ERRATUM FOR: NON-FREE CURVES ON FANO VARIETIES

BRIAN LEHMANN, ERIC RIEDL, AND SHO TANIMOTO

Abstract. We correct a mistake in Section 3 of "Non-free curves on Fano varieties"

The purpose of this note is to correct a mistake in [2, Section 3]. The effect of this change is small: it requires modifying several constants in the statements of results, particularly in Section 3. We will follow the numbering of [2]. We thank Eric Jovinelly for bringing the mistake to our attention.

The mistake arose from incorrectly stating [1, Corollary 6.11]. The correct statement is as follows:

Theorem 3.4 ([1, Corollary 6.11])**.** *Let Z be a smooth projective variety defined over $\mathbb{C}$ and let B be a complex smooth projective curve. Suppose that $W \subset \mathrm{Mor}(B, Z)$ is a locally closed reduced subvariety such that the maps parametrized by W dominate Z and the general map parametrized by W is birational onto its image. Let s be a general map parametrized by W. Then the image of the map $T_sW \to H^0(B, N_s)$ has vanishing intersection with $(N_s)_{tors}$.*

This result is only used in Proposition 3.5, which must be modified as follows.

Proposition 3.5. *Let Z be a smooth projective variety defined over $\mathbb{C}$ and let B be a complex smooth projective curve. Suppose that W is a locally closed subvariety of $\mathrm{Mor}(B, Z)$ such that the morphisms parametrized by W dominate Z. Let s be a general element of W.*

(1) If the general map s is birational onto its image then the length of the torsion of N_s is at most $3g(B) + \mathrm{codim}_{H^0(B,s^*T_Z)}(T_sW)$.

(2) If the general map s factors as the composition of a finite morphism of smooth curves $h : B \to B'$ followed by a morphism $s' : B' \to Z$ that is birational onto its image, then the length of the torsion of N_s is at most $3g(B')d + \mathrm{codim}_{H^0(B',s'^*T_Z)}(T_{s'}W')d + r$ where d is the degree of h, W' is the image of W in $\mathrm{Mor}(B', Z)$, and r is the total ramification degree of h. In particular, if $g(B') \geq 2$ then the length of the torsion of N_s is at most $6g(B) + \mathrm{codim}_{H^0(B',s'^*T_Z)}(T_{s'}W')d$.

Proof. (1) We have the short exact sequence

$$H^0(B, s^*T_Z) \to H^0(B, N_s) \to H^1(B, T_B).$$

By Theorem 3.4 we see the image of T_sW in $H^0(B, N_s)$ is disjoint from $H^0(B, (N_s)_{tors})$. This implies that the image of $H^0(B, s^*T_Z)$ in $H^0(B, N_s)$ intersects $H^0(B, (N_s)_{tors})$ in dimension at most $\mathrm{codim}_{H^0(B,s^*T_X)}(T_sW)$. Since the cokernel of $H^0(B, s^*T_Z) \to H^0(B, N_s)$ injects into $H^1(B, T_B)$, it follows that it has dimension at most $h^1(B, T_B) \leq 3g(B)$. Altogether this proves the statement.

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(2) Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_B & \longrightarrow & s^*T_Z & \longrightarrow & N_s & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{=} & & \psi\big\downarrow & & \\
0 & \longrightarrow & h^*T_{B'} & \longrightarrow & s^*T_Z & \longrightarrow & h^*N_{s'} & \longrightarrow & 0
\end{array}$$

Note that $(h^*N_{s'})_{tors}$ has length d times the length of $(N_{s'})_{tors}$ which by (1) is at most $3g(B') + \mathrm{codim}_{H^0(B',s'^*T_Z)}(T_{s'}W')$. By the snake lemma, ψ is surjective and its kernel is a torsion sheaf of length at most r. Altogether this proves the first statement. The final statement follows from the Riemann-Hurwitz formula $g(B) = dg(B') + \frac{r}{2} - d + 1$. Indeed, we have

$$\begin{aligned}
6g(B) = 6dg(B') + 3r - 6d + 6 &= 3dg(B') + r + (3dg(B') + 2r - 6d + 6) \\
&\geq 3dg(B') + r.
\end{aligned}$$

□

For the reader's convenience, we separate our a step in the proof of [2, Lemma 3.7] as its own lemma.

