GEOMETRIC MANIN'S CONJECTURE IN CHARACTERISTIC p

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ABSTRACT. Geometric Manin's conjecture for complex Fano varieties describes the structure of the moduli space of curves. We propose a version of this conjecture in characteristic p and describe its connection to the Batyrev–Manin–Peyre–Tschinkel conjecture over global fields. This is a survey paper written for a volume of the Summer Research Institute in Algebraic Geometry held at Colorado State University in 2025.

CONTENTS

1. INTRODUCTION

Geometric Manin's conjecture predicts the structure of the moduli space of curves on a Fano variety. This conjecture has its roots in several influential discoveries made approximately 40 years ago: the close relationship between curves and birational geometry ([Mor79], [Mor82]), asymptotic formula describing counts of rational points ([FMT89], [BM90]), and the topological properties of spaces of maps from Riemann surfaces into projective Fano manifolds ([Seg79]).

In this expository paper our goals are:

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(1) To give a conjectural description of families of curves on Fano varieties in characteristic p while accounting for pathological examples.
(2) To give a careful formulation of the Batyrev–Manin–Peyre–Tschinkel conjecture over a global function field with a clear explanation of Peyre's constant in this setting.
(3) To briefly explain the recent developments of [DLTT25] to solve new cases of the Batyrev–Manin–Peyre–Tschinkel conjecture over global function fields.

This paper is intended to supplement the material in the recent book [Tan25a]. Throughout, by a curve on a projective variety X we mean a morphism $s : C \to X$ from a smooth projective geometrically integral curve C.

1.1. **Historical background: curves on Fano varieties.** Over the past 50 years, rational curves have emerged as an essential tool in the study of complex Fano varieties. Some highlights are:

- Mori's solution of Hartshorne's conjecture using Bend-and-Break and the deformation properties of free curves ([Mor79]). Since Mori's ground-breaking work, rational curves have continued to play a key role in our understanding of positivity of the tangent bundle.
- The use of lines and conics in the classification of smooth Fano threefolds by [Isk77, Isk79, IP99, MM82, MM83, MM03].
- The proof of the boundedness of smooth Fano varieties by [KMM92].

Starting from the influential works [KP01] (for homogeneous spaces) and [HRS04] (for hypersurfaces), attention turned toward the explicit study of the moduli space of rational curves on Fano varieties: classifying the irreducible components, studying their dimension and singularities, etc. Some examples are [Tho98, KP01, CS09, Tes09, Bou12, Bou16, BK13, RY19, LT19a, LT21, LT24, LT22, BLRT22, BJ22, Oka24, Oka25, MTiB25, BV17, BS23]. For a long time there was no general expectation about how such curves should behave; the situation in characteristic 0 was clarified in the papers [LT19a], [LRT25] and extended to curves of arbitrary genus.

1.2. **Historical background: Manin's conjecture on rational points.** Yuri Manin and his collaborators initiated a program (later known as the *Manin program*) seeking a geometric explanation of various asymptotic formulas for the counting functions of rational points on smooth Fano varieties. This program led to the *Batyrev–Manin–Peyre–Tschinkel conjecture* (or *Manin's conjecture* for short) which predicts the precise asymptotic formula for the counting function of rational points of bounded height on smooth Fano varieties. This has been developed in a series of works [FMT89, BM90, Pey95, BT98a, Pey03, Pey17, LST22, LS24]. This conjecture has been confirmed for various examples including, but not limited to, various homogeneous spaces using harmonic analysis [FMT89, BT98b, CLT02, STBT07, ST16], complete intersections of low degree using the circle method [Bir62, BHB17, FM17], and various (singular) del Pezzo surfaces using the universal torsor method [dlB02, dlBBD07, DlBB11, dlBBP12, Der14, DP19, DP20, BD25].

There also have been many works developing Manin's Conjecture over global function fields. Versions of this conjecture were first formulated by Bourqui and Peyre; in [Bou03, Bou11b, Bou13] Bourqui studied toric varieties and varieties with simple Cox rings and in [Pey12] Peyre studied flag varieties. More recent examples include hypersurfaces [Lee11, BV17, BS23, Saw24, HL25] and del Pezzo surfaces [Bou13, DLTT25, Gla25, Tan25b] as well as many others.

