A CHARACTERIZATION OF PROJECTIVE SPACE VIA LENGTHS OF EXTREMAL RAYS

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Abstract. We prove a new characterization of complex projective space using lengths of extremal rays.

1. Introduction

We work over the complex number field $\mathbb{C}$. In this short note, our goal is to give a new characterization of projective space which relies on the anticanonical degrees of curves. The first statements in this direction are due to the influential paper [Mor79]. [CMSB02] and [Keb02] extended Mori's argument to show that $\mathbb{P}^n$ is the only smooth projective variety of dimension n such that every curve has anticanonical degree $\geq (n+1)$. [CT07] further improved this characterization of $\mathbb{P}^n$ by allowing isolated LCIQ singularities and proving that one only needs to find a single extremal ray of length $\geq n+1$. [Che17] addressed varieties with quotient singularities.

Our main result extends previous work on characterizations of $\mathbb{P}^n$ in several directions: it covers all MMP-type singularities, weakens the assumption on the anticanonical degree to the lowest possible value, and only requires knowledge of a single ray of the Mori cone. Thus it also contributes to the study of varieties with "long" extremal rays, which has been carried out in the smooth setting by [AO02, HN13, DH17].

Theorem 1.1. *Let X be an n-dimensional projective variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Assume that there exists a $(K_X+\Delta)$-negative extremal ray*

$$R \subset \overline{\mathrm{NE}}(X)$$

which is rational, relatively ample at infinity, and satisfies

$$l(R) := \inf\bigl\{-(K_X+\Delta)\cdot C \,|\, C \text{ is rational and } [C]\in R\bigr\} > n.$$

Then $X \simeq \mathbb{P}^n$, (X,Δ) is terminal, and Δ is a divisor of degree less than 1.

This result was known previously for toric pairs (X,Δ) by [Fuj03] (and generalized to toric foliations in [FS24] and [FS25]). The consequences of Theorem 1.1 for log canonical pairs are particularly striking. In this setting, the quantity $l(R)$ of Theorem 1.1 is known as the length of the extremal ray R (see Remark 3.4).

Corollary 1.2. *Let (X,Δ) be an n-dimensional projective log canonical pair such that $\overline{\mathrm{NE}}(X)$ admits a $(K_X+\Delta)$-negative extremal ray of length $> n$. Then $X\simeq\mathbb{P}^n$, (X,Δ) is terminal, and Δ is a divisor of degree less than 1.*

2020 *Mathematics Subject Classification.* Primary 14E30; Secondary 14J17.

Key words and phrases. lengths of extremal rays, cone theorem, Mori theory, projective space.

Corollary 1.3. *Let (X, Δ) be an n-dimensional projective log canonical pair such that $K_X + \Delta$ is not nef. Suppose that*

$$-(K_X + \Delta) \cdot C > n$$

for every rational curve C on X. Then $X \simeq \mathbb{P}^n$, (X, Δ) is terminal, and Δ is a divisor of degree less than 1.

There are similar statements for semi-log canonical pairs; for example, Corollary 3.5 establishes a semi-log canonical version of Corollary 1.2.

Remark 1.4. A celebrated result of Kobayashi and Ochiai characterizes the smooth Fano varieties of large index (see e.g. [KO82, Ion86, Fuj87] and [AD14, FM21] for generalizations to worse singularities). One might hope for an analogous characterization of varieties with "long" extremal rays. In the smooth setting this is accomplished in [DH17] and in the toric setting by [Fuj06]; it is natural to ask whether this classification can be extended to the singular setting.

Our proof of Theorem 1.1 relies on several ingredients. First, we need a classification result from [CMSB02]. Second, we use the recently improved bounds in Bend-and-Break established by [JLR25b] and the existence of very free curves on mildly singular Fano varieties proved by [JLR25a]. Third, we require the theory of quasi-log structures originally introduced by Ambro in [Amb03]; see [Fuj17, Chapter 6] and [Fuj23] for more background. More precisely, we will need the theory of extremal rays for quasi-log schemes as developed by [Fuj25].

In Section 2 we present several results on the cone of curves for quasi-log schemes obtained by combining the results of [Fuj25] and [JLR25b]. Throughout we adopt the notation and results established in [Fuj11], [Fuj17], [Fuj23], and [Fuj25]. In Section 3 we prove Theorem 1.1 and related results.

