MATH4426
Robert Gross
Homework 1
Due January 24, 2025

Please remember that your answers to these problems must be typed and stapled, and are due at the start of class.

1. Alice, Bob, Carol, Dave, Ed, Felix, George, Howie, Indira, and Janice are roommates who share dinners together. They are planning who will cook for the next 5 days (Monday, Tuesday, Wednesday, Thursday, Friday).

(a) Suppose that there is one cook assigned for each night, and no one will cook more than once. How many possible schedules are there? Assume that two schedules are different if the same five people cook in a different order. In other words, Alice–Bob–Carol–Dave–Ed is a different schedule from Bob–Alice–Carol–Dave–Ed.

(b) Suppose that Alice is out of town on Monday and Tuesday and is only available to cook on Wednesday, Thursday, and Friday. She does not have to cook on any of those three nights, but cannot cook on the first two nights. How many possible schedules are there?

(c) Suppose that Alice changes her plans so that she is available on all 5 nights, but Bob is only available on Monday, Tuesday, and Wednesday. How many possible schedules are there?

(d) Suppose that Alice and Bob are available each night, and suppose that the roommates decide that *two* people will cook dinner each night. All ten roommates will cook exactly once in the next five days. How many possible schedules are there?

(e) Suppose that Alice and Bob refuse to cook together on the same night. How many possible schedules are there?

2. Suppose that n and k are nonnegative integers, $n \geq k$. Show that

$$k\binom{n}{k} = (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1}.$$

3. A dance class consists of 24 students, 13 women and 11 men. How many ways are there to choose 6 women and 6 men, and then pair each woman with a man?

4. How many different arrangements are there of the letters REVERBERATE?

5. Suppose that 8 people named A, B, C, *etc.*, are lining up for a photo.

(a) How many different ways can they line up?

(b) How many ways can they line up if A and B must be next to each other in either order?

(c) Suppose instead that there are 4 men and 4 women lining up, and the genders alternate. How many ways can they line up? The first person in line can be either a man or a woman.

(d) Suppose instead that the 4 men and 4 woman are married, and they will line up so that spouses are adjacent to each other, in either order (so that now two men or two women can stand next to each other). How many ways can they line up?

6. A committee consists of 5 Republicans, 4 Democrats, and 6 independent voters.

(a) How many ways are there to pick a sub-committee of 6 voters?
(b) How many ways are there to pick a sub-committee of 6 voters without including any independents?
(c) How many ways are there to pick a sub-committee with 3 Republicans and 3 Democrats?
(d) How many ways are there to pick a sub-committee with 2 Republicans, 2 Democrats, and 2 independents?
(e) How many ways are there to pick a sub-committee of 6 voters with at least 1 Republican and 1 Democrat?

7. How many solutions are there to the equation

$$a + b + c + d + e = 20$$

if
(a) a, b, c, d, and e are nonnegative integers?
(b) a, b, c, d, and e are positive integers?
(c) a, b, c, d, and e are positive even integers?

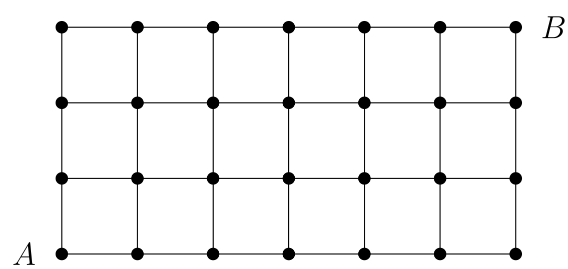
8. Suppose that n, m, and r are non-negative integers.
(a) Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}.$$

Hint: Consider a group of n men and m women. How many different ways are there to choose r of them?
(b) Show that

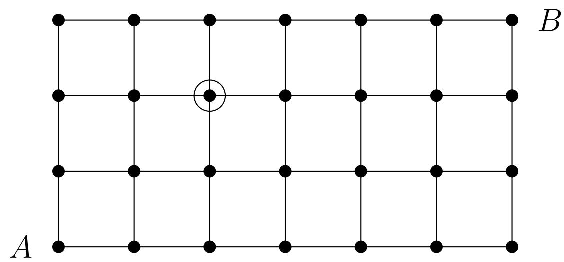
$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

9. Consider this grid:



Suppose that starting at the point A, you can go one step up or one step to the right, continuing until B is reached. How many different paths are there from A to B? Hint: Notice that to go from A to B, your path must contain 6 steps to the right, and 3 steps upwards.

10. Consider this grid:



Continuing with the previous problem, how many paths are there from A to B that pass through the circled point?