When one counts the number of rational points, it is important to consider the *exceptional set* and remove the contribution of the exceptional set from the counting function so that the asymptotic formula reflects the global geometry of the underlying pair. The geometric aspect of Manin's conjecture, particularly concerning the exceptional set, has been developed in [HTT15, LTT18, LT17, Sen21, LST22, LT19b, Gao23]. In [LST22] the authors and Sengupta proposed a conjectural description of the exceptional set in Manin's conjecture; using recent advances in higher dimensional algebraic geometry such as the minimal model program [BCHM10] and Birkar's solution to the BAB conjecture in [Bir19, Bir21] we verified that this proposed set is indeed a thin set as predicted by Peyre in [Pey03]. There is an alternative proposal using the notion of freeness of rational points combined with all heights approach; see [Pey17] and [Pey21].

1.3. **Batyrev's heuristic and extensions.** In his influential lecture notes [Bat88], Batyrev developed a heuristic for the asymptotic behavior of rational points on Fano varieties over global fields. The heuristic is based on the geometry of moduli spaces of curves. Suppose that X is a Fano variety defined over a finite field. Counting the number of degree d maps $s : \mathbb{P}^1 \to X$ is equivalent to counting the $\mathbb{F}_q$-points on the associated irreducible components of $\mathrm{Mor}(\mathbb{P}^1, X)$. We can estimate these numbers once we know the dimension and number of irreducible components in degree d. If we assume that these spaces have the "expected" geometry, we obtain the familiar predicted asymptotic formula $cq^d d^{\rho(X)-1}$ for the counting function in Manin's conjecture. This is further pioneered by David Bourqui and Emmanuel Peyre. See, e.g., [Pey12, Bou03, Bou11b, Bou13].

The geometric meaning of the leading constant was later clarified by Ellenberg and Venkatesh. They observed that such an asymptotic formula naturally follows from homological stability results via the Grothendieck-Lefschetz trace formula. Furthermore, over $\mathbb{C}$ the Cohen–Jones–Segal conjecture predicts that the stable homology of the moduli space of curves does indeed stabilize to the homology of the space of continuous maps, and such stable homology naturally leads to Peyre's constant.

[BT98a] was the first to recognize the importance of the minimal model program in establishing Batyrev's geometric heuristic. Building on recent advances in the MMP, [LT19a] and [LRT25] formulated a set of conjectures in characteristic 0 which translate Batyrev's heuristics into a precise description of the irreducible components of the spaces of curves on any Fano variety (or the spaces of sections of any Fano fibration). Also [LRT25] completed the first major step toward solving these conjectures by computing the exceptional set in the geometric setting.

An alternative approach to Batyrev's heuristic is to work in the Grothendieck ring. This motivic approach was pioneered particularly by Bourqui in [Bou09, Bou10, Bou11a] and also by [CLL16]. The recent development of the motivic Euler product by [Bil23] has opened the way for new advances; see e.g. [BDH22, Fai25].

Disclaimer: Geometric Manin's conjecture in characteristic p and Batyrev–Manin–Peyre–Tschinkel's conjecture over global function fields are still under development. This survey paper contains lots of speculative observations and conjectures; these may need further corrections in the future.

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2. Preliminaries

Throughout we will work over a ground field k. A variety is an integral separated scheme of finite type over the ground field. When the ground field has characteristic p, we denote by F^e the e-th iterate of the (absolute) Frobenius map on X which is a homeomorphism that raises all functions to the p^e-th power. If the ground field k is finite, then for any k-scheme X we denote by Fr_k the geometric Frobenius automorphism of $X_{\overline{k}}$, i.e. the morphism $X_{\overline{k}} \to X_{\overline{k}}$ induced by base change from the automorphism of $\overline{k}$ that is the inverse of $t \mapsto t^{|k|}$.

2.1. **Numerical spaces.**

Let X be a normal projective variety over a field. We denote the abelian group of Cartier divisors by $\mathrm{CDiv}(X)$. A $\mathbb{Q}$-Cartier divisor is an element of $\mathrm{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Two Cartier divisors D_1, D_2 are numerically equivalent if for every closed integral curve C on X we have $D_1.C = D_2.C$; numerical equivalence is written as $D_1 \equiv D_2$. We define $N^1(X)$ to be the $\mathbb{R}$-vector space of Cartier divisors up to numerical equivalence.

2.1.1. *Curves.* A 1-cycle on X is a formal sum of closed integral curves on X. Two 1-cycles α_1, α_2 are numerically equivalent if for every Cartier divisor D we have $D.\alpha_1 = D.\alpha_2$; numerical equivalence is written as $\alpha_1 \equiv \alpha_2$.