Acknowledgments. Osamu Fujino was partially supported by JSPS KAKENHI Grant Numbers JP21H04994 and JP23K20787. Eric Jovinelly was supported by an NSF postdoctoral research fellowship, DMS-2303335. Brian Lehmann was supported by Simons Foundation grant Award Number 851129. Eric Riedl was supported by NSF CAREER grant DMS-1945944 and Simons Foundation grants 00011850 and 00013673.

The authors thank Makoto Enokizono and Kento Fujita for useful discussions.

2. Extremal rays for quasi-log schemes

Classical studies on the lengths of extremal rays, for example [Kaw91], rely on the Bend-and-Break theorem of [MM86]. The bounds in these results can be improved by replacing [MM86, Theorem 5] with [JLR25b, Theorem 1.1]. This is equally true for the statements on extremal rays in the quasi-log setting presented below. We emphasize that such generalizations are not merely of technical interest but are quite useful, e.g. in proving Theorem 1.1 and Corollary 1.2.

Theorem 2.1 ([Fuj25, Theorem 1.12])**.** *Let $[X, \omega]$ be a quasi-log scheme and let $\varphi\colon X \to W$ be a projective morphism of schemes such that $-\omega$ is φ-ample. Let P be an arbitrary closed point of W. Let E be any positive-dimensional irreducible component of $\varphi^{-1}(P)$ such that $E \not\subset X_{-\infty}$. Then E is covered by (possibly singular) rational curves ℓ with*

$$0 < -\omega \cdot \ell \leq \dim E + 1.$$

In particular, E is uniruled.

Proof. In the final step of the proof of [Fuj25, Theorem 1.12] (see [Fuj25, p. 676] for details), we use

$$\begin{aligned}0 < -\nu^*\omega \cdot \Gamma &\leq 2\dim \overline{E} \cdot \frac{-\nu^*\omega \cdot C}{-K_{\overline{E}} \cdot C} \\ &\leq 2\dim \overline{E},\end{aligned}$$

which essentially goes back to [MM86]. By applying [JLR25a, Theorem 1.1] instead, we can replace $2\dim \overline{E}$ with $\dim \overline{E}+1$. This yields the desired inequality in Theorem 2.1. □

Theorem 2.2 (Lengths of extremal rational curves, [Fuj25, Theorem 1.13])**.** *Let $[X,\omega]$ be a quasi-log scheme, and let $\pi\colon X \to S$ be a projective morphism of schemes. Suppose that $R \subset \overline{\mathrm{NE}}(X/S)$ is an ω-negative extremal ray which is rational and relatively ample at infinity. Let $\varphi_R\colon X \to W$ be the contraction morphism over S associated to R. We put*

$$d = \min_E \dim E,$$

where E runs over positive-dimensional irreducible components of $\varphi_R^{-1}(P)$ for all $P \in W$. Then R is spanned by a (*possibly singular*) *rational curve ℓ with*

$$0 < -\omega \cdot \ell \leq d+1.$$

Proof. The proof of [Fuj25, Theorem 1.13] applies verbatim. The only change is to replace [Fuj25, Theorem 1.12] with Theorem 2.1 in the proof of [Fuj25, Theorem 1.13] (see [Fuj25, p. 676] for details). □

Corollary 2.3 ([Fuj25, Corollary 12.3])**.** *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X \to S$ be a projective morphism of schemes. Let R be a $(K_X+\Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ that is rational and relatively ample at infinity. Let $\varphi_R\colon X \to W$ be the contraction morphism over S associated to R. We put*

$$d = \min_E \dim E,$$

where E runs over positive-dimensional irreducible components of $\varphi_R^{-1}(P)$ for all $P \in W$. Then R is spanned by a (*possibly singular*) *rational curve ℓ with*

$$0 < -(K_X+\Delta) \cdot \ell \leq d+1.$$

Furthermore, if φ_R is birational and (X,Δ) is kawamata log terminal, then R is spanned by a (*possibly singular*) *rational curve ℓ with*

$$0 < -(K_X+\Delta) \cdot \ell < d+1.$$

Lastly, let V be an irreducible component of the degenerate locus

$$\{x \in X \mid \varphi_R \text{ is not an isomorphism at } x\}$$

of φ_R. Then V is uniruled.

