ERRATUM: RATIONAL CURVES ON PRIME FANO THREEFOLDS OF INDEX 1

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Abstract. In this erratum, we provide correct proofs of Theorem 7.4 and Theorem 7.6 in [LT21].

1. Introduction

The proofs of Theorem 7.4 and 7.6 of [LT21] are not correct. Indeed, the second and third authors claimed that the zero locus of a general section of a globally generated locally free sheaf is irreducible, but this is not correct in general. We need a stronger condition (such as ampleness) to claim this, but unfortunately in our situations we do not have this property. Thus the method of proof of Theorem 7.4 and 7.6 of [LT21] is not correct. In this erratum, we provide correct proofs of Theorem 7.4 and Theorem 7.6.

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2. The spaces of prime Fano threefolds of index 1.

Let X be a smooth Fano threefold over $\mathbb{C}$ of Picard rank 1 and index 1 such that $-K_X$ is very ample and $(-K_X)^3 = 2g-2$. Note that g takes the values between 3 and 10 or 12; we will focus on the cases $3 \leq g \leq 10$. For each g in this range we construct a space M_{2g-2} parametrizing smooth Fano threefolds X with the given genus as well as certain singular degenerations.

The case of $g = 3$: Let $\overline{M}_4 = \mathbb{P}^{69}$ be the projective space parametrizing quartic hypersurfaces in $\mathbb{P}^4$. Let $M_4 = \overline{M}_4$.

The case of $g = 4$: Let $\mathbb{P}^{20}$ be the projective space parametrizing quadric hypersurfaces in $\mathbb{P}^5$ and $\mathbb{P}^{55}$ be the projective space parametrizing cubic hypersurfaces in $\mathbb{P}^5$. We set $\overline{M_6} = \mathbb{P}^{20} \times \mathbb{P}^{55}$. Let $M_6 \subset \overline{M}_6$ be the Zariski open subset parametrizing pairs of quadric hypersurfaces and cubic hypersurfaces whose intersection is a complete intersection of dimension 3.

The case of $g = 5$: Let $\mathbb{A}^{28}$ be the vector space of quadric sections on $\mathbb{P}^6$. Let $\overline{M}_8 = \mathrm{Gr}(3, \mathbb{A}^{28})$ be the Grassmannian parametrizing nets of quadrics. Let $M_8 \subset \overline{M}_8$ be the Zariski open subset parametrizing nets of quadrics whose base locus is a complete intersection of dimension 3.

The case of $g = 6$: Let $\mathbb{P}^{54}$ be the projective space parametrizing quadric hypersurfaces in $\mathbb{P}^9$, $\mathbb{A}^{10}$ be the vector space of linear sections on $\mathbb{P}^9$, and $\mathrm{Gr}(2, \mathbb{A}^{10})$ be the Grassmannian

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parametrizing pencils of hyperplane sections. We set $\overline{M}_{10} = \mathbb{P}^{54} \times \mathrm{Gr}(2, \mathbb{A}^{10})$. Let $M_{10} \subset \overline{M}_{10}$ be the Zariski open subset parametrizing pairs of quadrics and pencils such that the intersection of the quadric, the base locus of the pencil, and $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ is a complete intersection of dimension 3 in $\mathrm{Gr}(2,5)$.

The case of $g = 7$: Let $\mathbb{A}^{16}$ be the vector space of linear sections on $\mathbb{P}^{15}$ and $\mathrm{Gr}(7, \mathbb{A}^{16})$ be the Grassmannian parametrizing 7-dimensional subspaces of hyperplane sections. We set $\overline{M}_{12} = \mathrm{Gr}(7, \mathbb{A}^{16})$. Let $M_{12} \subset \overline{M}_{12}$ be the Zariski open subset parametrizing subspaces of hyperplane sections such that the intersection of the base locus and $\mathrm{OGr}_+(5, 10) \subset \mathbb{P}^{15}$ is a complete intersection of dimension 3 in $\mathrm{OGr}_+(5, 10)$.

The case of $g = 8$: Let $\mathbb{A}^{15}$ be the vector space of linear sections on $\mathbb{P}^{14}$ and $\mathrm{Gr}(5, \mathbb{A}^{15})$ be the Grassmannian parametrizing 5-dimensional subspaces of hyperplane sections. We set $\overline{M}_{14} = \mathrm{Gr}(5, \mathbb{A}^{15})$. Let $M_{14} \subset \overline{M}_{14}$ be the Zariski open subset parametrizing subspaces of hyperplane sections such that the intersection of the base locus and $\mathrm{Gr}(2, 6) \subset \mathbb{P}^{14}$ is a complete intersection of dimension 3 in $\mathrm{Gr}(2, 6)$.