We define $N_1(X)$ to be the $\mathbb{R}$-vector space of 1-cycles up to numerical equivalence. This is a finite-dimensional vector space and it contains a natural lattice $N_1(X)_{\mathbb{Z}}$ generated by classes of curves. It contains two natural cones:

- $\overline{\mathrm{Eff}}_1(X)$ is the closure of the cone generated by all effective 1-cycles.
- $\mathrm{Nef}_1(X)$ is the nef cone, i.e. the cone of all numerical classes α such that $E.\alpha \geq 0$ for every effective Cartier divisor E.

Both cones are closed, convex, full-dimensional, and pointed.

2.2. **Brauer groups.** Since we will be working exclusively with quasiprojective schemes X over fields, the cohomological and Azumaya Brauer groups coincide; we denote this common group by $\mathrm{Br}(X)$. The algebraic part of the Brauer group $\mathrm{Br}_1(X)$ is the kernel of the map $\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s})$ induced by base change to the separable closure.

Let X be a smooth projective variety over a global field k. Then its adelic space is given by

$$X(\mathbb{A}_k) := \prod_{v \in \Omega_k} X(k_v),$$

as a topological space with the product topology, where Ω_k is the set of all places of k and k_v is the completion of k with respect to $v \in \Omega_k$. One can define the *Brauer–Manin set*

$$X(\mathbb{A}_k)^{\mathrm{Br}(X)}$$

which contains the set $X(k)$ of rational points. When X is geometrically rationally connected, the Colliot-Thélène conjecture predicts that $X(k)$ is non-empty as soon as the Brauer–Manin set is non-empty. Moreover the conjecture predicts that $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$. Readers interested in this construction should consult [CTS21].

2.3. **Thin sets.** Suppose X is a variety over a global field k. One possible notion of a "small" subset $Z \subset X(k)$ is a non-Zariski dense subset. However, this notion is not sufficiently flexible for working with arithmetic questions. Instead, we will need the notion of a thin set introduced by Serre in the context of the inverse Galois problem. See [Ser08] for more details.

Definition 2.1. Let X be a projective variety over a field k. A thin map is a morphism of projective varieties $f : Y \to X$ such that

(1) f is generically finite onto its image, and
(2) f is not birational.

A thin set in $X(k)$ is any subset of a finite union $\cup_{i=1}^r f_i(Y_i(k))$ where $\{f_i : Y_i \to X\}$ is a finite collection of thin maps.

The notion of a thin set is meaningless over some fields (e.g. an algebraically closed field), but over global fields it is an important and useful condition.

2.4. **Fujita invariants.** The following invariant plays a central role in Manin's conjecture:

Definition 2.2. Let X be a smooth projective variety defined over k and L be a big and nef $\mathbb{Q}$-divisor on X. The *Fujita invariant* or *a-invariant* is the following invariant:

$$a(X, L) := \min\{t \in \mathbb{R} \mid \text{the numerical class } t[L] + [K_X] \text{ is pseudo-effective}\}.$$

When L is nef but not big, we set $a(X, L) = +\infty$. By [HTT15, Proposition 2.7], this is a birational invariant under pullback via a birational morphism between smooth projective varieties. (See also [Tan25a, Proposition 4.1.3].)

In characteristic 0, [BDPP13] shows that when L is big and nef $a(X, L)$ is positive if and only if X is geometrically uniruled. In positive characteristic, the same statement holds by [Das20, Theorem 1.6].

Definition 2.3. Let X be a smooth projective variety defined over k and L be a big and nef $\mathbb{Q}$-divisor on X. Let $f : Y \to X$ be a dominant generically finite morphism from a smooth projective variety Y. We say f is an *a-cover* if $a(Y, f^*L) = a(X, L)$.

Next we define the b-invariants:

Definition 2.4. Let X be a smooth projective variety defined over k and L be a big and nef $\mathbb{Q}$-divisor on X. Assume that X is geometrically uniruled. The *face of* (k, X, L) is defined by

$$\mathsf{F}(X, L) := \mathrm{Nef}_1(X) \cap \{\alpha \in N_1(X) \mid (a(X, L)L + K_X).\alpha = 0\}.$$

The *b-invariant of* (k, X, L) is defined by

$$b(k, X, L) = \dim\langle \mathsf{F}(X, L)\rangle,$$

where $\langle \mathsf{F}(X, L)\rangle \subset N_1(X)$ is the subspace generated by $\mathsf{F}(X, L)$. By [HTT15, Proposition 2.10] as well as [Tan25a, Proposition 4.1.16], this is a birational invariant under pullback via a birational morphism between smooth projective varieties.