Lemma 0.1. *Let Z be a smooth projective variety defined over $\mathbb{C}$ and let $M \subset \mathrm{Mor}(B, Z)$ parametrize a dominant family of curves. For a general $s \in M$ we have $h^1(B, s^*T_Z) < g(B)(\dim(Z) + 2)$.*

Proof. Since the family of curves is dominant, the normal sheaf N_s is generically globally generated. Thus we have

$$\begin{aligned}
h^1(B, s^*T_Z) &\leq h^1(B, T_B) + h^1(B, N_s) \\
&\leq 3g(B) + g(B)(\dim(Z) - 1)
\end{aligned}$$

where the last inequality follows from [2, Lemma 2.4]. □

Finally, the previous changes affect the constants in [2, Theorem 3.8.(1,2)].

Theorem 3.8 (Grauert-Mulich)**.** *Let Z be a smooth projective variety defined over $\mathbb{C}$ and let $\mathcal{E}$ be a torsion free sheaf on Z of rank r. Let M be an irreducible component of $\mathrm{Mor}(B, Z)$ and let $ev : \mathcal{U} \to Z$ denote the evaluation map.*

(1) Assume that the composition of ev with the normalization map for $\mathcal{U}$ is dominant with connected fibers and that ev is flat on the preimage of some open subset of M_{red}. Assume that a general $s : B \to Z$ parametrized by M is birational onto its image. Then we have

$$\begin{aligned}
&\| \mathrm{SP}_{Z,[s(B)]}(\mathcal{E}) - \mathrm{SP}(s^*\mathcal{E}) \| \leq \\
&\qquad \frac{1}{2}\left(2g(B)^2\dim(Z)^2 + 10g(B)^2\dim(Z) + 5g(B)\dim(Z) + 15g(B) + 2\right)\mathrm{rk}(\mathcal{E})
\end{aligned}$$

(2) Assume that the composition of ev with the normalization map for $\mathcal{U}$ is dominant with connected fibers and that ev is flat on the preimage of some open subset of M_{red}. Assume that there is some curve B' of genus ≥ 2 such that the general map $s : B \to Z$

parametrized by M factors through a morphism $s' : B' \to Z$ that is birational onto its image. Then we have

$$\|\operatorname{SP}_{Z,[s(B)]}(\mathcal{E}) - \operatorname{SP}(s^*\mathcal{E})\| \leq \frac{1}{2}\left(2g(B)^2\dim(Z)^2 + 10g(B)^2\dim(Z) + 5g(B)\dim(Z) + 18g(B) + 2\right)\operatorname{rk}(\mathcal{E})$$

Proof. (1) Let t be the length of the torsion part of N_s, let $\mathcal{G}$ be the subsheaf of $(N_s)_{tf}$ generated by global sections, and let q be the dimension of the cokernel of the composition

$$V \to H^0(B, s^*T_Z) \to H^0(B, (N_s)_{tf})$$

where $V \subset H^0(B, s^*T_Z)$ is the tangent space to M_{red} at s.

[2, Theorem 3.3] shows that $\mu^{max}(M_{\mathcal{G}}^\vee) \leq 2g(B)\dim(Z) + 2$. Proposition 3.5.(1) shows that $t \leq 3g(B) + \operatorname{codim}_{H^0(B,s^*T_Z)}(V)$. Lemma 0.1 shows that the latter quantity is at most $g(B)(\dim(Z)+2)$. By [2, Lemma 3.7] we have $q \leq g(B)\dim(Z) + 5g(B)$. We then apply [2, Theorem 3.2] to obtain the desired statement.

(2) Proposition 3.5.(2) shows that in this situation $t \leq 6g(B) + \operatorname{codim}_{H^0(B',s'^*T_Z)}(V')$ and Lemma 0.1 shows that the latter quantity is at most $g(B')(\dim(Z)+2) \leq g(B)(\dim(Z)+2)$. Then we can obtain the desired bound by repeating the argument for (1) using this new estimate on t. □

Theorem 3.8 is used in the proof of [2, Theorem 4.3]; the constant γ in this statement must be updated to match the new bound from Theorem 3.8. In turn, Theorem 4.3 is used in other results but the constants are not mentioned in the statements, only in the proofs (which must be adjusted accordingly).

References

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Department of Mathematics, Boston College, Chestnut Hill, MA 02467
E-mail address: lehmannb@bc.edu

Department of Mathematics, University of Notre Dame, 255 Hurley Hall, Notre Dame, IN 46556
E-mail address: eriedl@nd.edu

Graduate School of Mathematics, Nagoya University, Furocho Chikusa-ku, Nagoya, 464-8602, Japan
E-mail address: sho.tanimoto@math.nagoya-u.ac.jp