Optimal Dynamic Matching under Local Compatibility: An Application to Kidney Exchange

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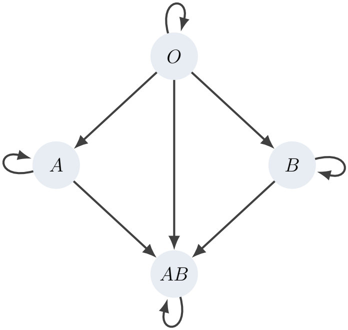
Kidney Exchange

- Kidney transplantation is the best treatment for end-stage renal disease, particularly when the kidney comes from a living donor.
- Patients often arrive with a willing donor who is incompatible.
    - Compatibility = ABO blood-type *and* tissue-type (cross-match).
- **Kidney exchange**: incompatible patient–donor pairs swap donors so that each patient gets a compatible kidney.
- Operating in the US, UK, Netherlands, Korea, Turkey, ...

Compatibility

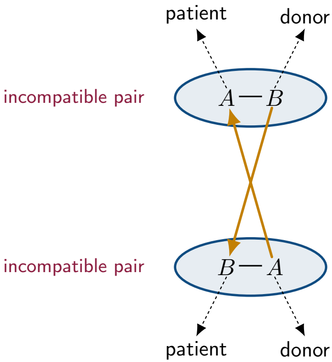
- Blood-type Compatibility:



$O \succ A, B \succ AB$ in donor-receiver compatibility.

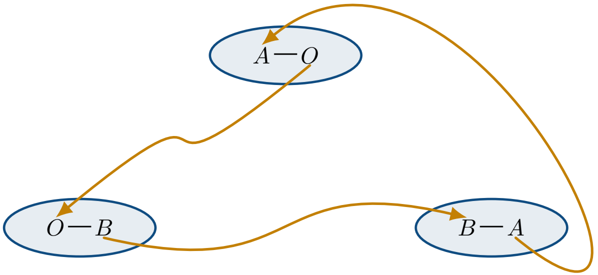
- Tissue-type Compatibility: Patient does not have pre-formed antibodies against donor's tissue-type DNA.

Two-way Exchange



Denoted $(A - B,\ B - A)$. Surplus: 2 transplants.

Three-way Exchange



Denoted $(A - O,\ O - B,\ B - A)$. Surplus: 3 transplants.

Dynamic Nature of Kidney Exchange

- Pairs arrive with a Poisson process; arrival rates λ_i, total rate λ.
- Patients incur a waiting cost; continuous discount rate ρ.
- Question. How should a centralized clearinghouse match dynamically to maximize total expected discounted surplus?

Core trade-off: match *now* for a smaller, sooner surplus vs. wait for a *larger but later* (discounted) surplus.

An Illustrative Trade-off

Suppose the pool contains a $B - A$ pair and an $A - O$ pair just arrived. A future $A - B$ arrives stochastically.

Option	Exchange now	Exchange later	Surplus
1	$(B - A,\ A - O,\ O - B)$	—	3 now
2	$(A - O,\ O - A)$	$(B - A,\ A - B)$	2 now + 2 later

Intuition. Hold the $B - A$ if and only if the marginal value of an additional $B - A$ in the pool exceeds 1. But that marginal value depends on the entire state.

Matching Environment

- $O-A$, $O-B$ are abundant (underdemanded pairs — matched on the spot at surplus 2, treated as use-it-or-lose-it).
- WLOG: no AB donor/patient (rare); no self-demanded pair $(A-A, B-B, O-O)$.