The most standard example of Fujita invariants and b-invariants is the following:

Example 2.5. Let X be a smooth weak Fano variety defined over k and let $L = -K_X$. Then we have $a(X, L) = 1$ and $b(k, X, L) = \dim N_1(X) = \rho(X)$.

Definition 2.6. Let X be a smooth Fano variety and let $f : Y \to X$ be an a-cover. We say f is *face-contracting* if the induced map

$$\mathsf{F}(Y, f^*L) \to \mathsf{F}(X, L),$$

is not injective.

2.5. **Globally F-regular varieties.** [Ray78] provided the first example of the failure of Kodaira vanishing for a smooth projective variety in characteristic p. This failure leads to many other pathologies. However, if we impose an F-splitting assumption then we can sometimes recover certain consequences of Kodaira vanishing.

Definition 2.7. Let X be a smooth projective variety over an algebraically closed field of characteristic p. We say that X is F-split if the natural map $\mathcal{O}_X \to F_*\mathcal{O}_X$ admits a splitting in the category of $\mathcal{O}_X$-modules.

We will mainly use the following stronger property.

Definition 2.8. Let X be a normal variety over an algebraically closed field of characteristic p. We say that X is globally F-regular if for every effective Cartier divisor D there is a positive integer e such that the natural map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$$

admits a splitting in the category of $\mathcal{O}_X$-modules.

Globally F-regular varieties share many important properties with Fano varieties in characteristic 0. A key advantage of globally F-regular varieties is the following vanishing result:

Theorem 2.9 ([SS10, Theorem 6.8])**.** *Let k be a field of characteristic p. Suppose X is a smooth projective geometrically integral k-variety that is geometrically globally F-regular. Then for any big and nef Cartier divisor L on X we have $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ for every $i > 0$.*

Corollary 2.10. *Let k be a field of characteristic p. Suppose X is a smooth geometrically integral Fano k-variety that is geometrically globally F-regular. Then $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Many techniques in Manin's conjecture (such as the formulation of Peyre's constant in [Pey95], [BT98a], and [CLT10]) require such a vanishing condition. Since this condition does not hold for arbitrary Fano varieties in characteristic p – [Mad16] gives the counterexample of regular del Pezzo surfaces over imperfect fields – we will rely on these vanishing results in our cases of interest.

Finally we have the following conjecture:

Conjecture 2.11. Let k be a field of characteristic p. Suppose X is a smooth geometrically integral Fano k-variety that is geometrically globally F-regular. Let $f : Y \to X$ be a dominant generically finite morphism from a smooth projective variety. Then we have $a(Y, -f^*K_X) \leq a(X, -K_X)$.

Note that in characteristic 0 this easily follows from the ramification formula.

3. Rational curves on Fano varieties in characteristic p

As discussed above, the work of Mori and his collaborators revolutionized the study of rational curves on Fano varieties. The following definition identifies the rational curves with the best possible deformation-theoretic properties; such rational curves play a key role in the theory.

Definition 3.1. Let X be a smooth projective geometrically integral variety over a field k. We say that a rational curve $s : \mathbb{P}^1 \to X$ is:

- free, if s^*T_X is nef;
- very free, if s^*T_X is ample.

More generally, suppose C is a smooth projective geometrically integral curve over k. For any $r \geq 0$, we say that $s : C \to X$ is r-free if every positive rank quotient of s^*T_X has slope at least $2g(C) + r$.

A famous question of Kollár asks whether every smooth Fano variety over an algebraically closed field of characteristic p carries a very free rational curve (or equivalently, is separably rationally connected). We review known results on this question in the next two subsections.

3.1. **Counterexamples.**

Although Kollár's question is still open for smooth Fano varieties, there are mildly singular Fano varieties which do not carry any free rational curves at all. The first examples were given in [Kol95]; the following example is a particular case of Kollár's construction presented by [Xu12, Section 5].

Example 3.2. We work over $\overline{\mathbb{F}}_2$. Let $\phi : X \to \mathbb{P}^2$ be the blow-up of the seven $\mathbb{F}_2$-points of $\mathbb{P}^2$. Then X is a weak del Pezzo surface of degree 2.

