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Homework 3
Mathematics 4470.01
Due September 23, 2022

All homework solutions longer than one page **must be stapled**. A paper clip is not acceptable.

Remember that all homework solutions must be typeset in some way. You may print your answers on both sides of the page if you want.

1. Define a sequence with $x_1 = 1$, $x_2 = 1$, and $x_n = x_{n-1} + x_{n-2} + 1$. There are at least two ways to find an explicit formula for x_n.

   (*a*) Write $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. We know that the Fibonacci numbers F_n can be written as $F_n = a\alpha^n + b\beta^n$ for some real numbers a and b.

   Because the recursion for x_n is so similar to the recursion for the Fibonacci numbers, it is reasonable to hope that there is a formula of the form $x_n = A\alpha^n + B\beta^n + C$ for real constants A, B, and C. Use the values of x_1, x_2, and x_3 to solve for A, B, and C, and then verify that your answer satisfies the recursion formula $x_n = x_{n-1} + x_{n-2} + 1$.

   (*b*) There is a way to get an explicit formula that does not involve guesswork. We have

$$x_n = x_{n-1} + x_{n-2} + 1$$
$$x_{n-1} = x_{n-2} + x_{n-3} + 1$$

   if $n > 3$. Subtract and rearrange to get $x_n = 2x_{n-1} - x_{n-3}$ for $n > 3$. This is a third-order recurrence relation, meaning that you must solve a cubic polynomial. In general, this is not easy, but in this case, it is easy to factor the cubic polynomial into a linear and a quadratic factor, and proceed from there to get the roots λ_1, λ_2, and λ_3. We then know that $x_n = D\lambda_1^n + E\lambda_2^n + F\lambda_3^n$ for some real constants D, E, and F. Using the known values of x_1, x_2, and x_3, you can solve for D, E, and F.

2. Suppose that you have a bank account that pays 0.1% interest monthly. Suppose as well that you deposit 100 every month right after the interest is paid, and that the initial balance is 0. We can model this situation:

$$x_0 = 0$$
$$x_n = 1.001x_{n-1} + 100$$

   (*a*) Find a non-recursive expression for x_n in terms of n.

   (*b*) How many months will it take for the bank balance to go over 10000?

   (*c*) Suppose instead that the interest is 0.1% each month, and you deposit 200 every month right after the interest is paid. Find a non-recursive expression for the monthly balance. How many months will it take for the balance to go over 10000?

   (*d*) Suppose instead that the interest is 0.2% each month, and you deposit 100 every month right after the interest is paid. Find a non-recursive expression for the monthly balance. How many months will it take for the balance to go over 10000?

3. As usual, we define an nth order recursion relation with the formula

$$x_m = a_{n-1}x_{m-1} + a_{n-2}x_{m-2} + \cdots + a_0x_{m-n}$$

Write $p(x) = x^n - a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_0$. Suppose that we can factor $p(x) = (x-\lambda)^2 q(x)$.

(*a*) Show that $p(\lambda) = p'(\lambda) = 0$.
(*b*) Show that $x_m = m\lambda^{m-1}$ satisfies the recursion relation.