On Characterizations of the Probabilistic Serial Mechanism[☆]

Onur Kesten

Carnegie Mellon University, Tepper Business School, Pittsburgh, PA, 15213 USA

Morimitsu Kurino

University of Tsukuba, Faculty of Engineering, Information and Systems, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8573, Japan

M. Utku Ünver

Boston College, Department of Economics, Chestnut Hill, MA, 02167, USA

Abstract

This paper studies the problem of assigning n indivisible objects to n agents when each agent consumes one object and monetary transfers are not allowed. Bogomolnaia and Moulin [4] prove that for $n = 3$, their proposed mechanism, the probabilistic serial, is characterized by the three axioms of ordinal efficiency, envy-freeness, and weak strategy-proofness. We show that this characterization does not extend to problems of arbitrary size of $n \geq 5$. Moreover, we show that weak strategy-proofness is logically independent of weak invariance which makes PS characterization possible together with ordinal efficiency and envy-freeness.

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Keywords: Probabilistic serial; ordinal efficiency; envy-freeness; weak strategy-proofness

1. Introduction

Many real-life problems such as school choice, organ transplantation, and on-campus housing involve the assignment of discrete indivisible objects without the use of monetary transfers. We consider the simplest discrete resource allocation problem in which n objects are assigned to n agents who have strict preferences over objects. A *mechanism* is a rule that specifies a stochastic assignment of objects to agents based on their reported preferences. The widely-used mechanism for this type of problems in practice is the *random serial dictatorship* (RSD) mechanism: randomly order the agents and let them sequentially choose their favorite objects. RSD is well-known for its

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Email addresses: okesten@andrew.cmu.edu (Onur Kesten), kurino@sk.tsukuba.ac.jp (Morimitsu Kurino), unver@bc.edu (M. Utku Ünver)

strategy-proofness and ex-post efficiency. In their seminal paper, Bogomolnaia and Moulin [4] (BM hereafter) showed that RSD is neither ordinally efficient nor envy-free, but is weakly envy-free.

BM introduced the *probabilistic serial* (PS) mechanism as a major competitor to RSD. The outcome of PS is computed via the *simultaneous eating algorithm* (SEA): Imagine that each object is a continuum of probability shares. Let agents simultaneously "eat away" from their favorite objects at the same speed; once the favorite object of an agent is gone, she turns to her next favorite object, and so on. We interpret the share of an object eaten away by an agent throughout the process as the probability PS assigns her that object.

PS is ordinally efficient and envy-free. This surprising observation in turn led to much attention being devoted to PS and its various extensions[1] and characterizations. Unlike RSD which is strategy-proof, PS is weakly strategy-proof. BM provided a first characterization of PS through ordinal efficiency, envy-freeness, and weak strategy-proofness with the added condition that there are three agents. Recently several papers provide various PS characterizations for more general settings with arbitrary number of agents and possibly for multiple copies of objects. A common theme in these characterizations is the use of ordinal efficiency and envy-freeness along with an invariance/monotonicity type property that requires the robustness of the assignment to certain perturbations of agents' preferences.[2] Nevertheless, a generalization of the original BM characterization to an arbitrary number of agents/objects has thus far been elusive. We specifically ask whether the BM characterization result holds for problems of arbitrary size and give a negative answer to this question. In particular, we construct a highly non-trivial mechanism for the case of five agents, different from PS, which satisfies ordinal efficiency, envy-freeness, and weak strategy-proofness. This counterexample provides justification to the recent characterization approaches that rely on stronger properties than weak strategy-proofness.

Section 2 describes the formal model and Section 3 provides the main result. Section 4 concludes.

2. Model

A discrete resource allocation problem [cf. 9, 13], or simply a **problem**, is a list $(N, A, \succ)$ where $N = \{1, \ldots, n\}$ is a finite set of agents; A is a finite set of objects with $|A| = |N| = n$; and $\succ = (\succ_i)_{i \in N}$ is a preference profile where $\succ_i$ is the strict preference relation of agent i on A. Let $\mathbf{P}$ be the set of preferences for any agent. Let $\succeq_i$ denote the weak preference relation induced by $\succ_i$. We assume that preferences are linear orders on A, i.e., for all $a, b \in A$ and all $i \in N$, $a \succeq_i b \Leftrightarrow a = b$ or $a \succ_i b$. We sometimes represent $\succ_i$ as an ordered list beginning with the

[1] See, for example, Kojima [11], Özgür Yılmaz [12], Athanassoglou and Sethuraman [1].

[2] Kesten et al. [10] proposed an upper invariance property, which was later refined by Bogomolnaia and Heo [3] and Hashimoto and Hirata [7]. The weakest property among these auxiliary properties, which characterizes PS along with ordinal efficiency and envy-freeness is the weak invariance property by Hashimoto et al. [8].

most preferred object of agent i and continuing to her least. For example, given $A = \{a, b, c\}$, we interpret $\succ_i = (b, c, a)$ as $b \succ_i c \succ_i a$. A centralized authority shall assign objects to agents such that each agent receives exactly one object.

A **random allocation** for agent i is a vector $P_i = (p_{i,a})_{a \in A}$ where $p_{i,a} \in [0, 1]$ denotes the probability that agent i receives object a, and $\sum_{a \in A} p_{i,a} = 1$. A **random assignment**, denoted as $P = [P_i]_{i \in N} = [p_{i,a}]_{i \in N, a \in A}$, is a bistochastic matrix, i.e., $\sum_{a \in A} p_{i,a} = 1$ for all $i \in N$ and $\sum_{i \in N} p_{i,a} = 1$ for all $a \in A$. Let $\mathcal{R}$ be the set of random assignments.