There are exactly seven (-2)-curves on X corresponding to the strict transforms of the seven $\mathbb{F}_2$-lines on $\mathbb{P}^2$. The contraction of these (-2)-curves yields a birational map $\phi : X \to X'$ to a degree 2 log del Pezzo surface X'. The anticanonical linear series defines a degree 2 finite morphism $g : X' \to \mathbb{P}^2$ that turns out to be purely inseparable. Using the theory of p-closed foliations, one sees that g corresponds to a surjection $\psi : T_{X'} \to g^*\mathcal{O}_{\mathbb{P}^2}(-1)$.

This quotient obstructs the existence of (very) free rational curves in the smooth locus of X'. Indeed, if there were such a curve $s : \mathbb{P}^1 \to X'$, then a general deformation would be contained in the locus where ψ is a surjective map of locally free sheaves. Then $s^*\psi$ would define a negative quotient of $s^*T_{X'}$, a contradiction.

The previous example is a specific instance of an interesting class of weak del Pezzo surfaces X described by the following lemma. The surfaces which satisfy the equivalent conditions are classified explicitly by [KN22].

Lemma 3.3 ([BLRT23, Theorem 1.1])**.** *Let X be a weak del Pezzo surface over an algebraically closed field of characteristic p. Then the following properties are equivalent:*

- *Every element in $|-K_X|$ is singular.*
- *X admits a dominant family of rational curves with larger than expected dimension.*

In fact, in all such examples there is a finite purely inseparable morphism $f : Y \to X$ such that $K_{Y/X}$ is not pseudo-effective. This implies that there are many dominant families of rational curves on X (coming from Y) which have dimension larger than expected. In many cases (such as the example of [Xu12] discussed above) there is a birational model that carries no free curves at all; such examples will necessarily be poorly behaved for Manin's conjecture.

3.2. **Positive examples.** There are several classes of Fano varieties over algebraically closed fields for which the existence of (very) free rational curves is known. In some cases, one needs to include an F-splitting assumption to obtain the best behavior of curves.

Example 3.4. Suppose X is a smooth Fano hypersurface in $\mathbb{P}^n$. It is known that X is separably rationally connected when X is general ([Zhu24]), or even when X is a general Fano complete intersection ([CZ14]).

The case of an arbitrary smooth Fano hypersurface is still open. [ST19] shows separable rational connectedness of all Fano hypersurfaces of index ≥ 2 with degree less than the characteristic (and a similar statement for Fano complete intersections). However, in general it can be difficult to find a very free rational curve: [Che25] shows that the minimal degree of a very free rational curve on a Fano hypersurface cannot be bounded above by a linear function in the dimension or the degree. There is an alternative approach to this problem using the circle method. See [BS23, Theorem 1.5].

[CS22] proves similar results for complete intersections in certain homogeneous varieties.

Example 3.5. Suppose that X is a smooth del Pezzo surface. Since X is rational, it is separably rationally connected. Furthermore the result of [BLRT23] mentioned above shows that every dominant family of rational curves on X has the expected dimension.

If we assume that X has degree ≥ 2, or X has degree 1 and the characteristic satisfies $p \geq 11$, then [BLRT23] shows that the following conditions are equivalent:

- X is F-split.
- Every irreducible component of $\mathrm{Mor}(\mathbb{P}^1, X)$ representing a nef class will generically parametrize free rational curves.

Example 3.6. Suppose that X is a normal projective threefold of Fano type in characteristic ≥ 5. Then [GNT19, Theorem 1.5] shows that X is rationally chain connected.

If X is a smooth projective threefold that is globally F-regular in characteristic ≥ 11 and X admits a Mori contraction to a curve or surface, then X is separably rationally connected by [GLP$^+$15, Theorem 0.2].

As in the previous examples, it is reasonable to expect better behavior for Fano varieties under an F-splitting assumption:

Conjecture 3.7. Let X be a globally F-regular smooth Fano variety over an algebraically closed field of characteristic p. Then X is separably rationally connected.

In fact, we expect more to be true: every rational ray in the interior of $\mathrm{Nef}_1(X)$ should be represented by the class of a very free rational curve. Thus such varieties are suitable candidates for Manin's conjecture.

4. Sections of Fano fibrations in characteristic p

Manin's conjecture addresses the behavior of rational points on a Fano variety over a number field. To obtain the best geometric analogue, we should study rational points on a Fano variety over the function field of a curve (particularly over a finite base field). We will always pass directly to an integral model and focus on the equivalent problem of understanding sections of a Fano fibration.

