

## Revenue (GDP) and Profit Functions

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September 2006

### **Profit Function**

$$(1) \quad \pi^j(p, w, k_j) = \max_{y^j, v^j} \{ p y^j - w v^j \mid (y^j, v^j) \in T^j(y^j, v^j, k_j) \}$$

$T^j$  is a convex set, and conditional on the fixed factor  $k_j$  the set is strictly convex.

reasoning: If  $T^j$  is a crs technology, the profit function is unbounded, hence strict convexity is imposed on the conditional production set. The nonzero profit  $\pi$  in firm  $j$  is received by some residual claimant --- an owner, a manager, or a set of workers with power to appropriate rents.  $\pi^j = \pi_k^j k^j$  can be taken to define the amount of  $k$ . It is often convenient to drop the notation for  $k$ .

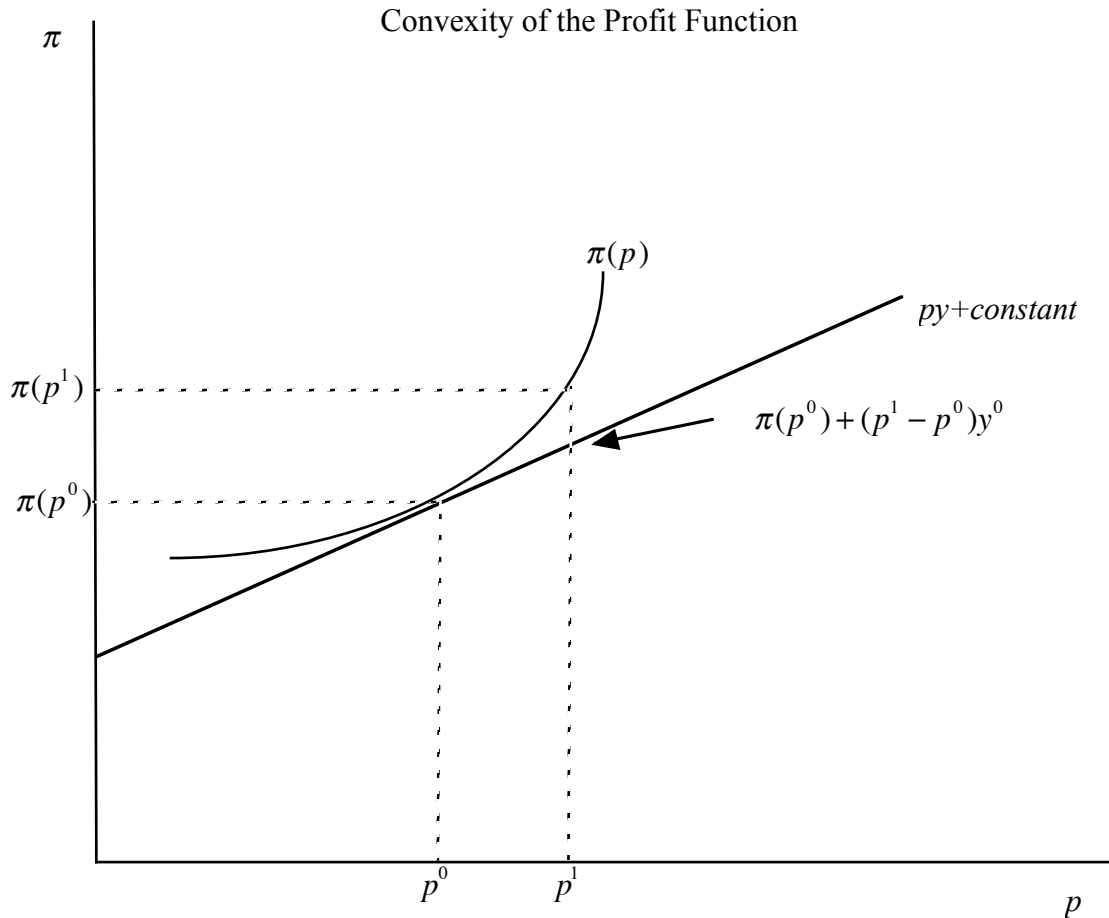
### **properties of the profit function**

#### convex in $p$ and $w$

From maximization  $\pi(p^1, w^1) \geq p^1 y^0 - w^1 v^0$ , hence

$$(2) \quad \pi(p^0, w^0) - \pi(p^1, w^1) \geq [p^0 - p^1] y^0 - [w^0 - w^1] v^0. \text{ Then } \pi \text{ must be convex.}$$

*A Diagram can illustrate this in  $p$ , with a similar diagram for  $w$ .*



### Hotelling's Lemma

$\pi_p = y$  and  $\pi_w = -v$ . (The latter relationship is essentially Shephard's lemma since the negative of the cost function appears in the profit function.) This follows from (2) and infinitesimal changes.