Observe that a random assignment gives only the marginal probability distribution according to which each agent will be assigned an object. It does not specify the distribution according to which objects should jointly be assigned to agents. To define this joint probability distribution, we first need to define (deterministic) assignments and probability distributions over them. An **assignment** is an element $P \in \mathcal{R}$ such that $p_{i,a} \in \{0, 1\}$ for all $i \in N$ and all $a \in A$. Let $\mathcal{A}$ be the set of assignments. A **lottery** $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$ is a probability distribution over assignments, i.e., $\lambda_\alpha \in [0, 1]$ for all $\alpha \in \mathcal{A}$ and $\sum_{\alpha \in \mathcal{A}} \lambda_\alpha = 1$.

Clearly, each lottery induces a random assignment. Let $P^\lambda \in \mathcal{R}$ be the random assignment induced by lottery λ, i.e., $p^\lambda_{i,a} = \sum_{\alpha \in \mathcal{A}: \alpha_{i,a}=1} \lambda_\alpha$ for all $i \in N$ and all $a \in A$. It turns out that the converse statement is also true: For each $P \in \mathcal{R}$ there exists a lottery λ that induces it, i.e., $P^\lambda = P$ [2, 14]. Thus, the centralized authority can simply restrict attention to random assignments rather than lotteries.[3]

Let $i \in N$, $a \in A$, $\succ_i \in \mathbf{P}$, and $P, R \in \mathcal{R}$ be given. Let $U(\succ_i, a) = \{b \in A \mid b \succeq_i a\}$ be the upper contour set of object a under at $\succ_i$. Let $F(\succ_i, a, P_i) = \sum_{b \in U(\succ_i, a)} p_{i,b}$ be the probability that i is assigned an object at least as good as a under P_i. Moreover, P_i **stochastically dominates** R_i at $\succ_i$ if $F(\succ_i, a, P_i) \geq F(\succ_i, a, R_i)$ for all $a \in A$. In addition, P **stochastically dominates** R at $\succ$ if P_i stochastically dominates R_i at $\succ_i$ for all $i \in N$.

We are now ready to introduce a powerful efficiency notion. A random assignment is **ordinally efficient** at $\succ$ if it is not stochastically dominated by another random assignment at $\succ$. BM characterizes ordinal efficiency by acyclicity as follows. A random assignment P is ordinal efficient at $\succ$ if and only if there is no cycle $(a_1, i_1, a_2, i_2, \ldots, a_m, i_m, a_{m+1})$ such that $a_1 = a_{m+1}$, and for each $\ell \in \{1, \ldots, m\}$, $a_\ell \succ_{i_\ell} a_{\ell+1}$ and $p_{i, a_{\ell+1}} > 0$.

Our fairness property is a fundamental principle in mechanism design theory originally proposed by Foley [6]. A random assignment is envy-free if each agent, regardless of her vNM utilities, weakly prefers her random allocation to that of any other agent. Formally, given $\succ \in \mathbf{P}^N$, $P \in \mathcal{R}$ is **envy-free** at $\succ$ if for all $i \in N$, P_i stochastically dominates P_j for all $j \in N$ at $\succ_i$.

A mechanism is a systematic way of finding a random assignment for a given problem. Formally,

[3]Once a random assignment is determined, finding a lottery that induces it is computationally easy. [UTKU: PLEASE ANSWER THE REVIEW 1'S COMMENT HERE: Be more specific about what "computationally easy" means, and give some references.]

a **mechanism** is an allocation rule $\varphi : \mathbf{P}^N \to \mathcal{R}$. A mechanism is said to satisfy a property if its outcome, for any problem, satisfies that property. A mechanism φ is **weakly strategy-proof** if no agent ever stochastically gains by misreporting her preferences, i.e., for all $\succ$ and all $i \in N$, there is no $\succ_i'$ such that $\varphi_i(\succ_i', \succ_{-i})$ stochastically dominates $\varphi_i(\succ_i, \succ_{-i})$ at $\succ_i$ and $\varphi_i(\succ_i', \succ_{-i}) \neq \varphi_i(\succ_i, \succ_{-i})$. We next define weak invariance introduced by Hashimoto et al. [8]. Let $\succ_i |_B$ be the restriction of $\succ_i \in \mathbf{P}$ to $B \subseteq A$, i.e., $\succ_i |_B$ is a preference relation over B such that for all $a, b \in B$, $a \succ_i |_B\ b \Leftrightarrow a \succ_i b$. Then a mechanism φ is called **weakly invariant** if for each $\succ \in \mathbf{P}^N$, each $i \in N$, each $a \in A$, and each $\succ_i' \in \mathbf{P}$, when $U(\succ_i', a) = U(\succ_i, a)$ and $\succ_i' |_{U(\succ_i', a)} = \succ_i |_{U(\succ_i', a)}$, we have $\varphi_{i,a}(\succ) = \varphi_{i,a}(\succ_i', \succ_{-i})$.

BM introduced the **probabilistic serial mechanism** (**PS**),[4] the outcome of which can be computed via the following simultaneous eating algorithm (SEA):

Given a problem $\succ$, think of each object a as an infinitely divisible good with supply of 1.

Step 1: Each agent "eats away" from her favorite object at the same unit speed. Proceed to the next step when an object is completely exhausted.

⋮

Step s, for $s \in \{2, \ldots, S\}$: Each agent eats away from her remaining favorite object at the same speed. Proceed to the next step when an object is completely exhausted.

The procedure terminates after $S \leqslant |N|$ steps when each agent has eaten exactly 1 total unit of objects (i.e., at time 1). The random allocation of an agent i by PS is then given by the amount of each object she has eaten until the algorithm terminates. Let $PS(\succ) \in \mathcal{R}$ denote the outcome of PS for problem $\succ$.