Geometric Manin's conjecture is the study of the asymptotic behavior of sections of Fano fibrations. In this section, we give a precise formulation of this conjecture, focusing on the geometric aspects. This section is adapted from [LRT25]. Some of the conjectures in this section have been established in characteristic 0 using birational geometry, but currently the characteristic p versions seem out of reach.

Definition 4.1. Let k be a field of characteristic p. A *good Fano fibration* is a morphism $\pi : \mathcal{X} \to B$ with the following properties:

(1) $\mathcal{X}$ is a smooth projective geometrically integral variety.
(2) B is a smooth projective geometrically integral curve.
(3) π is flat and $\mathcal{O}_B \cong \pi_* \mathcal{O}_{\mathcal{X}}$.
(4) The generic fiber X_η is a smooth geometrically integral Fano variety that is geometrically globally F-regular.
(5) The generic fiber X_η is a Mori dream space.
(6) π admits a section.

We denote the moduli space of sections of $\pi : \mathcal{X} \to B$ by $\mathrm{Sec}(\mathcal{X}/B)$. For a curve class $\alpha \in N_1(\mathcal{X})$, $\mathrm{Sec}(\mathcal{X}/B, \alpha)$ denotes the finite type subscheme parametrizing sections with numerical class α.

Remark 4.2. For a projective smooth family over an uncountable algebraically closed field being globally F-regular is an open condition. Indeed, let $f : \mathcal{Y} \to T$ be a projective smooth family and let $\mathcal{L}$ be an f-ample divisor on $\mathcal{Y}$. Then by [SS10, Proposition 5.3(1)], each fiber Y_t is globally F-regular if and only if the section ring

$$R_t := \bigoplus_m H^0(Y_t, \mathcal{O}(mL)),$$

is strongly F-regular. Then being strongly F-regular is an open condition by [ST25, Theorem B]. Thus our claim.

It is natural to wonder whether some of the properties in Definition 4.1 can be weakened. However we will focus only on this case where a positive answer seems most likely.

The following key definition identifies the analogue of a free curve in the relative setting. Just as with freeness, relative freeness is determined by the positivity of the pullback of the tangent bundle. Note that a section will automatically be contained in the locus where $T_{\mathcal{X}/B}$ is locally free.

Definition 4.3. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. A section $s : B \to \mathcal{X}$ is relatively r-free if every positive rank quotient of $s^*T_{\mathcal{X}/B}$ has slope at least $2g(B) + r$.

4.1. **Numerical classes.** Our first task is to describe the set of numerical classes of sections of π. This set will be contained in a translate of the subspace

$$V = \{\alpha \in N_1(\mathcal{X}) \,|\, \alpha.F = 0\}$$

where F is a general fiber of π. Thus we first focus our attention on V.

Note that we have an injective linear map $N_1(X_\eta) \to N_1(\mathcal{X})$ whose image is contained V. (This map is dual to the surjective restriction map $N^1(\mathcal{X}) \to N^1(X_\eta)$.) From now on we will identify $N_1(X_\eta)$ with its image in V.

Lemma 4.4 ([LRT25, Lemma 5.2])**.** *Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. Then we have*

$$\mathrm{Nef}_1(X_\eta) = V \cap \mathrm{Nef}_1(\mathcal{X}).$$

We define $\mathrm{Nef}_{vert,\mathbb{Z}}$ *to be* $\mathrm{Nef}_1(X_\eta) \cap V_{\mathbb{Z}}$*.*

Proof. Since we are assuming that $\mathcal{X}_\eta$ is a Mori dream space, the pseudo-effective cone of divisors on $\mathcal{X}_\eta$ is equal to the effective cone. Then the proof of [LRT25, Lemma 5.2] applies. □

Remark 4.5. There is a minor subtlety: although $N_1(\mathcal{X}_\eta)_{\mathbb{Z}}$ is contained in $V_{\mathbb{Z}} \cap N_1(\mathcal{X}_\eta)$, the two lattices may not coincide; the difference reflects the monodromy of π. In particular, the monoid $\mathrm{Nef}_{vert,\mathbb{Z}}$ may be strictly larger than the monoid $\mathrm{Nef}_1(\mathcal{X}_\eta) \cap N_1(\mathcal{X}_\eta)_{\mathbb{Z}}$.