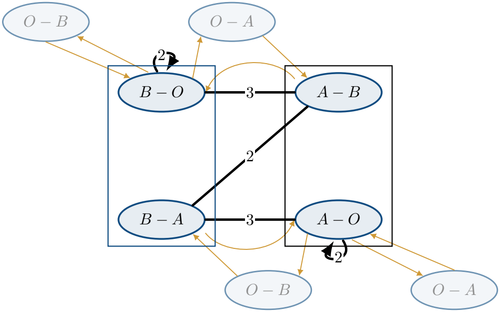
Four pair types control the state:

- $A-B$ and $B-A$: reciprocal pairs.
- $\mathbb{O}_{A-B} = \{B-O,\ AB-O,\ AB-A\}$: 3-way $(A-B,\ \mathbb{O}_{A-B},\ O-A)$, surplus 3.
- $\mathbb{O}_{B-A} = \{A-O,\ AB-O,\ AB-B\}$: symmetric.

State

$s = (s_1, s_2, s_3, s_4) \in \mathbb{N}^4 = (|\mathbb{O}_{A-B}|,\ |A-B|,\ |B-A|,\ |\mathbb{O}_{B-A}|)$.

Linear Compatibility Structure



A local linear compatibility graph with surplus weights $(2, 3, 2, 3, 2)$.

1d Was Easy. 4d Is Not.

1d case (Ünver, 2010): overdemanded pairs matched on arrival, so $s_1 = s_4 = 0$ and the state collapses to $s_2 - s_3 \in \mathbb{Z}$.

- A threshold mechanism is optimal. Driver: concavity in a one-dim state.

General 4d problem: all four state variables matter; state space is $\mathbb{N}^4$.

- Concavity alone is not propagated by the optimality operator.
- Even concavity plus sub-/supermodularity / strong superconcavity is not propagated.
- Prior queueing-theory methods (Koole, 1998) don't apply: their *queueing basis* $e_j - e_{j-1}$ has no economic meaning in matching.

Our Contributions

- Methodology: a tailored notion of $\mathcal{D}^M$-multimodularity (with a new "matching basis" $\mathcal{D}^M$) that
  - is propagated by the optimality operator,
  - is just strong enough to imply all required second-order properties.
- Structural result: characterize the second-order shape of V^*.
- Optimal mechanism: in the (numerically generic) unbalanced regime, the optimal policy is a multi-dimensional threshold mechanism with three monotone thresholds.
- Extension: methodology applies to any dynamic matching with a local linear compatibility graph.

Model

- Types $\mathcal{T} = \{1, \ldots, n\}$,
  local linear compatibility:
  type i can match $i \pm 1$ & endpoints with themselves.
- Poisson arrival rates λ_i in continuous time; let
  overall arrival rate $\lambda := \sum \lambda_i$,
  prob. for next pair's type i $p_i := \lambda_i / \lambda$,
  (discrete time $\equiv$) discount $\delta := \lambda/(\lambda + \rho) \in (0, 1)$.
- Surplus a_{ij}
  $\in \{2, 3\}$ for kidney exchange,
  $\in \mathbb{R}$ for other problems.
- State $s \in \mathbb{N}^n$.

Bellman Equation

$\mathcal{M}(s, s')$: max surplus along any exchange sequence $s \to s'$.

Bellman equation for optimal value function V^*:

$$V^*(s) \; = \; \delta \sum_i p_i \max_{s+e_i \to s'} \Big\{ \mathcal{M}(s + e_i, s') + V^*(s') \Big\}.$$

Matching Vectors

A single feasible exchange transitions s by $-w$ where

$$w \in \mathcal{D}^* := \{\, e_1,\ e_1 + e_2,\ e_2 + e_3,\ e_3 + e_4,\ e_4 \,\}.$$

- $w = e_1$: $\mathbb{O}_{A-B}$ matched with itself (surplus 2).
- $w = e_1 + e_2$: 3-way $(\mathbb{O}_{A-B}, A - B)$ (surplus 3).
- $w = e_2 + e_3$: 2-way $(A - B, B - A)$ (surplus 2).
- $w = e_3 + e_4$: 3-way $(B - A, \mathbb{O}_{B-A})$ (surplus 3).
- $w = e_4$: $\mathbb{O}_{B-A}$ matched with itself (surplus 2).