The case of $g = 9$: Let $\mathbb{A}^{14}$ be the vector space of linear sections on $\mathbb{P}^{13}$ and $\mathrm{Gr}(3, \mathbb{A}^{14})$ be the Grassmannian parametrizing 3-dimensional subspaces of hyperplane sections. We set $\overline{M}_{16} = \mathrm{Gr}(3, \mathbb{A}^{14})$. Let $M_{16} \subset \overline{M}_{16}$ be the Zariski open subset parametrizing subspaces of hyperplane sections such that the intersection of the base locus and $\mathrm{LGr}(3, 6) \subset \mathbb{P}^{13}$ is a complete intersection of dimension 3 in $\mathrm{LGr}(3, 6)$.

The case of $g = 10$: Let $\mathbb{A}^{14}$ be the vector space of linear sections on $\mathbb{P}^{13}$ and $\mathrm{Gr}(2, \mathbb{A}^{14})$ be the Grassmannian parametrizing 2-dimensional subspaces of hyperplane sections. We set $\overline{M}_{18} = \mathrm{Gr}(2, \mathbb{A}^{14})$. Let $M_{18} \subset \overline{M}_{18}$ be the Zariski open subset parametrizing subspaces of hyperplane sections such that the intersection of the base locus and $G_2/P \subset \mathbb{P}^{13}$ is a complete intersection of dimension 3 in G_2/P.

Lemma 2.1. *For each* $3 \leq g \leq 10$, M_{2g-2} *is smooth and simply connected.*

Proof. Since $\overline{M}_{2g-2}$ is smooth and simply connected, it suffices to show that the codimension of $\overline{M}_{2g-2} \setminus M_{2g-2}$ is at least 2. We will demonstrate this proof when $g = 6$; the other cases are similar. Let $\mathcal{X} \subset \mathbb{P}^9 \times \overline{M}_{10}$ be the family formed by intersecting $\mathrm{Gr}(2, 5)$ with the quadric and codimension 2 linear space defined by the corresponding point in $\overline{M}_{10}$. We denote the projections by $\pi : \mathcal{X} \to \overline{M}_{10}$ and $ev : \mathcal{X} \to \mathrm{Gr}(2, 5)$. Then every fiber of ev is isomorphic to $\mathbb{P}^{53} \times \mathrm{Gr}(2, 9)$. In particular $\mathcal{X}$ is irreducible. However, if $\overline{M}_{10} \setminus M_{10}$ has codimension 1, then $\mathcal{X}$ cannot be irreducible: every fiber of π over $\overline{M}_{10} \setminus M_{10}$ has dimension at least 4, so in this situation the π-preimage of $\overline{M}_{10} \setminus M_{10}$ would contain an additional irreducible component of $\mathcal{X}$ by a dimension count. Our assertion follows. □

Let $D_{2g-2} \subset M_{2g-2}$ be the locus parametrizing singular objects; we call this locus the discriminant locus. Regarding this locus, we have

Lemma 2.2. *For each* $3 \leq g \leq 10$, D_{2g-2} *is irreducible in* M_{2g-2}.

Proof. We demonstrate this analysis when $g = 6$. Let $\pi : \mathcal{X} \to M_{10}$ be the universal family of complete intersections of one quadric section and two hyperplane sections on $\mathrm{Gr}(2, 5)$ with the evaluation map $ev : \mathcal{X} \to \mathrm{Gr}(2, 5)$. Then for any $x \in \mathrm{Gr}(2, 5)$, the fiber $ev^{-1}(x)$ is isomorphic to a Zariski open subset of $\mathbb{P}^{53} \times \mathrm{Gr}(2, 9)$ parametrizing complete intersections containing x. Among them, complete intersections which are singular at x form an irreducible locus of dimension 63. Indeed, this follows from the fact that in the space of matrices, the locus of non-full rank matrices is irreducible. Thus as x varies, singular threefolds form an irreducible locus of dimension at most 69 in M_{10}.

The other cases are similar. Every family is defined by taking hypersurface sections of a homogeneous space $\mathbb{G}$. The sublocus parametrizing threefolds which are singular at a fixed point $x \in \mathbb{G}$ corresponds to choices of hypersurface sections whose derivatives are linearly dependent in $\mathcal{T}_x\mathbb{G}$. This is a cone over the locus of non-full rank matrices $A : \mathcal{T}_x\mathbb{G} \to k^{\dim \mathbb{G}-3}$, which is always irreducible. Letting x vary, we see that D_{2g-2} is irreducible. □

Lemma 2.3. *For each $3 \leq g \leq 10$, a general variety parametrized by D_{2g-2} is a terminal Gorenstein Fano threefold embedded via its anticanonical linear series.*