Proof. All we need to do is to replace [Fuj25, Theorem 1.12] and [Fuj25, Theorem 1.13] with Theorem 2.1 and Theorem 2.2, respectively, in the proof of [Fuj25, Corollary 12.3] (see [Fuj25, p. 677] for details). The desired statement then follows. □

The following corollary is an immediate consequence of the cone and contraction theorem for quasi-log schemes (see [Fuj17, Theorems 6.7.3 and 6.7.4]) together with Theorem 2.2.

Corollary 2.4 ([Fuj22, Corollary 1.8])**.** *Let $[X, \omega]$ be a quasi-log canonical pair and let $\pi \colon X \to S$ be a projective morphism onto a scheme S. Let $\mathcal{L}$ be a π-ample line bundle on X. We put*

$$d := \max_{s \in S} \dim \pi^{-1}(s).$$

Then $\omega + (d+1)\mathcal{L}$ is π-nef. In particular, $\omega + (\dim X + 1)\mathcal{L}$ is always π-nef.

Theorem 2.1 and [FH23, Theorem 1.8] (see Conjectures 1.15 and 1.21 in [Fuj23]) represent the optimal results on this topic in full generality. For the sake of completeness, we also record here some remarks on other statements in [Fuj25].

Remark 2.5. In the proof of [Fuj25, Proposition 9.1] (see [Fuj25, p. 665] for details), the bound

$$0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$$

was obtained by using [Fuj11, Theorem 1.1], [Fuj25, Theorem 1.12], or [Fuj25, Corollary 12.3]. By replacing these results with Theorem 2.1 or Corollary 2.3, we can improve the bound to

$$0 < -(K_X + \Delta) \cdot C \leq \dim X + 1$$

in the statement of [Fuj25, Proposition 9.1].

Since [Fuj25, Theorem 1.8] is an application of [Fuj25, Proposition 9.1], the same improvement applies; namely, we may replace

$$0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$$

with

$$0 < -(K_X + \Delta) \cdot C \leq \dim X + 1$$

in [Fuj25, Theorem 1.8] (see [Fuj25, p. 667–668] for details).

Consequently, because [Fuj25, Theorem 9.2] is derived from [Fuj25, Theorem 1.8], we can similarly replace

$$0 < -\mathcal{P} \cdot C \leq 2 \dim X$$

with

$$0 < -\mathcal{P} \cdot C \leq \dim X + 1$$

(see [Fuj22, pp. 668–669] for details).

Applying this once more to [Fuj25, Theorem 1.6 (iii)], which depends on [Fuj25, Theorem 9.2], we obtain the improved inequality

$$0 < -\omega \cdot C_j \leq \dim U_j + 1$$

in place of

$$0 < -\omega \cdot C_j \leq 2 \dim U_j$$

(see [Fuj25, p. 670] for details).

Finally, since [Fuj25, Theorem 1.5] is a special case of [Fuj25, Theorem 1.6], we may also replace

$$0 < -(K_X + \Delta) \cdot C_j \leq 2 \dim U_j$$

with the sharper bound

$$0 < -(K_X + \Delta) \cdot C_j \leq \dim U_j + 1.$$

Moreover, Theorem 2.1 also implies [Fuj25, Conjecture 1.21] (see also [Fuj25, Remark 1.22]).

3. A Characterization of Projective Space

In this section we first prove a weaker analogue of Theorem 1.1 for terminal varieties. We then reduce the general case of Theorem 1.1 to the terminal case using the theory of quasi-log schemes.

3.1. **Terminal varieties.**

This section focuses on the special case of Theorem 1.1 when X has terminal singularities. We will need the following definition which extends the notion of "very free" from rational curves to curves of arbitrary genus. The key property of very free curves is that they deform with the expected dimension.

Definition 3.1. Let X be a projective variety. A morphism $s : C \to X$ from a (smooth projective integral) curve C is said to be very free if $s(C)$ lies in the smooth locus of X and every non-zero quotient of s^*T_X has slope at least $2g(C) + 1$.