We next turn from V to the possible numerical classes of nef sections (contained in a translate of V). We can put restrictions on the numerical classes of sections as follows. Suppose $\mathcal{X}_b$ is a reducible fiber of π over a closed point b of B. Every section of π must intersect $\mathcal{X}_b$ at a smooth point, and in particular, will intersect a unique irreducible component of $\mathcal{X}_b$.

Definition 4.6. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. An *intersection profile* λ consists of a choice of one generically smooth and geometrically integral irreducible component in each fiber of π. The (finite) set of all intersection profiles is denoted by Λ.

Alternatively, we can identify an intersection profile with the affine subspace of $N_1(\mathcal{X})$ that consists of classes α which have intersection 1 against the irreducible component in each fiber identified by λ and intersection 0 against all other components. Henceforth we will not distinguish between these two different ways of thinking about intersection profiles.

The following definition summarizes the above discussion by identifying the possible classes of nef sections.

Definition 4.7. We define $\mathsf{S}_{\mathcal{X}/B}$ to be the convex hull of the set of nef $\mathbb{Z}$-classes which have intersection 1 against a general fiber F of π.

For each intersection profile λ, we let S_λ denote the convex hull of the set of nef $\mathbb{Z}$-classes lying in the affine subspace of $N_1(\mathcal{X})$ corresponding to λ.

Remark 4.8. We emphasize that S_λ can be strictly contained in the set of nef $\mathbb{R}$-curve classes which have intersection profile λ.

Note that $\mathsf{S}_{\lambda,\mathbb{Z}}$ is preserved by adding nef curve classes in $V_\mathbb{Z}$. In particular, Lemma 4.4 shows that $\mathsf{S}_{\lambda,\mathbb{Z}}$ naturally carries the structure of a module over the monoid $\mathrm{Nef}_{vert,\mathbb{Z}}$. The following conjecture predicts that this module structure essentially controls $\mathsf{S}_{\lambda,\mathbb{Z}}$.

Conjecture 4.9 (char 0: [LRT25, Corollary 5.8])**.** Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. For each intersection profile λ, S_λ is a rational polyhedral convex set whose recession cone is $\mathrm{Nef}_1(\mathcal{X}_\eta)$.

One consequence of this conjecture is that $\mathsf{S}_{\lambda,\mathbb{Z}}$ is "sandwiched" between two translates of $\mathrm{Nef}_{vert,\mathbb{Z}}$. In other words, any sum indexed over $\mathsf{S}_{\lambda,\mathbb{Z}}$ will only differ from a similar sum indexed over $\mathrm{Nef}_{vert,\mathbb{Z}}$ by an asymptotically negligible amount.

4.2. **Exceptional set.** It is well-known that in Manin's conjecture one must remove an "exceptional set" to obtain the correct count. [LT17] and [LST22] predict that the exceptional set comes from certain types of maps $f : \mathcal{Y} \to \mathcal{X}$.

Definition 4.10. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. Suppose that $\mathcal{Y}$ is a smooth projective B-scheme such that $\mathcal{Y}_\eta$ is geometrically integral. A thin B-morphism $f : \mathcal{Y} \to \mathcal{X}$ is an *exceptional map* if either

- f is non-dominant and $a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta}) \geq 1$;
- f is dominant and $a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta}) > 1$;
- f is an a-cover and $\kappa(-f^*K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta} + K_{\mathcal{Y}_\eta}) > 0$;
- f is an a-cover with $\kappa(-f^*K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta} + K_{\mathcal{Y}_\eta}) = 0$ and is geometrically non-Galois, or;
- f is an a-cover with $\kappa(-f^*K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta} + K_{\mathcal{Y}_\eta}) = 0$, geometrically Galois, and face-contracting.

Remark 4.11. We expect that certain pathologies of $\mathrm{Sec}(\mathcal{X}/B)$ – such as the existence of infinitely many irreducible components of $\mathrm{Sec}(\mathcal{X}/B)$ with too large dimension – should be accounted for by exceptional maps, in the sense that such irreducible components should be the images of sections on some $\mathcal{Y}$. See [LRT25, Theorem 1.3] for a statement of this type over $\mathbb{C}$.

Definition 4.12. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration.