It is known that PS is ordinally efficient, envy-free, and weakly strategy-proof (BM), and moreover is weakly invariant [8].

3. The Main Results

BM provide a complete characterization of PS by ordinal efficiency, envy-freeness, and weak strategy-proofness when there are three agents and three objects. Our main result shows that this characterization no longer holds with five or more agents:

Proposition 1. *When there are five or more agents, there exists a mechanism, different from PS, satisfying ordinal efficiency, envy-freeness, and weak strategy-proofness.*

We prove this proposition through a counterexample, i.e., by providing a mechanism, different from PS, that satisfies all three properties. First, in Lemma 1, we look at the case of five agents and construct the mechanism that differs from PS only at one preference profile out of 120^5 profiles. Then we extend it to the case of more than five agents to prove Proposition 1.

[4] PS was initially proposed by Crès and Moulin [5] for a model where agents have the same rankings over objects.

Lemma 1. *When there are five agents, there exists a mechanism, different from PS, satisfying ordinal efficiency, envy-freeness, and weak strategy-proofness.*

Proof of Lemma 1. Suppose that there are five agents $N = \{1, 2, ..., 5\}$ and five objects $A = \{a, b, c, d, e\}$. Let $\succ^*$ be defined as follows:

$$
\begin{aligned}
&a \succ_i^* c \succ_i^* d \succ_i^* e \succ_i^* b \quad \text{for } i \in \{1, 2, 3\} \\
&b \succ_4^* c \succ_4^* d \succ_4^* e \succ_4^* a \\
&b \succ_5^* a \succ_5^* c \succ_5^* e \succ_5^* d
\end{aligned}
$$

The PS outcome for this problem is

$$
PS(\succ^*) = \frac{1}{720} \begin{pmatrix} 240 & 0 & 192 & 180 & 108 \\ 240 & 0 & 192 & 180 & 108 \\ 240 & 0 & 192 & 180 & 108 \\ 0 & 360 & 72 & 180 & 108 \\ 0 & 360 & 72 & 0 & 288 \end{pmatrix}.
$$

Define

$$
P^* = \frac{1}{720} \begin{pmatrix} 220 & 0 & 210 & 185 & 105 \\ 220 & 0 & 210 & 185 & 105 \\ 220 & 0 & 210 & 185 & 105 \\ 0 & 360 & 75 & 165 & 120 \\ 60 & 360 & 15 & 0 & 285 \end{pmatrix}.
$$

Construct the mechanism φ as follows:

$$
\varphi(\succ) = \begin{cases} P^* & \text{if } \succ = \succ^*, \\ PS(\succ) & \text{otherwise.} \end{cases}
$$

We show that mechanism φ is ordinally efficient, envy-free, and weakly strategy-proof.

We first show ordinal efficiency of φ. It suffices to prove that P^* is ordinally efficient at $\succ^*$, because φ coincides with PS in any other preference profile than $\succ^*$, and PS is ordinally efficient (BM). Suppose to the contrary that P^* is not ordinally efficient at $\succ^*$. By the BM's characterization of ordinal efficiency, there is a cycle $(a_1, i_1, a_2, i_2, \ldots, a_m, i_m, a_{m+1})$ such that $a_1 = a_{m+1}$, and for each $\ell \in \{1, \ldots, m\}$, $a_\ell \succ_{i_\ell} a_{\ell+1}$ and $p^*_{i, a_{\ell+1}} > 0$.

Claim 1. $5 \in \{i_1, \ldots, i_m\}$.

Proof. It suffices to show that $\{i_1, \ldots, i_m\} \not\subseteq \{1, 2, 3, 4\}$. Suppose not. Note that for each $\ell \in \{1, \ldots, m\}$, $a_\ell \succ_{i_\ell} a_{\ell+1}$ and $p^*_{i, a_{\ell+1}} > 0$. Then, by the definition of P^*, for each $\ell \in \{1, \ldots, m\}$,

$a_\ell \succ_{i_\ell} a_{\ell+1}$ and $PS_{i,a_{\ell+1}}(\succ^*) > 0$. This means that cycle $(a_1, i_1, \ldots, a_m, i_m, a_{m+1})$ is also a cycle for $PS(\succ^*)$, i.e., $PS(\succ^*)$ is not ordinally efficient at $\succ^*$, a contradiction. □

Thus, let $i_1 = 5$ without loss of generality.

Claim 2. $h_1 = a$ or $h_2 = a$.

Proof. Suppose not. Then $\{h_1, h_2\} \subseteq \{b, c, d, e\}$. Similarly to the proof of Claim 1, the cycle $(a_1, i_1, \ldots, a_m, i_m, a_{m+1})$ is also a cycle for $PS(\succ^*)$, i.e., $PS(\succ^*)$ is not ordinally efficient at $\succ^*$, a contradiction. □

By Claim 2 we have the following two cases to consider.

Case 1: $h_1 = a$. Then $a \succ_5 h_2$, $p^*_{5,h_2} > 0$, $h_m \succ_{i_m} a$, and $p^*_{i_m,a} > 0$. Since $a \succ_5 h_2$, $h_2 \in \{c, e, d\}$. Then, since $p^*_{5,h_2} > 0$, $h_2 \in \{c, e\}$. On the other hand, since $h_m \succ_{i_m} a$, we have $i_m = 4$. However, $p^*_{i_m,a} = p^*_{4,a} = 0$ – a contradiction.