[JLR25a, Theorem 1.3] guarantees that every terminal Fano variety admits a very free curve in its smooth locus; furthermore, we can ensure that the degree is arbitrarily large compared to the genus. We prove Theorem 1.1 for terminal varieties by studying how very free curves deform and break into rational curves.

The following lemma shows that in our situation a general very free curve of high degree passes through many general points of X.

Lemma 3.2. *Let X be a terminal Fano variety of dimension n and suppose $c > 0$ is chosen so that cK_X is Cartier. Suppose every rational curve on X has anticanonical degree $> n$. Let $g, d > 0$ be positive integers and define*

$$t = \max\left\{ \left\lceil \frac{d}{n} \right\rceil - (cn+1)g - cn, 0 \right\}$$

Then every genus g very free curve $f : C \to X$ of anticanonical degree d has a deformation that maps a general set of t points in C to a general set of t points in X.

Proof. Since cK_X is a Cartier divisor, our assumption implies that every rational curve on X has anticanonical degree $\geq n + \frac{1}{c}$.

Let M_0 denote the irreducible component of the moduli space $\mathrm{Mor}(C, X)$ containing our original very free curve $f : C \to X$. We inductively construct a family M_k of curves in the following way. Suppose we have constructed a family M_i sending general points $q_1, \ldots, q_i \in C$ to fixed points $p_1, \ldots, p_i \in X$. Select a general point $q_{i+1} \in C$ and let $Y_{i+1} \subset X$ be the subvariety swept out by $f(q_{i+1})$ as f varies over M_i. Fix some general $p_{i+1} \in Y_{i+1}$ and let $M_{i+1} \subset M_i$ be the family of curves that send q_{i+1} to p_{i+1}. By construction, $\dim M_{i+1} = \dim M_i - \dim Y_{i+1}$.

Define $\theta(i)$ to be the number of indices $j \leq i$ for which $\dim Y_j < n$. Because $\dim M_i$ is eventually 0 and $\theta(i)$ is eventually positive, there is a largest integer k such that $\dim M_k \geq \theta(k)$. The family M_k sends $q_1, \ldots, q_k$ to $p_1, \ldots, p_k$ and has dimension between $\theta(k)$ and $\theta(k) + n$. For ease of notation, set $k_1 = \theta(k)$ and $k_0 = k - \theta(k)$.

Because we chose each q_i generally, $\dim Y_i \geq \dim Y_{i+1}$ for all $i \geq 1$. Thus the set $\{p_1, \ldots, p_{k_0}\}$ is a general collection of k_0 points in X. Our goal is to show that k_0 is at least t. To do so, consider a resolution of singularities $\pi : X' \to X$ that is an isomorphism over the smooth locus. Each point p_i lies in the smooth locus of X because it is a general point on a curve that meets the smooth locus.

We apply [JLR25b, Lemma 2.1] to the strict transform on X' of the curves parametrized by M_k. This produces a stable map $f' : C' \to X'$ where the stabilization of C' is C and the preimages under $C' \to C$ of $q_{k-k_1+1}, \ldots, q_k \in C$ are trees of rational curves. For the rest of the proof, we consider C as the unique irreducible component of C' with positive genus and $q_i \in C \subset C'$ as points of C'.

To fix notation, let $p'_i \in X'$ be the unique point that maps to $p_i \in X$ under π. Let $I \subset \{1, \dots, k\}$ be the set of indices i such that C' contains a rational tree attached to C at q_i. Set $h = |I|$. For each $i \in I$, there is a distinct rational component $R_i \subset C'$ that is not contracted by f' and whose f'-image contains p'_i. These rational curves cannot be contracted by π, so our hypothesis on X implies

$$-K_X \cdot \Big(\sum_{i \in I} \pi_* f'_* R_i \Big) \geq h \left(n + \frac{1}{c} \right).$$