Proof. We first show that each D_{2g-2} actually parametrizes a single threefold of the desired type. Let X be a smooth Fano threefold of genus $g \geq 5$. Let $\widetilde{X}$ denote the blow-up of X along a line and consider the anticanonical model $\phi : \widetilde{X} \to \bar{X}$. By [IP99, Proposition 4.3.1], $\bar{X}$ is a terminal Gorenstein Fano threefold of index 1 and genus $g' = g-2$. This constructs singular Fano threefolds of genus $3 \leq g' \leq 8$ and $g' = 10$. By [IP99, Lemma 4.1.1, Proposition 4.4.1], when $g = 12$ we may obtain a terminal Gorenstein Fano threefold of genus $g' = 9$ by blowing up a conic instead of a line. In each case, $\phi^*(-K_{\bar{X}}) = -K_{\widetilde{X}}$ is not expressible as the sum of two moving Weil divisors. Thus each $\bar{X}$ is BN-general in the sense of [Muk02, Proposition 7.8]. Hence, the anticanonical linear series embeds $\bar{X}$ as a complete intersection of the desired type [Muk02, Theorem 6.5(2)].

We next verify that our desired properties spread out to a general threefold parametrized by the (irreducible) variety D_{2g-2}. Since the property of being a Gorenstein terminal threefold is open in flat families ([KM98, Corollary 5.44]), we conclude that a general member of D_{2g-2} has these properties. Once we know the Gorenstein terminal property for the general singular threefold, it is clear from the construction that each such threefold is embedded by the anticanonical linear series. □

3. Spaces of Curves

For each moduli space M_{2g-2} parametrizing Fano threefolds as constructed earlier, let $\pi : \mathcal{X}_{2g-2} \to M_{2g-2}$ be the parametrized family of complete intersections. Let $X_{2g-2} \subset \mathcal{X}_{2g-2}$ be a general fiber of π. Every component of the Kontsevich space $\overline{M}_{0,0}(X_{2g-2})$ which generically parametrizes free curves extends to a component of $\overline{M}_{0,0}(\mathcal{X}_{2g-2})$ parametrizing free curves on $\mathcal{X}_{2g-2}$. To prove irreducibility of moduli spaces of free curves on X_{2g-2}, we analyze the corresponding components of $\overline{M}_{0,0}(\mathcal{X}_{2g-2})$.

Lemma 3.1. *The locus of $\overline{M}_{0,0}(\mathcal{X}_{2g-2})$ parameterizing free cubics (resp. very free quartics) contracted by π is irreducible.*

Proof. Let $\mathbb{G}$ be the homogenous space containing each fiber of π as a complete intersection and consider the evaluation map $ev : \mathcal{X}_{2g-2} \to \mathbb{G}$. By [LT21, Lemma 7.3 and Lemma 7.5], a general free cubic (resp. very free quartic) on X_{2g-2} embeds as a rational normal curve in $\mathbb{G}$. Rational normal curves of degree $d = 3, 4$ in $\mathbb{G}$ form an irreducible family ([Tho98]). Moreover, for each such curve the sublocus of M_{2g-2} parametrizing complete intersections which contain the curve is irreducible. Thus the corresponding family in $\mathcal{X}_{2g-2}$ is also irreducible. □

Let $Z_3, Z_4 \subset \overline{M}_{0,0}(\mathcal{X}_{2g-2})$ be the closure of the locus of free cubics (resp. very free quartics) contracted by π. The following lemma verifies the validity of the statements of [LT21, Theorem 7.4 and Theorem 7.6].

Lemma 3.2. *For $d = 3, 4$, the natural map $\pi_* : Z_d \to M_{2g-2}$ has irreducible general fiber.*

Proof. Let $\tilde{Z}$ be a resolution of Z_d and assume the Stein factorization $f : \tilde{Z} \to Y$, $g : Y \to M_{2g-2}$ of $\pi_* : \tilde{Z} \to M_{2g-2}$ is nontrivial. Since M_{2g-2} is smooth and simply connected, the branch locus $B \subset M_{2g-2}$ of the finite part of the Stein factorization is non-empty and has codimension one. Over any point $b \in B$, the fiber of $\tilde{Z}$ has an everywhere non-reduced connected component N. By upper semicontinuity of fiber dimension applied to the image of the family of curves parametrized by $\tilde{Z}$ in $Y \times_{M_{2g-2}} \mathcal{X}_{2g-2}$, some irreducible component $N_1 \subset N$ parametrizes a dominant family of curves of dimension $\geq d$ on the complete intersection $X_b = \pi^{-1}(b)$. When $d = 4$, by a similar upper semicontinuity argument and [LT21, Lemma 8.1], we may further assume N_1 parameterizes curves passing through two general points which are general in $X_b^{\times 2}$. Since B has codimension 1, Lemma 2.3 shows that for $b \in B$ general X_b has at worst Gorenstein terminal singularities.