Case 2: $h_2 = a$. Then $h_1 = b$ and $b \succ_5 a$. Also $h_m \succ_{i_m} b$ and $p^*_{i_{m+1},b} > 0$. Thus, by the definition of P^*, $i_m = 4$. But we have $h_m \succ_{i_m} b$, which contradicts that b is the top choice of $i_m = 4$.

Hence P^* is ordinally efficient at $\succ^*$.

Second, we show that φ is envy-free. Since PS is envy-free (BM), all that remains is to show that P^* is envy-free at $\succ^*$. Let $sd(\succ_i)$ be the stochastic dominance relation induced by preference $\succ_i$. Hence, we need to show, for example for agent 1 that $P^*_1 sd(\succ^*_1) P^*_4$ and $P^*_1 sd(\succ^*_1) P^*_5$. Similarly, analogous conditions must hold for agents 4 and 5. Consider the following table:

$\succ^*_1$	a	c	d	e	b
P^*_1	220	430	615	720	720
P^*_4	0	75	240	360	720
P^*_5	60	75	75	360	720

$(\times \frac{1}{720})$

The first row indicates the objects following the preference ordering $\succ^*_1$ of agent 1. The second row shows $F(\succ^*_1, x, P^*_1)$ corresponding to each object x given in the first row. Similarly, the third and fourth rows, respectively, show $F(\succ^*_1, x, P^*_4)$ and $F(\succ^*_1, x, P^*_5)$ corresponding to each object x given in the first row. The following tables are similarly obtained for agents 4 and 5:

$\succ^*_4$	b	c	d	e	a
P^*_4	360	435	600	720	720
P^*_1	0	210	395	500	720
P^*_5	360	375	375	660	720

$\succ^*_5$	b	a	c	e	d
P^*_5	360	420	435	720	720
P^*_1	0	220	430	535	720
P^*_4	360	360	435	555	720

$(\times \frac{1}{720})$

Based on the above tables, we conclude that P^* is envy-free at $\succ^*$.

Finally, we next show that φ is weakly strategy-proof. For notational simplicity, given $\succ_i \neq \succ^*_i$ let $P = \varphi(\succ_i, \succ^*_{-i})$. Recall that $P^* = \varphi(\succ^*)$. We need to show for all $i \in N$, $\succ_i$, if $P_i \neq P^*_i$, then $P_i \, sd(\succ^*_i) \, P^*_i$ is not possible, and $P^*_i \, sd(\succ_i) \, P_i$ is not possible. By symmetry, it suffices to verify the above two conditions for $i \in \{1, 4, 5\}$. To illustrate, for example, consider the case where $i = 1$ and her preference is $\succ_1 = (e, b, a, c, d) \neq \succ_1^*$. We denote $P = \varphi(\succ_1, \succ_{-1}^*) \equiv PS(\succ_1, \succ_{-1}^*)$, and recall $P^* \equiv \varphi(\succ_1^*, \succ_{-1}^*)$. To verify the first condition for this case, we use the following table:

$\succ_1^*$	a	c	d	e	b
P_1^*	$F(\succ_1^*, a, P_1^*)$	$F(\succ_1^*, c, P_1^*)$	$F(\succ_1^*, d, P_1^*)$	$F(\succ_1^*, e, P_1^*)$	$F(\succ_1^*, b, P_1^*)$
P_1	$F(\succ_1^*, a, P_1)$	$F(\succ_1^*, c, P_1)$	$F(\succ_1^*, d, P_1)$	$F(\succ_1^*, e, P_1)$	$F(\succ_1^*, b, P_1)$

Here the first row indicates the objects following the preference ordering $\succ_1^*$. To verify the first condition, it suffices to find an object under which the entry in the second row is strictly greater than that in the third row. In the following tables, for brevity, we stop at the first object for which this requirement is met. Similarly, to verify the second condition, we use the following table:

$\succ_1$	e	b	a	c	d
P_1	$F(\succ_1, e, P_1)$	$F(\succ_1, b, P_1)$	$F(\succ_1, a, P_1)$	$F(\succ_1, c, P_1)$	$F(\succ_1, d, P_1)$
P_1^*	$F(\succ_1, e, P_1^*)$	$F(\succ_1, b, P_1^*)$	$F(\succ_1, a, P_1^*)$	$F(\succ_1, c, P_1^*)$	$F(\succ_1, d, P_1^*)$

Based on the table, to verify the second condition, it suffices to find an object under which the entry in the second row is strictly greater than that in the third row. Similar to above, for brevity, we stop at the first object for which this requirement is met. Next we consider each case in turn. All tables below have a factor of 1/720 for simplification.

Consider agent 1. Take any preference $\succ_1$ ($\neq \succ_1^*$).

Case 1-1: $\succ_1 = (a, c, d, b, e)$ or (a, c, b, d, e). Then $P = PS(\succ^*)$ (Recall that $P = PS(\succ_1, \succ_{-1}^*)$) and thus $P_1 = \frac{1}{720}(240, 0, 192, 180, 108)$. Hence

$\succ_1^*$	a	c	d	$\succ_1$	a
P_1^*	220	430	615	P_1	240
P_1	240	432	612	P_1^*	220

Case 1-2: $\succ_1 = (a, c, b, e, d)$ or $(a, c, e, \cdots)$. Then $P_1 = \frac{1}{720}(240, 0, 192, 0, 288)$. Thus

$\succ_1^*$	a	c	d	$\succ_1$	a
P_1^*	220	430	615	P_1	240
P_1	240	432	432	P_1^*	220

Case 1-3: $\succ_1 = (a, b, c, \cdots)$. Then $P_1 = \frac{1}{720}(240, 80, 112, \cdots)$. Hence