### homogeneous of degree one

Follows from the linear objective function in (1).

### Concave in k

$\pi(p, w, \lambda k^0 + (1 - \lambda)k^1) \geq \lambda \pi(p, w, k^0) + (1 - \lambda)\pi(p, w, k^1)$  by the maximum value property of the profit function, noting that the right hand side is always feasible under the assumption that the technology is convex.

### Revenue (GDP) Function

$$(3) \quad r(p, v, \{k^j\}) = \min_w \{ \sum \pi^j(p, w, k^j) + w \cdot v \}$$

The expression on the right hand side under the curly brackets is the sum of factor payments, including profits, hence it is GDP. Intuitively, the minimization problem says that the objective is to minimize the resource bill for goods produced at price vector  $p$ . One can think of the Invisible Hand as doing the minimization. This program has a unique solution. The first order condition is equivalent to market clearance for the variable factors of production.

The mathematical programming logic of (3) is revealed by combining profit maximization with the general equilibrium resource allocation problem:

$$r(p, v, \{k^j\}) = \max_{\{y^j, v^j\}} \min_w p \sum_j y^j - w [\sum_j v^j - v] \mid (y^j, v^j) \in T(\{y^j\}, \{v^j\}, \{k^j\})$$

The  $w$  vector is recognized as the vector of Lagrange multipliers for the factor endowment constraints. A competitive economy has equilibrium factor prices equal to the Lagrange multipliers.

This derivation of the GDP function is an alternative and rather more revealing derivation of the revenue or GDP function in the case where the technology is strictly convex. Dixit and Norman instead define

$$(3') \quad r(p, v) = \max_y \{ p \cdot y \mid y \in T(y, v) \}.$$

Here,  $T$  is the union of the firm technologies  $T^j$ .

In the case where the aggregate technology exhibits constant returns to scale, we can build up from the profit function to the GDP function by imposing a further smoothness condition. If we regard  $k^j$  as divisible and allocable across all firms subject to  $\sum k^j = k$ , then

$$\begin{aligned} r(p, v, k) &= \max_{\{k^j\}} \min_w [ \sum \pi^j(p, w, k^j) + w \cdot v \mid \sum k^j = k ] \\ &= \max_{\{k^j\}} [ r(p, v, \{k^j\}) \mid \sum k^j = k ]. \end{aligned}$$

Obviously the additional allocation of the  $k^j$  results in an equal return in every firm. Pure firm profits disappear. The aggregate technology now exhibits constant returns to scale, due to the divisibility assumption. Firms are simply replicated at the appropriate scale to produce sectoral output or combinations of output. We can subsume  $k$  into the vector  $v$  for more compact notation.

The new trade literature on firm heterogeneity, if it maintains convex technology, goes back to profit function (1) and aggregates to GDP function (3).

Notice that the notation for the index  $j$  encompasses both firms and sectors.

### properties of the GDP or revenue function

convex in  $p$  and concave in  $v$

Convexity in  $p$  is inherited from convexity of  $\pi$ . Concavity in  $v$  follows from the minimum value structure of (3), using an exactly parallel proof.

#### Homogeneous of degree one in $p$

The first order condition for (3) implies  $-\Sigma\pi'_w = v$ . The first derivatives of  $\pi$  are homogeneous of degree zero in  $p$  and  $w$ , hence the optimal  $w$  implied by the first order condition is homogeneous of degree one in  $p$ . Then  $r$  is homogeneous of degree one in  $p$ .

#### Hotelling's lemma in general equilibrium

$r_p = y$ , a property inherited from  $\pi$ .

$r_v = w$ .

An important complication is that  $r$  need not be differentiable in  $p$ . A set of points  $y \in T$  can solve the maximization problem. This issue becomes clear by considering important special cases, beginning with the Ricardian model where non-differentiable  $r$  obtains. In contrast,  $r_v$  is defined except for pathological cases.

By convexity, if  $r_p$  is defined, then  $r_{pp}$  is positive semidefinite. The generalized law of supply: supply curves slope upward. By homogeneity,  $r_{pp}p = 0$ . Cross effects are not restricted otherwise except in special cases.

Similarly, by concavity,  $r_{vv}$  is negative semidefinite. This is the generalized law of inversed demand: demand curves slope downward. With constant returns to scale,  $r_{vv}v = 0$ . Cross effects are not restricted otherwise except in special cases.

An important set of cross effects concern those between  $v, p$ . Where  $r_p$  is defined we have  $r_{pv} = r'_{vp}$ , the reciprocity theorem of Samuelson derived in perhaps his greatest paper (1953). This means  $x_v = w'_p$  in general equilibrium, a remarkable and non-obvious result due to duality.