Rob Gross
Homework 6
Mathematics 4470.01
Due October 28, 2022

All homework solutions longer than one page **must be stapled**. A paper clip is not acceptable.

Remember that all homework solutions must be typeset in some way. You may print your answers on both sides of the page if you want.

1. Sometimes transition probabilities are derived from observations. A few weeks ago, we considered a 7-state model with the observations below (which, remember, are transposed from what we need for the transition matrix). Now (unlike in the previous problem), assume that states 6 and 7 are absorbing: $p_{66} = p_{77} = 1$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 7 | 9 | 2 | 0 | 0 | 1 | 21 |
| 2 | 0 | 3 | 19 | 17 | 1 | 0 | 2 | 42 |
| 3 | 0 | 2 | 20 | 11 | 10 | 4 | 23 | 70 |
| 4 | 0 | 0 | 10 | 26 | 26 | 6 | 24 | 92 |
| 5 | 0 | 0 | 0 | 5 | 22 | 2 | 30 | 59 |

If the system starts in state $i$ (for each $i$ from 1 to 5), compute how long on average it takes before the system ends in one of the two absorbing states, and compute the probability in each case of ending in state 6 and state 7. Hint: There is a similar problem in the text that should be helpful.

2. Consider a system with two independent controls, $A$ and $B$, that can prevent the system from being destroyed. The system is under control if either (or both) are working, and the system is destroyed if both fail simultaneously.

Assume that if either fails upon activation, the repair is made on the following day.

Assume for simplicity that the probability that a control fails is 0.05, so that the probability that a control works is 0.95. Assume as well that this is true for a newly repaired control (unlike the example that we worked in class).

Model this using a four-state Markov chain. We know that inevitably, the system must fail. Compute the mean time to failure assuming that both $A$ and $B$ are working at the start of the simulation.

3. Repeat the previous problem if we have three independent controls, $C$, $D$, and $E$, and the system is functioning if any one of them is working. In this problem, assume that the probability that a control fails is 0.2. Model with an eight-state Markov chain (though many of the states are similar), and compute the mean time to failure. Are you surprised by the result?

4. This problem is project 5.1 in chapter 5 of our text.

The *Michaelis–Menton* equation models the processing of drugs in the body. In one setting, the equation is

$$\frac{dx}{dt} = \frac{-Kx}{A + x}$$

where $x = x(t)$ is the concentration of a drug in the body at time $t$, and $K$ and $A$ are positive constants. The equation is often simplified further:

($a$) For the drug cocaine, $A$ is much larger than $x$; one example is $A = 6$ and $x(0) = 0.0025$. Under these circumstances, the equation can be approximated:

$$\frac{dx}{dt} = \frac{-Kx}{A}.$$

This can be solved in terms of $K$, $A$, and the unknown constant $x(0)$.

($b$) For alcohol, the reverse holds: $x(t)$ is much larger than $A$ (one example is $x(0) = 0.025$ and $A = 0.005$). Under these circumstances, the equation can be approximated as

$$\frac{dx}{dt} = \frac{-Kx}{x}.$$

($c$) Solve the equation without any approximations, giving an equation relating $x$ and $t$. Do not try to solve for $x$ as a function of $t$.