We claim that $-K_X \cdot \pi_* f'_* C' \geq (k-1)n + \frac{1}{c} k_1$. If $h = k$, this follows from the previous paragraph because X is Fano. Otherwise, if $h < k$, observe that any index $i \notin I$ satisfies $i \leq k_0$ by construction. In particular, $\{p'_i\}_{i \notin I}$ is a general set of $k - h$ points in X'. As $f'(q_i) = p'_i$ for each $i \notin I$, the twist down of $f'^*(T_{X'})|_C$ by all but one of these $k - h$ points is generically globally generated. Since a generically globally generated vector bundle on C has degree ≥ 0, we conclude that $-K_{X'} \cdot f'_* C \geq (k - h - 1)n$. Terminality of X also ensures $-K_X \cdot \pi_* f'_* C \geq -K_{X'} \cdot f'_* C$. Using the fact that $h \geq k_1$, we obtain

$$\begin{aligned} -K_X \cdot \pi_* f'_* C' &\geq -K_X \cdot \left(\pi_* f'_* C + \sum_{i \in I} \pi_* f'_* R_i \right) \\ &\geq (k - h - 1)n + h \left(n + \frac{1}{c} \right) \\ &\geq (k-1)n + \frac{1}{c} k_1. \end{aligned}$$

This proves our claim.

Since the dimension of M_0 is $d - (g-1)n$, the dimension of M_k is at most $k_1 + n$, and the codimension of M_k in M_0 is at most $n(k - k_1) + (n-1)k_1$, it follows that

$$d \leq (g-1)n + k_1 + n + nk - k_1 = (g+k)n.$$

Putting this together with $(k-1)n + \frac{1}{c} k_1 \leq d$, we see that

$$(k-1)n + \frac{1}{c} k_1 \leq gn + kn$$

which shows that $k_1 \leq (g+1)nc$. Using the bound $d \leq (g+k)n$ from above, we find that $k_0 = k - k_1 \geq \lceil \frac{d}{n} \rceil - g - (g+1)cn$ as desired. □

Using Lemma 3.2, we carefully select a degeneration of a very free curve of large degree. We show this degeneration breaks our curve into two pieces: a very free rational curve of degree $n+1$ and a higher-genus very free curve. The very free rational curve allows us to apply [CMSB02] and conclude that X is $\mathbb{P}^n$.

Theorem 3.3. *Let X be a terminal Fano variety of dimension n such that every rational curve in X has anticanonical degree $> n$. Then $X \simeq \mathbb{P}^n$.*

Proof. We first prove that X contains a very free rational curve of degree $n+1$ in its smooth locus. Choose a positive integer c so that cK_X is Cartier. Let $\pi : X' \to X$ be a resolution of singularities that is an isomorphism over the smooth locus. By [JLR25a, Theorem 1.3] there is a genus $g \geq 2$ such that X admits a very free curve C of genus g

and arbitrarily large anticanonical degree. In particular, setting $\gamma = (1+nc)(g+1)$ we may suppose that the anticanonical degree d of C satisfies

$$d > n(1+nc)(3g+\gamma). \tag{3.1}$$

Define $s = \lceil \frac{d}{n} \rceil - \gamma$ and $\delta = d - (g-1)n - (s+1)n$. By construction $s \geq 1$ and $\delta \geq 1$.

Let $q_0, q_1, \ldots, q_s, r_1, \ldots, r_{\delta-1}$ be general points of C. Let $p_0, p_1, \ldots, p_s$ be general points of X and let $D_1, \ldots, D_{\delta-1}$ be general divisors in a very ample linear series on X. By Lemma 3.2 there is a δ-parameter family of maps $f : C \to X$ with $f(q_i) = p_i$. Thus, we can find a 1-parameter subfamily M of maps $f : C \to X$ with $f(q_i) = p_i$ and $f(r_i) \in D_i$. We take the strict transforms of these maps and apply Bend-and-Break on X' to obtain a limiting stable map $f' : C' \to X'$ where C' has a rational tail attached to C at q_0. Since the image of q_0 does not lie in the π-exceptional locus, we see that the rational tail is not contracted by $\pi \circ f'$. For the rest of the proof, we consider C as the unique irreducible component of C' with positive genus and consider $q_i, r_j \in C \subset C'$ as points of C'.

We study the map $f' : C' \to X'$. Let R_i be the fiber of the stabilization map $C' \to C$ over q_i and let T_j be the fiber of the stabilization map $C' \to C$ over r_j. Let $I \subset \{1, \ldots, s\}$ be the set of integers i for which R_i is a curve and let $J \subset \{1, \ldots, \delta - 1\}$ be the set of integers for which T_j is a curve. Set $h = |I|$.