$\succ_1^*$	a	c	$\succ_1$	a
P_1^*	220	430	P_1	240
P_1	240	352	P_1^*	220

Case 1-4: $\succ_1 = (a, b, d, \cdots)$ or $(a, b, e, \cdots)$. Then $P_1 = \frac{1}{720}(240, 80, 0, \cdots)$. Hence

$\succ_1^*$	a	c	$\succ_1$	a
P_1^*	220	430	P_1	240
P_1	240	240	P_1^*	220

Case 1-5: $\succ_1 = (a, d, \cdots)$ or $(a, e, \cdots)$. Then $P_1 = \frac{1}{720}(240, 0, 0, \cdots)$. Hence

$\succ_1^*$	a	c	$\succ_1$	a
P_1^*	220	430	P_1	240
P_1	240	240	P_1^*	220

Case 1-6: $\succ_1 = (b, \cdots)$. Obviously $p_{1,b} = 1/3 = 240/720$. Hence,

$\succ_1$	b
P_1	240
P_1^*	0

Note that $p_{1,a}$ is the largest if $\succ_1 = (b, a, \cdots)$. Suppose $\succ_1 = (b, a, \cdots)$. Then $P_1 = \frac{1}{720}(60, 240, \cdots)$. Hence, the other table is

$\succ_1^*$	a
P_1^*	220
P_1	60

Thus, for any preference $\succ_1$, we have the desired result.

Case 1-7: $\succ_1 = (c, \cdots)$. Then $P_1 = \frac{1}{720}(0, 0, 432, \cdots)$. Hence

$\succ_1^*$	a	$\succ_1$	c
P_1^*	220	P_1	432
P_1	0	P_1^*	210

Case 1-8: $\succ_1 = (d, \cdots)$. Then $P_1 = \frac{1}{720}(0, 0, 0, 585, 135)$. Hence

$\succ_1^*$	a	$\succ_1$	d
P_1^*	220	P_1	585
P_1	0	P_1^*	185

Case 1-9: $\succ_1 = (e, \cdots)$. Then $P_1 = \frac{1}{720}(0, 0, 0, 90, 630)$.

$\succ_1^*$	a	$\succ_1$	e
P_1^*	220	P_1	630
P_1	0	P_1^*	105

Next, consider agent 4. Take any preference $\succ_4$ $(\neq \succ_4^*)$. We denote $P = \varphi(\succ_4, \succ_{-4}^*) \equiv PS(\succ_4, \succ_{-4}^*)$, and recall $P^* \equiv \varphi(\succ_4^*, \succ_{-4}^*)$.

Case 4-1: $\succ_4 = (b, c, d, a, e)$, (b, c, a, d, e), or (b, a, c, d, e). Then $P = PS(\succ^*)$. Thus

$\succ_4^*$	b	c
P_4^*	360	435
P_4	360	432

$\succ_4$	b	(a)	c	(a)	d
P_4	360	(360)	432	(432)	612
P_4^*	360	(360)	435	(435)	600

Case 4-2: $\succ_4 = (b, c, a, e, d)$, $(b, c, e, \cdots)$, or (b, a, c, e, d). Then $P_4 = \frac{1}{720}(0, 360, 72, 0, 288)$. Hence

$\succ_4^*$	b	c
P_4^*	360	435
P_4	360	432

$\succ_4$	b	(a)	c	(a)	e
P_4	360	(360)	432	(432)	720
P_4^*	360	(360)	435	(435)	555

Case 4-3: $\succ_4 = (b, a, d, \cdots)$ or $(b, d, \cdots)$. Then $P_4 = \frac{1}{720}(0, 360, 0, 247.5, 112.5)$. Hence

$\succ_4^*$	b	c	$\succ_4$	b	(a)	d
P_4^*	360	435	P_4	360	(360)	607.5
P_4	360	360	P_4^*	360	(360)	525

Case 4-4: $\succ_4 = (b, a, e, \cdots)$ or $(b, e, \cdots)$. Then $P_4 = \frac{1}{720}(0, 360, 0, 0, 360)$. Hence

$\succ_4^*$	b	c	$\succ_4$	b	(a)	e
P_4^*	360	435	P_4	360	(360)	720
P_4	360	360	P_4^*	360	(360)	480

Case 4-5: $\succ_4 = (a, \cdots)$. Obviously $p_{4,a} = 1/4 = 180/720$. Thus

$\succ_4$	a
P_4	180
P_4^*	0

Note that $p_{4,b}$ is the largest if $\succ_4 = (a, b, \cdots)$. Suppose $\succ_4 = (a, b, \cdots)$. Then $P_4 = \frac{1}{720}(180, 270, 0, \cdots)$. And the other table is

$\succ_4^*$	b
P_4^*	360
P_4	270

Thus, for any preference, we have the desired result.

Case 4-6: $\succ_4 = (c, \cdots)$. We calculate $p_{4,c} = 1/2 = 360/720$. Thus

$\succ_4$	c
P_4	360
P_4^*	75

Note that $p_{4,b}$ is the largest if $\succ_4 = (c, b, \cdots)$. Suppose $\succ_4 = (c, b, \cdots)$. Then $P_4 = \frac{1}{720}(0, 180, 360, \cdots)$. And the other table is

$\succ_4^*$	b
P_4^*	360
P_4	180

Thus, for any preference, we have the desired result.

Case 4-7: $\succ_4 = (d, \cdots)$. Obviously $p_{4,d} \geq 1/3 = 240/720$. Then

$\succ_4$	d
P_4	at least 240
P_4^*	165

Note that $p_{4,b}$ is the largest if $\succ_4 = (d, b, \cdots)$. Suppose $\succ_4 = (d, b, \cdots)$. Then $P_4 = \frac{1}{720}(0, 90, \cdots)$. And the other table is

$\succ_4^*$	b
P_4^*	360
P_4	90

Thus, for any preference, we have the desired result.

