey paper, but it includes several new results and examples as well.

4. Recent miscellaneous results

My recent miscellaneous results have centered around the geometry of curves.

4.1. **Results.**

(13) On the asymptotic enumerativity property for Fano manifolds (w/ R. Beheshti, C. Lian, E. Riedl, J. Starr, S. Tanimoto)
submitted, arXiv:2310.15252 [math.AG]

(14) Approximating rational points on surfaces (joint with D. McKinnon, M. Satriano)
submitted, arXiv:2403.02480 [math.AG]

(15) Restricted tangent bundles for general free rational curves (w/ E. Riedl)
Int. Math. Res. Not. (2023), no. 12, 9901-9949

(13) analyzes when we can expect a Gromov-Witten invariant on a Fano variety to be enumerative. It gives some counterexamples to a question of Lian and Pandharipande and also provides some new (positive) examples.

(14) describes a geometric approach to understanding the approximation constants of McKinnon and Roth using tools from the MMP.

(15) analyzes the behavior of the restricted tangent bundle for rational curves. In particular, it asks: when does the restricted tangent bundle exactly match the "expected value"?

5. Research Area III: Positivity and convexity

5.1. **Background.** Toric varieties provide a concrete link between algebraic geometry and combinatorics. In this dictionary, the theory of convex polytopes yields results about the positivity of divisors. Many features of this dictionary extend to arbitrary projective varieties via the theory of positivity of divisors.

My work has centered around constructing a "dual" theory for curves. Using abstract convexity techniques, one can define a dual volume function on the cone of curves. Surprisingly, the dual volume contains real geometric information closely related to the mobility counts defined below.

5.2. **Results.**

(16) Correspondences between convex geometry and complex geometry (w/ J. Xiao)
EpiGA 1 (2017), Art. 6

(17) Positivity functions for curves on algebraic varieties (w/ J. Xiao)
Algebra Number Theory 13 (2019), no. 6, 1243-1279

(18) Convexity and Zariski decomposition structure (w/ J. Xiao)
Geom. Funct. Anal. 26 (2016), no. 4, 1135-1189

(18) recasts the famous Zariski decomposition as a result in convexity theory. Indeed, a key feature of the Zariski decomposition (and its higher dimensional analogues) is its interaction with the concavity of the volume function. The main results show that curve classes on arbitrary varieties admit a "Zariski decomposition" structure that is closely analogous to the classical case of surfaces. Again, the key property is concavity with respect to a "dual volume".

(17) compares the dual volume function with the volume functions defined using geometric invariants (such as the mobility count). Conjecturally the two are equal; the paper shows that this would follow from a natural statement about complete intersection curves.

(16) extends the dictionary between convex geometry and divisor theory in algebraic geometry.

6. Research Area IV: Positivity of subvarieties

6.1. **Background.** The theory of ample divisors – divisors which are "positive" in several equivalent ways – is central to algebraic geometry. Over the last fifty years, it has become clear that other weaker notions of "positivity" are similarly well-behaved. A key perspective is that the geometry of a line bundle $\mathcal{L}$ is best captured by studying the asymptotic behavior of sections of $\mathcal{L}^{\otimes m}$ as m increases.

I have worked on extending the notions of "positivity" to subvarieties of higher codimension. In contrast to divisors, there is no longer a close tie between subvarieties and vector bundles and it is important to separate the two concepts. A main theme in my work is that the best way to understand a cycle Y is by looking at the asympotic behavior of mY as m increases.

The results below fall into two categories. The first is the abstract study of numerical equivalence and formal versions of positivity. These ideas have later been used in a fundamental way in the study of dynamical degrees. The second is the relationship between positivity and asymptotic behavior. While these papers describe a conjectural picture that I find quite appealing, unfortunately it seems very difficult to understand examples.