We claim that $s - h \geq 3g$. If not, then $h > s - 3g$ and the total $-K_X$-degree of $\pi_* f'_* C'$ must be at least

$$\begin{aligned}(h+1)\left(n + \frac{1}{c}\right) > (s-3g)\left(n+\frac{1}{c}\right) &= sn + \frac{s}{c} - 3g\left(n + \frac{1}{c}\right) \\ &\geq d - \gamma n + \frac{s}{c} - 3g\left(n+\frac{1}{c}\right) \geq d + \frac{d}{nc} - (3g+\gamma)\left(n + \frac{1}{c}\right).\end{aligned}$$

By the inequality (3.1) the right hand side is larger than d and we obtain a contradiction. Thus, $s - h \geq 3g$ and we see that the spine curve $C \subset C'$ is not contracted by $\pi \circ f'$.

Note that the deformations of $f'|_C : C \to X'$ which satisfy $f(q_i) = p_i$ for all but one value of $i \in \{1, \ldots, s\} \backslash I$ define a dominant family of curves. Thus the twist down of $f'^* T_{X'}$ by the sum of the corresponding points $\{q_i\}$ is generically globally generated. Since $s - h - 1 \geq 3g - 1 \geq 2g + 1$ we conclude that every positive rank quotient of $f'^* T_{X'}$ has slope at least $2g+1$. Thus $f'|_C$ is very free and deforms with the expected dimension.

We now carefully calculate $-K_{X'}$ degrees. Because $f'|_C$ sends $s - h$ chosen points of C to $s - h$ general points in X' and sends $\delta - 1 - |J|$ chosen points of C to general divisors in X, we see that $h^0(f'^* T_{X'}|_C)$ is at least $(s-h)n + \delta - 1 - |J|$. Since $f'|_C$ is very free, we know that $h^1(f'^* T_{X'}|_C) = 0$. Hence, by Riemann-Roch the $(-K_{X'})$-degree of $f'_* C$ must be at least $(s-h)n + \delta - 1 - |J| + (g-1)n$.

We next apply [JLR25a, Lemma 4.3] with $S = R_i$. Since S only meets the rest of C' along the spine curve, which is not contained in the π-exceptional locus, [JLR25a, Lemma 4.3] shows that $-K_{X'} \cdot f'_* R_i \geq -\pi^* K_X \cdot f'_* R_i > n$. Since it must be an integer, $-K_{X'} \cdot f'_* R_i \geq n + 1$. Similarly, Lemma [JLR25a, Lemma 4.3] shows $-K_{X'} \cdot f'_* T_j \geq 1$. Thus, by [JLR25a, Lemma 4.3] again the total $-K_{X'}$ degree of $f'_* C'$ is at least

$$\begin{aligned} -K_{X'} \cdot f'_* C + \sum_{i \in I \cup \{0\}} -K_{X'} \cdot f'_* R_i + \sum_{j \in J} -K_{X'} \cdot f'_* T_j &\geq -K_{X'} \cdot f'_* C + (h+1)(n+1) + |J| \\ &\geq (s-h)n + \delta - 1 - |J| + (g-1)n + (h+1)(n+1) + |J| \\ &= (s+1)n + (g-1)n + \delta + h \\ &= d + h. \end{aligned}$$

It follows that $h = 0$ and every inequality above is an equality. Thus, $-K_{X'} \cdot f'_* R_0 = n+1$ and $-K_{X'} \cdot f'_* C = sn + (g-1)n + \delta - 1 - |J|$.

The map $f' : C' \to X'$ we obtain depends on the choice of each q_i, p_i, r_j, and D_j, but for general choices the stable map f' will lie in a fixed stratum of the Kontsevich space. Since $f'|_C$ is very free, the space of deformations of $f'|_C$ has dimension $sn+\delta-1-|J|$. Thus there are only finitely many possibilities for the map $f'|_C$ which satisfy the incidence conditions $\pi \circ f'(q_i) = p_i$ for $1 \leq i \leq s$ and $\pi \circ f'(r_j) \in D_j$ for $j \in \{1, \dots \delta - 1\} \setminus J$. This means that for general choices of the points $\{p_i\}$ and divisors $\{D_j\}$, the map $f'|_C$ will be a general element of the component $M \subset \mathrm{Mor}(C, X')$ it lies in. Moreover, the aforementioned incidence conditions do not depend on the choice of $q_0 \in C$. As M parameterizes a dominant family of maps and $f'|_C \in M$ is general, the point $f'|_C(q_0) \in X'$ will be a general point. In fact, setting $p'_0 = \pi^{-1}(p_0)$, the pair of points $(p'_0, f'|_C(q_0)) \in X' \times X'$ is general as well, once more because the incidence conditions above are independent of the choice of p_0.