Case 4-8: $\succ_4 = (e, \cdots)$. Then $P_4 = \frac{1}{720}(0, 0, 0, 0, 720)$.

$\succ_4^*$	b	$\succ_4$	e
P_4^*	360	P_4	720
P_4	0	P_4^*	120

Finally, we consider agent 5. Take any preference $\succ_5$ $(\neq \succ_5^*)$. We denote $P = \phi(\succ_5, \succ_{-5}^*) \equiv PS(\succ_5, \succ_{-5}^*)$, and recall $P^* \equiv \varphi(\succ_5^*, \succ_{-5}^*)$.

Case 5-1: $\succ_5 = (b, a, c, d, e)$. Then $P_5 = \frac{1}{720}(0, 360, 72, 144, 144)$. Hence

$\succ_5^*$	b	a	$\succ_5$	b	a	c	d
P_5^*	360	420	P_5	360	360	432	576
P_5	360	360	P_5^*	360	420	435	435

Case 5-2: $\succ_5 = (b, a, d, \cdots)$. Then $P_5 = \frac{1}{720}(0, 360, 0, 216, 144)$. Hence

$\succ_5^*$	b	a	$\succ_5$	b	a	d
P_5^*	360	420	P_5	360	360	576
P_5	360	360	P_5^*	360	420	420

Case 5-3: $\succ_5 = (b, a, e, \cdots)$. Then $P_5 = \frac{1}{720}(0, 360, 0, 0, 360)$. Hence

$\succ_5^*$	b	a	$\succ_5$	b	a	e
P_5^*	360	420	P_5	360	360	720
P_5	360	360	P_5^*	360	420	705

Case 5-4: $\succ_5 = (b, c, \cdots)$. Then $P_5 = \frac{1}{720}(0, 360, 72, \cdots)$. Hence

$\succ_5^*$	b	a	$\succ_5$	b	c
P_5^*	360	420	P_5	360	432
P_5	360	360	P_5^*	360	375

Case 5-5: $\succ_5 = (b, d, \cdots)$. P coincides with the one in Case 5-2, i.e., $P_5 = \frac{1}{720}(0, 360, 0, 216, 144)$. Hence

$\succ_5^*$	b	a	$\succ_5$	b	d
P_5^*	360	420	P_5	360	576
P_5	360	360	P_5^*	360	360

Case 5-6: $\succ_5 = (b, e, \cdots)$. P coincides with the one in Case 5-3, i.e., $P_5 = \frac{1}{720}(0, 360, 0, 0, 360)$. Hence

$\succ_5^*$	b	a	$\succ_5$	b	e
P_5^*	360	420	P_5	360	720
P_5	360	360	P_5^*	360	645

Case 5-7: $\succ_5 = (a, \cdots)$. Obviously $p_{5,a} = 1/4 = 180/720$. Thus

$\succ_5$	a
P_5	180
P_5^*	60

Note that $p_{5,b}$ is the largest if $\succ_5 = (a, b, \cdots)$. Suppose $\succ_5 = (a, b, \cdots)$. Then $P_5 = \frac{1}{720}(180, 270, 0, \cdots)$. Thus

$\succ_5^*$	b
P_5^*	360
P_5	270

Thus, for any preference, we have the desired result.

Case 5-8: $\succ_5 = (c, \cdots)$. Obviously $p_{5,c} \geq 1/3 = 240/720$. Thus,

$\succ_5$	c
P_5	at least 240
P_5^*	15

Note that $p_{5,b}$ is the largest if $\succ_5 = (c, b, \cdots)$. Suppose $\succ_5 = (c, b, \cdots)$. Then $P_5 = \frac{1}{720}(0, 180, 360, \cdots)$.

$\succ_5^*$	b
P_5^*	360
P_5	180

Thus, for any preference, we have the desired result.

Case 5-9: $\succ_5 = (d, \cdots)$. Obviously $p_{5,d} \geq 1/3 = 240/720$. Thus

$\succ_5$	d
P_5	at least 240
P_5^*	0

Note that $p_{5,b}$ is the largest if $\succ_5 = (d, b, \cdots)$. Suppose $\succ_5 = (d, b, \cdots)$. Then $P_5 = \frac{1}{720}(0, 90, 0, 540, 90)$.

$\succ_5^*$	b
P_5^*	360
P_5	90

Thus, for any preference, we have the desired result.

Case 5-10: $\succ_5 = (e, \cdots)$. Then $P_5 = \frac{1}{720}(0, 0, 0, 0, 720)$. Hence

$\succ_5^*$	b	$\succ_5$	e
P_5^*	360	P_5	720
P_5	0	P_5^*	285

□

Proof of Proposition 1. Suppose that there are more than five agents, $N = \{1, \ldots, n\}$, and more than five objects, $A = \{a, b, c, d, e, h_6, \ldots, h_n\}$ where $n > 6$. The proof strategy is to embed the mechanism, φ, constructed in Lemma 1 into a mechanism whose restriction to agents $\{1, \ldots, 5\}$ and objects $\{a, \ldots, e\}$ coincides with φ on some set of preference profiles, and does with PS in the complement. Thus we use the results in the proof of Lemma 1. In doing so, we overline the corresponding notations so that no confusion arises. For example, a preference of agent i is denoted $\overline{\succ}_i$ whose corresponding preference is denoted $\succ_i$ so that $\succ_i = \overline{\succ}_i|_{\{a,\ldots,e\}}$.