The Matching Operator and Other Event Operators

- Joint matching operator $T_M : \mathcal{V} \to \mathcal{V}$:

$$T_M f(s) \;=\; \max_{\substack{(\#_w)_{w\in\mathcal{D}^*}\in\{0,\dots,q\}^{|\mathcal{D}^*|}\\ s-\sum_w \#_w\, w\geq 0}} f\Big(s - \sum_{w\in\mathcal{D}^*} \#_w\, w\Big) + \sum_{w\in\mathcal{D}^*} \#_w\, a_w.$$

$\mathcal{V}$: set of functions from $\mathbb{N}^n$ to $\mathbb{R}$.

- Discount: $(T_\delta f)(s) := \delta f(s)$.
- Arrival of i: $(T_{\text{arr},i} f)(s) := f(s + e_i)$.
- Uniformization: $T_p(f_1, \dots, f_m)(s) := \sum_i p_i f_i(s)$.

Decomposition and Propagation

Observation (decomposition of dynamic programming operator T^*)

$$\begin{aligned} V^*(s) &= T^*V^*(s) \\ &= \Big(T_\delta \circ T_p\big(T_M \circ T_{\mathrm{arr},1}V^*, \ldots, T_M \circ T_{\mathrm{arr},n}V^*\big)\Big)(s). \end{aligned}$$

Theorem (Fundamental Theorem of Dynamic Programming)

Define for each state s, $V_k(s) := T^*V_{k-1}(s)$ from arbitrary $V_0 : \mathcal{V} \to \mathcal{V}$. Then for each s

$$V^*(s) = \lim_{k\to\infty} V_k(s),$$

such that V^* exists and is unique.

Our approach. We only need to propagate a property called "multimodularity" through T_M; the other operators are linear and propagate *any* $P(u, v)$-style property. So we can characterize properties of V^* at the limit and the optimal policy.

From Concavity to a Family of Properties

Example

For a 1-d operator $(Tf)(x) = \max\{f(x), f(x - e_i) + 1\}$:

$$\text{match} \iff f(x) - f(x - e_i) \leq 1.$$

- LHS is the marginal value of a type-i agent.
- Concavity in $e_i \implies$ LHS is decreasing in $s_i \implies$ threshold.

In higher dimensions we need more: substitution / complementarity between types, and "second-derivative" comparisons across coordinates.

A Unified Framework: Property $P(u, v)$

Definition

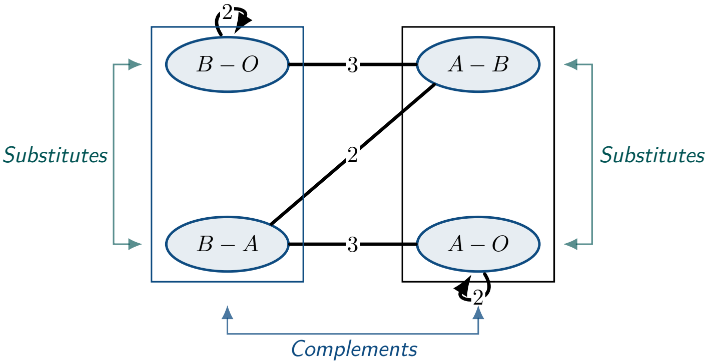
$f \in \mathcal{V}$ satisfies $P(u, v)$ if, for every x with $x + u,\ x + v,\ x + u + v \in \mathbb{N}^n$,

$$f(x + u) + f(x + v) \leq f(x) + f(x + u + v).$$

Named special cases (set $u = e_i$):

- $v = -e_i$: concavity in i.
- $v = -e_j$: ij-submodularity — i and j are substitutes.
- $v = e_j$: ij-supermodularity — i and j are complements.
- $v = \pm e_j - e_i$: strong ij-superconcavity — nothing is a closer substitute to i than i itself.

Substitutes and Complements in Kidney Exchange



Side B
('B' referring to B patients)

Side A
('A' referring to A patients)

Closedness

Closedness condition: $\mathcal{P} = \{P(u,v) : (u,v) \in U\}$ is closed under T_w if

$$P(u,v) \in \mathcal{P} \implies P(u,v+w) \in \mathcal{P}.$$

In 1d/2d: {concavity} alone is not closed under matching operators; closing it requires strong superconcavity (and, with $w = e_1 + e_2$, also 12-supermodularity).