By functoriality, we may identify N_1 with a non-reduced component of $\overline{M}_{0,0}(X_b, d)$. Since X_b has terminal Gorenstein singularities and N_1 pararameterizes a dominant family of curves, [LT23, Lemma 2.3] shows that if the general curve C parametrized by N_1 is irreducible then a general point of N_1 parametrizes a free curve. Since N_1 is non-reduced, we conclude that the general curve C parameterized by N_1 must be reducible. We will prove our statement by studying the locus of reducible curves in X_b of low degrees and verifying that there can be no such component N_1 parametrizing reducible curves and satisfying the other properties described above.

From now on we assume that the general curve C parametrized by N_1 is reducible. As we vary C there is (at least) one component $C_1 \subset C$ which dominates X_b and therefore belongs to a component of $\overline{M}_{0,0}(X)$ that parametrizes free curves and has the expected dimension. The other components of C must have total degree between one and two.

We first claim that each irreducible non-contracted component of $C \setminus C_1$ belongs to an irreducible component of $\overline{M}_{0,0}(X)$ that parametrizes a non-dominant family of curves. Otherwise C would be a quartic curve with two conics as irreducible components. The space of such conics through any fixed point $x \in X$ has dimension at most 1, and will have dimension equal to 1 at only finitely many points. Thus the locus of reducible deformations of C has dimension three. However, the irreducible component of $\overline{M}_{0,0}(X)$ parametrizing C must have dimension at least four. This verifies the claim.

We show that there can be no N_1 as described above by considering separately the cases of anticanonical degree 3 and 4. If $-K_{X_b} \cdot C = 4$, recall that we may assume N_1 parameterizes a quartic curve passing through two general points of X_b. This is impossible if the general curve parameterized by N_1 is reducible, a contradiction.

Suppose $-K_{X_b} \cdot C = 3$ instead. It follows that $C = C_1 \cup C_2$, where $-K_{X_b} \cdot C_i = 3 - i$. Let $W \subsetneq X_b$ denote the closed subvariety swept out by the deformations of C_2. As $-K_{X_b}$ is Cartier, the argument of [LT19a, Proposition 4.2] shows that W has a-invariant strictly greater than 1:

$$a(W, -K_{X_b}|_W) \geq 1 + \frac{q+1}{-K_{X_b} \cdot C_2} > 1.$$

where q is the difference between the actual dimension of deformations and the expected dimension of deformations of C_2. Note that if $q > 0$ then $a(W, -K_{X_b}|_W) > 2$. Appealing to the classification of surfaces with large a-invariant (see [LT19b, Proposition 3.17]) we see that one of the following two conditions hold:

(1) Deformations of C_2 have the expected dimension $-K_{X_b} \cdot C_2 = 1$;
(2) Letting $\phi : \widetilde{W} \to W$ denote a resolution of singularities, a run of the $(K_{\widetilde{W}} - 3\phi^* K_{X_b})$-MMP yields a birational morphism $\psi : \widetilde{W} \to \mathbb{P}^2$ and $\psi^*\mathcal{O}(1)$ is linearly equivalent to $-\phi^* K_{X_b}$.

Since $-K_{X_b}$ is very ample, every (-1)-curve contracted by ψ is also contracted by ϕ. Thus we may set $\widetilde{W} = \mathbb{P}^2$ so that ϕ is a finite birational morphism. Furthermore, since no sublinear series of $|\mathcal{O}(1)|$ is basepoint free, we conclude that ϕ must be an isomorphism.

Recall that as we vary C the images of the component C_1 dominate X. Since C_1 is a conic, if we fix a finite set of points in the divisor swept out by C_2 then the general C_1 will meet C_2 away from this finite set. Hence, the dimension of the space of stable maps formed by attaching C_1 to C_2 has the expected dimension. In particular if C_2 satisfies (1) then $\dim N_1 < -K_{X_b} \cdot C$, giving our contradiction. If C_2 satisfies (2) instead, then for general C in N_1 both C_1 and C_2 are general in moduli. Note that since $(W, -K_{X_b})$ is isomorphic to $(\mathbb{P}^2, \mathcal{O}(1))$, the normal sheaf of W in X restricted to C_2 is isomorphic to $\mathcal{O}(-2)$ because C_2 avoids any singular point on X. It follows from this that the normal bundle of C_2 is given by

$$\mathcal{O}(1) \oplus \mathcal{O}(-2).$$

Moreover it follows from [BLRT22, Proposition 2.9] that C_1 meets with W transversally. Thus we conclude by [GHS03, Lemma 2.6] that the first cohomology of the normal bundle of C vanishes. Therefore, $[C]$ is a smooth (and in particular, reduced) point of N_1, a contradiction. □

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