We first define the set of preferences for each agent:

- $\overline{\mathbf{P}}_i^* = \{\overline{\succ}_i^* \in \mathbf{P} \mid \overline{\succ}_i^* = (a, c, d, e, b, \ldots)\}$ for $i \in \{1, 2, 3\}$,
- $\overline{\mathbf{P}}_4^* = \{\overline{\succ}_4^* \in \mathbf{P} \mid \overline{\succ}_4^* = (b, c, d, e, a, \ldots)\}$,
- $\overline{\mathbf{P}}_5^* = \{\overline{\succ}_5^* \in \mathbf{P} \mid \overline{\succ}_5^* = (b, a, c, e, d, \ldots)\}$,
- $\overline{\mathbf{P}}_i^* = \{\overline{\succ}_i^* \in \mathbf{P} \mid \overline{\succ}_i^* = (h_i, \ldots)\}$ for $i \in \{6, \ldots, n\}$.

Note that for each agent $i \in \{1, \ldots, 5\}$, her preference over the top five objects in $\overline{\succ}_i^* \in \overline{\mathbf{P}}_i^*$ is the same as the one in $\succ_i^*$ considered in the proof of Lemma 1. We next define the random assignment, $\overline{P}^*$, as

$$\overline{P}^* = \begin{pmatrix} & & 0 & \ldots & 0 \\ & P^* & \vdots & & \vdots \\ & & 0 & \ldots & 0 \\ 0 & \ldots \ 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & \ldots \ 0 & & & 1 \end{pmatrix} = \begin{pmatrix} P^* & PS(\overline{\succ}^*)|_{\{1,\ldots,5\}\times\{h_6,\ldots,h_n\}} \\ PS(\overline{\succ}^*)|_{\{6,\ldots,n\}\times\{a,\ldots,e\}} & PS(\overline{\succ}^*)|_{\{6,\ldots,n\}\times\{h_6,\ldots,h_n\}} \end{pmatrix}$$

where $\overline{\succ}^* \in \prod_{i\in N} \overline{\mathbf{P}}_i^*$. That is,

$$p_{i,h}^{**} = \begin{cases} p_{i,h}^* & \text{if } i \in \{1, \ldots, 5\} \text{ and } h \in \{a, \ldots, e\}, \\ PS_{i,h}(\overline{\succ}^*) & \text{otherwise.} \end{cases}$$

Construct the mechanism $\overline{\varphi}$ as follows. For each $\succ \in \mathbf{P}^N$,

$$\overline{\varphi}(\succ) = \begin{cases} P^{**} & \text{if } \succ \in \prod_{i\in N} \overline{\mathbf{P}}_i^*, \\ PS(\succ) & \text{otherwise.} \end{cases}$$

By construction of $\overline{P}^*$ from P^* in the proof of Lemma 1, it is straightforward to check ordinal efficiency and envy-freeness of $\overline{P}^*$ at $\succ \in \prod_{i\in N} \overline{\mathbf{P}}_i^*$. Thus, since PS is ordinally efficient and envy-free, so is mechanism $\overline{\varphi}$.

It remains to show weak strategy-proofness of $\overline{\varphi}$. Note that for each agent $i \in \{6, \ldots, n\}$ and each preference profile $\succ \in \mathbf{P}^N$, we have $\overline{\varphi}_i(\succ) = PS_i(\succ)$. Thus, since PS is weakly strategy-proof (BM), no agent $i \in \{6, \ldots, n\}$ has incentives to manipulate mechanism $\overline{\varphi}$ in terms of weak strategy-proofness. Thus, we need to show it, by symmetry, for each agent $i \in \{1, 4, 5\}$. Let $i \in \{1, 4, 5\}$, $\succ \in \mathbf{P}^N$, and $\overline{\succ}^* \in \prod_{i\in N} \overline{P}_i^*$. Denote $\overline{P} = \overline{\varphi}(\succ_i, \overline{\succ}_i^*)$. We show that when $\overline{P}_i \neq \overline{P}_i^*$,

$$\overline{P}_i \ sd(\overline{\succ}_i^*) \ \overline{P}_i^* \text{ is impossible} \tag{1}$$

and

$$\overline{P}_i^* \ sd(\overline{\succ}_i) \ \overline{P}_i \text{ is impossible.} \tag{2}$$

<u>Case 1:</u> $\overline{\succ}_i \in \overline{P}_i^*$. Then $\overline{P}_i = \overline{P}_i^*$ and thus there is nothing to check.
<u>Case 2:</u> for each $h \in \{h_6, \ldots, h_n\}$, $\overline{\succ}_i$ satisfies $\overline{p}_{i,h} = 0$. Then $PS(\overline{\succ}_i, \overline{\succ}_{-i}^*) = PS(\succ_i, \succ_{-i}^*)$. The

proof of Lemma 1 can be applied to this case in proving statements (1) and (2).

Case 3: for some $h \in \{h_6, \ldots, h_n\}$, $\overline{\succ_i}$ satisfies $\overline{p}_{i,h} > 0$. We first show statement (1). Let f be the least preferred object among those in $\{a, \ldots, e\}$ under $\overline{\succ}_i$. By the construction of h, f, and $\overline{\succ}^*_i$, we have $f \mathrel{\overline{\succ}^*_i} h$. We know that $F(\overline{\succ}^*_i, f, \overline{P}^*_i) = \sum_{k\in\{a,\ldots,e\}} \overline{p}^*_{i,k} = 1$. Then, since $\sum_{k\in A} \overline{p}_{i,k} = 1$, $\overline{p}_{i,h} > 0$, and $f \mathrel{\overline{\succ}^*_i} h$, we have $F(\overline{\succ}^*_i, f, \overline{P}_i) = \sum_{k\in\{a,\ldots,e\}} \overline{p}_{i,k} < 1$. Thus, since $F(\overline{\succ}^*_i, f, \overline{P}^*_i) > F(\overline{\succ}^*_i, f, \overline{P}_i)$, statement (1) holds.