Beyond 3 dimensions: closure requires properties that are not concavity, sub-, super-, or strong superconcavity.

$\mathcal{D}$-Multimodularity (Hajek, 1985)

Definition

- A set $\mathcal{D} \subseteq \mathbb{Z}^n$ is a multimodular basis if $\sum_{v \in \mathcal{D}} v = 0$ and $\mathrm{span}(\mathcal{D}) = \mathbb{Z}^n$.
- f is $\mathcal{D}$-multimodular if $P(u, v)$ holds for every $u, v \in \mathcal{D}$.

Lemma (disjoint-sum lemma)

for any two disjoint $U, V \subseteq \mathcal{D}$ and admissible x,

$$f\big(x + \textstyle\sum_{u \in U} u\big) + f\big(x + \textstyle\sum_{v \in V} v\big) \;\leq\; f(x) + f\big(x + \textstyle\sum_{u \in U \cup V} u\big).$$

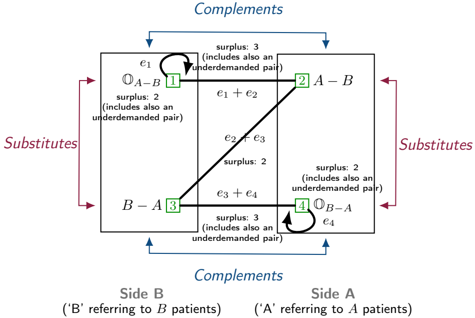
Sums of disjoint subsets of $\mathcal{D}$ are complements — exactly what closedness needs.

The Matching Basis $\mathcal{D}^M$

Definition (matching basis)

$$\mathcal{D}^M := \{\, e_1,\ -(e_1+e_2),\ e_2+e_3,\ -(e_3+e_4),\ e_4 \,\}.$$

- Same vectors as $\mathcal{D}^*$, with alternating signs so they sum to 0.
- Each element corresponds to a feasible matching transition.
- Concavity, sub-/supermodularity and strong superconcavity all follow as corollaries.



Main Propagation Result

Proposition (propagation under matching operator)

If $f \in \mathcal{V}$ is $\mathcal{D}^M$-multimodular, then so is $T_M f$.

Properties of the Value Function

Theorem

The optimal value function V^* is $\mathcal{D}^M$-multimodular.

Corollary

- *componentwise concave;*
- *strongly superconcave;*
- *ij-submodular for i, j of the same parity: $(\mathbb{O}_{A-B}, B-A)$ and $(A-B, \mathbb{O}_{B-A})$ are substitutes;*
- *ij-supermodular for i, j of different parity: $(\mathbb{O}_{A-B}, A-B)$, $(A-B, B-A)$, $(B-A, \mathbb{O}_{B-A})$, $(\mathbb{O}_{A-B}, \mathbb{O}_{B-A})$ are complements.*

Unbalanced Dynamic Exchange

Unbalanced: $A - B$ pairs arrive sufficiently more often than $B - A$ pairs $\implies$ at every reachable state,

- no $\mathbb{O}_{B-A}$ pair waits in the pool ($s_4 = 0$),
- the marginal value of an $A - B$ pair is below 1.

Numerically generic: holds whenever $p_{A-B} \neq p_{B-A}$ in the simulations of Ünver (2010).