Since R_0 attaches to C at q_0, $f'(R_0)$ passes through the general pair of points p'_0 and $f'|_C(q_0)$ of X'. Note that $\pi \circ f'(R_0)$ must be irreducible, because otherwise it would have two components both of $(-K_X)$-degree greater than n, but

$$n + 1 = -K_{X'} \cdot f'_* R_0 \geq -K_X \cdot \pi_* f'_* R_0$$

by [JLR25a, Lemma 4.3]. It follows that R_0 must contain an irreducible component F_0 such that $f'|_{F_0}$ passes through both p'_0 and $f'|_C(q_0)$; generality of these points implies that $f'|_{F_0}$ is a very free rational curve on X' by [Deb01, 4.10]. In particular $-K_{X'} \cdot f'_* F_0 \geq n+1$. If R_0 had more than one irreducible component, we could apply [JLR25a, Lemma 4.3] with S equal to a connected component of the complement of F_0 in R_0 to see that $-K_{X'} \cdot f'_* R_0$ is strictly larger than $n+1$, a contradiction. We conclude that R_0 is irreducible. By [JLR25a, Lemma 4.2], it follows that $f'(R_0)$ is disjoint from the π-exceptional locus.

Thus we have established the existence of a very free rational curve of degree $n+1$ (which by definition means that the image is contained in the smooth locus of X). In particular, the irreducible component of $\mathrm{Mor}(\mathbb{P}^1, X)$ containing such curves is doubly dominant (in the sense of [CMSB02]). Furthermore, the degree assumption implies that the curves on X cannot break as we deform them. Thus this irreducible component of $\mathrm{Mor}(\mathbb{P}^1, X)$ is also closed, irreducible, maximal, and everywhere unsplitting. By [CMSB02, Theorem 4.2] we conclude that X is isomorphic to $\mathbb{P}^n$. □

3.2. **General case.** We are now prepared to prove Theorem 1.1:

Proof of Theorem 1.1: Let $\varphi_R \colon X \to W$ be the extremal contraction associated to R (see, e.g., [Fuj11, Theorem 1.1]). By Corollary 2.3 together with the assumption $l(R) > \dim X$, the target W is a point. Consequently, $\rho(X) = 1$, the divisor $-(K_X + \Delta)$ is ample, and the non-lc locus $\mathrm{Nlc}(X, \Delta)$ satisfies $\dim \mathrm{Nlc}(X, \Delta) \leq 0$.

Suppose that (X, Δ) is not kawamata log terminal. Assume that there exists a positive-dimensional log canonical center V of (X, Δ). By adjunction (see, e.g., [Fuj17, Theorem 6.3.5 (i)]), the pair $[V, \omega]$, where $\omega := (K_X + \Delta)|_V$, is a quasi-log scheme, and $-\omega$ is ample. By Theorem 2.1, we obtain a rational curve $\ell \subset V$ such that

$$0 < -\omega \cdot \ell \leq \dim V + 1.$$

This yields a rational curve $\ell \subset X$ satisfying

$$0 < -(K_X + \Delta) \cdot \ell \leq \dim X,$$

which contradicts the assumption $l(R) > \dim X$. Therefore (X, Δ) has no positive-dimensional log canonical centers, and the non-klt locus $\mathrm{Nklt}(X, \Delta)$ is zero-dimensional since $\dim \mathrm{Nlc}(X, \Delta) \leq 0$. By [Fuj25, Theorem 1.17] (see also [FH23, Theorem 1.7]), there exists a rational curve $\ell \subset X$ such that

$$0 < -(K_X + \Delta) \cdot \ell \leq 1,$$

which again contradicts $l(R) > \dim X$. Hence (X, Δ) is kawamata log terminal.