We next show statement (2). We introduce notations. Let h be the most preferred object in $\{h_6, \ldots, h_n\}$ under $\succ_i$, and $\succ_i$ be represented by

$$\succ_i = (k_0, k_1, \ldots, k_M, h, \ldots) = \begin{cases} (h, \ldots) & \text{if } M = 1, \\ (k_1, \ldots, k_M, h, \ldots) & \text{if } M \geq 2. \end{cases}$$

We use the following two claims for the proof.

Claim 3. $1 \leq M \leq 4$, and thus $k_1, \ldots, k_M \in \{a, \ldots, e\}$.

Proof. Note that by construction $M \leq 5$. Suppose for a contradiction that $M = 5$. Then $k_1, \ldots, k_5$ are $a, \ldots, e$. In this case, $PS_i(\overline{\succ}_i, \overline{\succ}^*_{-i}) = PS_i(\succ_i, \succ^*_{-i})$, i.e., $\overline{P}_i = P_i$. Thus, since $\sum_{\ell\in\{a,\ldots,e\}} p_{i,\ell} = 1$, we have $\sum_{m=1}^{5} \overline{p}_{i,k_m} = 1$. Thus, since $\overline{p}_{i,h} > 0$ by assumption, $\sum_{\ell\in U(\overline{\succ}_i,h)} \overline{p}_{i,\ell} > 0$. This is a contradiction. □

Claim 4. If there is $m \in \{1, \ldots, M\}$ such that $F(\succ_i, k_m, P_i) \geq F(\succ_i, k_m, P^*_i)$, then statement (2) holds.

Proof. For each $\ell \in \{1, \ldots, M\}$, by Claim 3 and the simultaneous eating algorithm of PS, we have $PS_{i,k_\ell}(\overline{\succ}_i, \overline{\succ}^*_{-i}) = PS_{i,k_\ell}(\succ_i, \succ^*_{-i})$, i.e., $\overline{p}_{i,k_\ell} = p_{i,k_\ell}$, and moreover, by definition, have $\overline{p}^*_{i,k_\ell} = p^*_{i,k_\ell}$. Thus, since $\sum_{\ell=1}^{m} p_{i,k_\ell} \geq \sum_{\ell=1}^{m} p^*_{i,k_\ell}$, by the hypothesis of the claim, we have $\sum_{\ell=1}^{m} \overline{p}_{i,k_\ell} \geq \sum_{\ell=1}^{m} \overline{p}^*_{i,k_\ell}$. Note that $\overline{p}_{i,h} > 0$ by assumption and $\overline{p}^*_{i,h} = 0$ by construction. Hence, $\sum_{k\in U(\overline{\succ}_i,h)} \overline{p}_{i,k} > \sum_{k\in U(\overline{\succ}_i,h)} \overline{p}^*_{i,k}$, i.e., $F(\overline{\succ}_i, h, \overline{P}_i) > F(\overline{\succ}_i, h, \overline{P}^*_i)$, so that statement (2) holds. □

Now we complete the proof of Proposition 1. By Claim 3, $\succ_i$ corresponding to $\overline{\succ}_i$ falls into one of cases considered in the proof of Lemma 1. Looking at the second tables in all cases of the proof of Lemma 1, we can verify the hypothesis of Claim 4. Hence, statement (2) holds. □

Several papers have recently provided various PS characterizations by replacing weak strategy-proofness with invariance type properties while keeping ordinal efficiency and envy-freeness. It is tempting to think that these characterizations are made possible with stronger notions of invariance type properties than weak strategy-proofness. We conclude this paper by pointing out that such a claim is not correct, i.e.,

Proposition 2. *Weak strategy-proofness and weak invariance is logically independent.*

Proof. First, the mechanism φ constructed in the proof of Lemma 1 was shown to be weakly strategy-proof. It is straightforward to verify that mechanism $\varphi$is not weakly invariant.

We next provide an example in which a mechanism is not weakly strategy-proof but weakly invariant. Let $N = \{1, 2\}$ and $A = \{a, b\}$. For each agent $i \in N$, there are two preferences, $\succ_i = (a, b)$ and $\succ_i' = (b, a)$. Let a mechanism, φ, satisfy

$$\varphi(\succ_1, \succ_2) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \text{ and } \varphi(\succ_1', \succ_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $F(\succ_1, a, \varphi_1(\succ_1, \succ_2)) = 1/2$ and $F(\succ_1, b, \varphi_1(\succ_1, \succ_2)) = 1$. On the other hand, $F(\succ_1, a, \varphi_1(\succ_1', \succ_2)) = 1$ and $F(\succ_1, b, \varphi_1(\succ_1', \succ_2)) = 1$. Thus, for agent 1, $\varphi_1(\succ_1', \succ_2)$ stochastically dominates $\varphi_1(\succ_1, \succ_2)$ at $\succ_1$. Hence, φ is not weakly strategy-proof. On the other hand, its weak invariance trivially holds, because for $|N| = 2$, there is no other change of preferences from any preference needed for the hypothesis of weak invariance. □

4. Concluding Remarks

The characterization of PS by ordinal efficiency, envy-freeness, and weak strategy-proofness is proved not to hold for the case of five or more agents, though it is true for three agent case as BM showed. We admit that we have attempted for four agent case with failure. We leave it as an open question for future investigation.

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