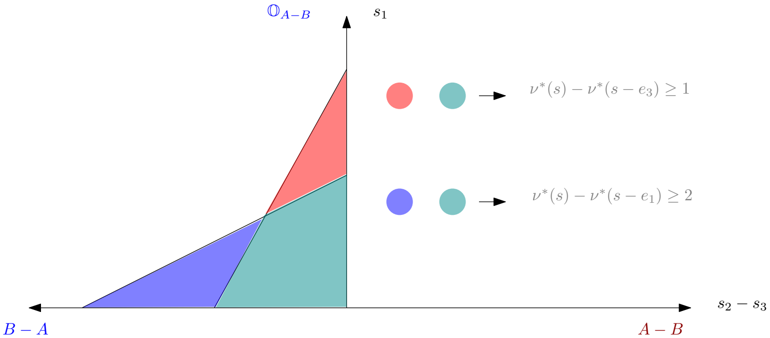
Reachable state space collapses to a 2-D picture in $(s_2 - s_3,\ s_1)$.

Four Decisions, Three Thresholds

When the pool has no $A - B$ pairs, the optimal mechanism faces four decisions, governed by three thresholds $t^{1,3}$, t^2, t^4:

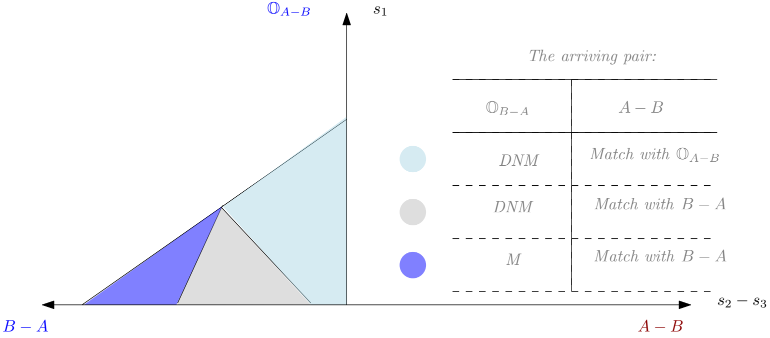
- $\mathbb{O}_{A-B}$ arrives. Keep if $s_1 + 1 \leq t^{1,3}(s_3)$, else match 2-way.
- $B - A$ arrives. Keep if $s_1 \leq t^{1,3}(s_3 + 1)$, else match an existing $\mathbb{O}_{A-B}$ (keeping the $B - A$).
- $\mathbb{O}_{B-A}$ arrives. 2-way if $s_3 \leq t^4(s_1)$, else 3-way with $B - A$.
- $A - B$ arrives with $s_1, s_3 > 0$. Match $\mathbb{O}_{A-B}$ if $s_3 \leq t^2(s_1)$, else match $B - A$.

Optimal Mechanism: Picture



- Triangular reachable region: existence follows from concavity.
- Boundary slope decreases: 1-3 submodularity.
- Slope in $(-1, 0)$: strong superconcavity.

All Three Thresholds in One Picture



The arriving pair:

$\mathbb{O}_{B-A}$	$A-B$
DNM	*Match with* $\mathbb{O}_{A-B}$
DNM	*Match with* $B-A$
M	*Match with* $B-A$

Main Theorem (Optimal Mechanism)

Theorem

For any unbalanced kidney exchange there exist threshold functions $t^{1,3}, t^2, t^4 : \mathbb{N} \to \mathbb{N}$ such that the induced multi-dimensional threshold mechanism is dynamically optimal. Moreover,

- $t^{1,3}$ *is non-increasing, with* $t^{1,3}(k+1) \geq t^{1,3}(k) - 1$;
- t^2 *is non-decreasing;*
- t^4 *is non-increasing, with* $t^4(k+1) \geq t^4(k) - 1$.

Each shape property maps to a second-order property of V^*, hence to $\mathcal{D}^M$-multimodularity.

Concluding Remarks

- Methodological contribution: $\mathcal{D}^M$-multimodularity — a closed family of second-order properties for dynamic matching with a local linear compatibility graph.
- Core technical insight: define T_M as a single joint maximization, not a composition of T_w's.
  $\Longrightarrow$ propagation of all required second-order properties *simultaneously*.
- Economic structure: in the (empirically generic) unbalanced regime, the optimal mechanism is a multi-dimensional threshold mechanism with three monotone thresholds.
- Generality: framework extends to any dynamic matching with a local linear compatibility graph.

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