By [BCHM10, Discussion after Corollary 1.4.3], there exists a projective birational morphism $f\colon \widetilde{X} \to X$ such that $(\widetilde{X}, \widetilde{\Delta})$ is a $\mathbb{Q}$-factorial terminal pair and

$$K_{\widetilde{X}} + \widetilde{\Delta} = f^*(K_X + \Delta).$$

Note that $-(K_{\widetilde{X}} + \widetilde{\Delta})$ is nef and big. Thus we may take a $(K_{\widetilde{X}} + \widetilde{\Delta})$-negative extremal ray $\widetilde{R} \subset \overline{\mathrm{NE}}(\widetilde{X})$ with $f_*\widetilde{R} = R$. Because f is log crepant $l(\widetilde{R}) > \dim \widetilde{X}$.

Let $\varphi_{\widetilde{R}}\colon \widetilde{X} \to \widetilde{W}$ be the associated contraction. By Corollary 2.3 again, $\widetilde{W}$ is a point, and hence $\rho(\widetilde{X}) = 1$. Therefore $f\colon \widetilde{X} \to X$ is an isomorphism and X is a $\mathbb{Q}$-factorial terminal Fano variety. Furthermore Δ is nef so that $-K_X \cdot C > n$ for every rational curve C on X. By Theorem 3.3 $X \simeq \mathbb{P}^n$. All the remaining desired properties can be verified easily. □

To deduce Corollary 1.2 and Corollary 1.3 from Theorem 1.1, we will require the following important remark concerning the value $l(R)$ in Theorem 1.1.

Remark 3.4. Let (X, Δ) be a projective log canonical pair and let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X)$. It is well known that

$$l(R) = \min\bigl\{-(K_X + \Delta) \cdot C \,|\, C \text{ is rational and } [C] \in R\bigr\}, \tag{3.2}$$

that is, there exists a rational curve C on X with $[C] \in R$ such that

$$l(R) = -(K_X + \Delta) \cdot C.$$

If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then the equality (3.2) is immediate. When $K_X + \Delta$ is only $\mathbb{R}$-Cartier, the same conclusion follows by adapting the argument in the proof of [Fuj17, Theorem 4.7.2 (1)].

Proof of Corollary 1.2: Since (X, Δ) is log canonical, R is relatively ample at infinity and [Fuj11, Theorem 1.1] shows that R is rational. The statement follows from Theorem 1.1 and Remark 3.4. □

Proof of Corollary 1.3. This follows from Corollary 1.2 and the cone theorem for log canonical pairs. □

Finally, we extend Corollary 1.2 to semi-log canonical pairs.

Corollary 3.5. *Let (X, Δ) be a connected n-dimensional projective semi-log canonical pair such that $\overline{\mathrm{NE}}(X)$ admits a $(K_X + \Delta)$-negative extremal ray R of length greater than n. Then $X \simeq \mathbb{P}^n$, (X, Δ) is terminal, and Δ is a divisor of degree less than* 1.

Proof. By the cone and contraction theorem for semi-log canonical pairs, there exists a contraction morphism

$$\varphi_R\colon X \to W$$

associated with the extremal ray R. Let $\nu\colon Z \to X$ be the normalization and write

$$K_Z + \Delta_Z := \nu^*(K_X + \Delta).$$

Let

$$Z = \bigsqcup_{i \in I} Z_i$$

be the decomposition into irreducible components and set $\Delta_{Z_i} := \Delta_Z|_{Z_i}$ for each $i \in I$. By Theorem 2.1, there exists an index $i_0 \in I$ such that

$$(\varphi_R \circ \nu)(Z_{i_0})$$

is a point. Applying Corollary 1.3 to the pair $(Z_{i_0}, \Delta_{Z_{i_0}})$, we conclude that

$$Z_{i_0} \simeq \mathbb{P}^n \quad \text{and} \quad \lfloor \Delta_{Z_{i_0}} \rfloor = 0.$$

This forces Z to be irreducible, hence $Z = Z_{i_0}$, and the normalization map $\nu\colon Z \to X$ is an isomorphism. The assertion now follows from Corollary 1.2. □

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