6.2. Results.

(19) Iitaka dimension for cycles
Trans. Amer. Math. Soc. 371 (2019), no. 7, 4815-4835
(20) Positivity of the diagonal (w/ J.C. Ottem)
Adv. Math. 335 (2018), 664-695
(21) Zariski decompositions of numerical cycle classes (w/ A.M. Fulger)
J. Alg. Geom. 26 (2017), no. 1, 43-106
(22) Asymptotic behavior of the dimension of the Chow variety
Adv. Math. 308 (2017), 815-835
(23) Morphisms and faces of pseudo-effective cones (w/ A.M. Fulger)
Proc. London Math. Soc. 112 (2016), no. 4, 651-676
(24) Positive cones of dual cycle classes (w/ A.M. Fulger)
Algebraic Geometry 4 (2017), no. 1, 1-28
(25) Kernels of numerical pushforwards (w/ A.M. Fulger)
Adv. Geom. 17 (2017), no. 3, 373-378
(26) Volume-type functions for numerical cycle classes
Duke Math. J. 165 (2016), no. 16, 3147-3187

(26) introduces a notion of volume for higher codimensional cycles. We define the "mobility count" of a cycle Y to be the number of general points in X that can be contained in cycles numerically equivalent to Y. Just as the volume of a divisor measures the growth in the dimension of the space of sections, the volume of a higher codimension cycle Y reflects the growth of the mobility count of mY. This function turns out to have nice analytic properties, solving a conjecture of Debarre-Ein-Lazarsfeld-Voisin.

(19) continues the analogy between mobility counts and spaces of sections by defining and studying the Iitaka dimension of a cycle. Surprisingly, it seems that the Iitaka dimension only takes integer values; the paper verifies this conjecture in several natural examples.

(21) continues this analogy yet further by defining a "Zariski decomposition" for higher codimensional cycles. This work combines the geometric properties of the mobility count with formal convexity/analytic properties of volume functions.

(22) provides a different take on the asymptotic behavior of cycles: instead of looking at cycles through general points, we ask how much they move in families. This paper gives a fairly complete theoretical description of this phenomenon.

(20) studies the positivity of the diagonal $\Delta_X \subset X \times X$. This paper states many conjectures describing the relationship between positivity of Δ_X and "motivic" properties of X, and establishes several of these conjectures in low dimensions.

(24) develops a theory of "nef" cycle classes in arbitrary dimension. Unlike for divisors and curves, there are many different non-equivalent notions, some of which have subsequently been found useful for other applications.

(23) analyzes how cycles behave with respect to morphisms; (25) is a minor side project in the same direction.

7. Research Area V: Numerical positivity of divisors

7.1. **Background.** One of the main principles of the Minimal Model Program is that sectional positivity of the canonical divisor (encoded in the canonical ring) should match with intersection-theoretic positivity of the canonical divisor. In my thesis I studied the abstract theory of numerical positivity and its applications to the minimal model program.

7.2. **Results.**

(27) A snapshot of the Minimal Model Program
Proc. of Symp. in Pure Math. 95 (2017), AMS, 1-32
(28) Volume and Hilbert function of R-divisors (w/ A.M. Fulger, J. Kollár)
Mich. Math. J. 65 (2016), no. 2, 371-387
(29) Numerical triviality and pullbacks
J. Pure Appl. Algebra 219 (2015), no. 12, 5637-5649
(30) Algebraic bounds on analytic multiplier ideals
Ann. Inst. Fourier 64 (2014), no. 3, 1077-1108
(31) Reduction maps and the minimal model program (w/ Y. Gongyo)
Compos. Math. 149 (2013), no. 2, 295-308
(32) On Eckl's pseudo-effective reduction map
Trans. of the A.M.S. 366 (2014), no. 3, 1525-1549
(33) Comparing numerical dimensions
Algebra Number Theory 7 (2013), no. 5, 1065-1100
(34) A cone theorem for nef curves
J. Algebraic Geom. 21 (2012), no. 3, 473-493

(27) is a survey paper providing a "snapshot" of the MMP at the time it was written.

(28) proves a useful property of Zariski decompositions in the maximal possible generality.

The papers (29) - (33) develop the theory of numerical positivity of divisors and derive several consequences concerning the main conjectures of the MMP.

(34) proves an analogue of the famous Cone Theorem for classes of nef curves using the (at the time) recent advances of BCHM.