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Homework 5  
Mathematics 4470.01  
Due October 21, 2022

All homework solutions longer than one page **must be stapled**. A paper clip is not acceptable.

Remember that all homework solutions must be typeset in some way. You may print your answers on both sides of the page if you want.

1. Suppose that  $A$  is a square  $n \times n$  matrix, and each column of  $A$  sums to 1. Suppose that  $\mathbf{v} \in \mathbb{R}^n$  is a column vector and the sum of the entries in  $\mathbf{v}$  is  $C$ . Let  $\mathbf{w} = A\mathbf{v}$ . Show that the entries in  $\mathbf{w}$  sum to  $C$ .

2. Suppose as in the previous problem that  $A$  is a square  $n \times n$  matrix, and each column of  $A$  sums to 1. Suppose that  $\mathbf{v}$  is an eigenvector for  $A$  with eigenvalue  $\lambda$ . Say as much as possible about  $\mathbf{v}$  and  $\lambda$ . Note that Perron–Frobenius does not apply to this situation, because  $A$  might contain negative entries.

3. The book tells us that most of the water (and hence most of the pollution) flowing into Lake Ontario is from Lake Erie. If we assume that the pollution flowing into those lakes could be stopped, this problem models how long it would take for the levels of pollution to drop. Set

$$a_n = \text{pollution level in Lake Ontario in year } n$$

$$b_n = \text{pollution level in Lake Erie in year } n$$

The model tells us that

$$a_{n+1} = \alpha a_n + (1 - \beta)b_n$$

$$b_{n+1} = \beta b_n$$

Suppose for simplicity that  $a_0 = b_0 = 1$ , and that  $\alpha$  and  $\beta$  are unequal real numbers between 0 and 1.

(a) Find closed form expressions for  $a_n$  and  $b_n$  in terms of  $n$ ,  $\alpha$ , and  $\beta$ .

(b) Suppose that  $\alpha = 0.9$  and  $\beta = 0.95$ . Using your expressions in the previous part of the problem, find the smallest value of  $n$  for which  $a_n < 0.75$  and  $b_n < 0.75$ . I suggest that verify your answer using a computer or calculator.

4. One simple ecological model of reforestation after a fire consists of three stages:

$b$  = bare earth

$g$  = grass

$t$  = trees

A transition matrix for these three states might be:

	$b$	$g$	$t$
$b$	0.3	0.2	0.1
$g$	0.6	0.5	0.1
$t$	0.1	0.3	0.8

Using this matrix and an eigenvalue-eigenvector analysis, decide what the eventual distribution of bare earth, grass, and trees will be.

5. The text tells us that there are two types of squirrels in Scotland, red and gray. In any acre of land, the possible states are

$r$  = only red squirrels

$g$  = only gray squirrels

$b$  = both red and gray squirrels

$n$  = neither red nor gray squirrels

A possible transition matrix for the squirrels is

	$r$	$g$	$b$	$n$
$r$	0.874	0.095	0.078	0.059
$g$	0.015	0.723	0.008	0.193
$b$	0.109	0.095	0.905	0.143
$n$	0.002	0.087	0.009	0.605

Say as much as possible about the long-term behavior of squirrel populations in Scotland.