For a numerical class $\alpha \in \mathsf{S}_{\mathcal{X}/B}$, an irreducible component M of $\mathrm{Sec}(\mathcal{X}/B, \alpha)$ is an *exceptional component* if there is an exceptional map $f : \mathcal{Y} \to \mathcal{X}$ and a component N of $\mathrm{Sec}(\mathcal{Y}/B)$ such that f induces a dominant map

$$f_* : N \to M.$$

A component M is a *Manin component* if it is not exceptional.

4.3. **Manin components.** The final step is to understand the structure of the set of Manin components M_α. The key task facing us to identify precisely which classes in $\mathsf{S}_{\mathcal{X}/B}$ represent Manin components and how many Manin components there are in each numerical class.

First of all we expect that Manin components parametrize free sections.

Conjecture 4.13 (char 0: [LRT25, Geometric Manin's conjecture 3])**.** Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. For each intersection profile λ, there is an $\alpha_0 \in \mathsf{S}_{\lambda,\mathbb{Z}}$ such that every Manin component M_α representing a class $\alpha \in \alpha_0 + \mathrm{Nef}_{vert,\mathbb{Z}}$ will generically parametrize free sections.

The following conjecture allows us to count the number of Manin components.

Conjecture 4.14 (char 0: [LRT25, Geometric Manin's conjecture 4])**.** Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. For each intersection profile λ, there is an $\alpha_0 \in \mathsf{S}_{\lambda,\mathbb{Z}}$ such that every *algebraic* equivalence class of curves contained in $\alpha_0 + \mathrm{Nef}_{vert,\mathbb{Z}}$ is represented by exactly one Manin component.

We emphasize that it is algebraic and not numerical equivalence that appears in Conjecture 4.14. (See [Oka25] for an example demonstrating the difference.) Currently the best known results which relate algebraic equivalence and the existence of families of curves are due to [Tia25] and [KT25]. Extending these results to solve Conjecture 4.14 is an important but challenging problem.

Remark 4.15. To identify the number of Manin components in a given *numerical* equivalence class of curves, one must understand the difference between algebraic and numerical equivalence for curves. Over $\mathbb{C}$, conjecturally there are $|\mathrm{Br}(\mathcal{X})|$ algebraic equivalence classes representing each numerical equivalence class. Indeed, this would be a consequence of triviality of the Griffiths group and the integral Hodge conjecture for Fano fibrations as developed by [Voi06] and others.

Remark 4.16. The main geometric structure carried by spaces of sections is the breaking-and-gluing structure. That is, using Bend-and-Break one can break off π-vertical rational curves from any section that has sufficiently many deformations. Conversely, starting from any section one can obtain a new section by gluing on sufficiently many π-vertical free rational curves and smoothing.

In practice, the validity of Conjecture 4.14 depends on whether the general fibers of π admit sufficiently many very free rational curves. This is one reason we have restricted our attention to fibrations whose generic fiber is globally F-regular.

5. Manin's conjecture over global function fields

In this section, we state two versions of Manin's conjecture over global function fields. The first version is the standard one based on ideas of [FMT89], [BM90], [Pey95], [BT98a], [Pey03], [LST22], and [LS24] in characteristic 0. The second version is based on the language of Manin components and the all-height-approach of Peyre [Pey17].

5.1. The standard formulation.

Let us describe the set up: let $k = \mathbb{F}_q$ be a finite field and let B be a smooth geometrically integral projective curve defined over k with function field $K(B)$. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. We are interested in counting rational points of the set

$$\mathcal{X}_\eta(K(B)).$$

To this end, we discuss how to define the correct counting function for this set. First, unlike over number fields, the height function only takes the values of q^n where n is an integer. For this reason it is necessary to consider the following definitions:

Definition 5.1. We define the *minimal degree* by

$$m(\pi) := \min\{-K_{\mathcal{X}/B}.C \mid [C] \in \mathrm{Sec}(\mathcal{X}/B)(k)\}.$$

By the Northcott property as in [LT24, Lemma 2.6] and [LT22, Lemma 2.2], this minimum is well-defined. Next we define the *index* by

$$r(\pi) := \min\{-K_{\mathcal{X}/B}.\alpha \,|\, \alpha \in N_1(\mathcal{X})_{\mathbb{Z}}, \mathcal{X}_b.\alpha = 0, -K_{\mathcal{X}/B}.\alpha > 0\}.$$

Note that for any section $[C] \in \mathrm{Sec}(\mathcal{X}/B)(k)$, we have

$$-K_{\mathcal{X}/B}.C = m(\pi) + r(\pi)d,$$

for some non-negative integer $d \geq 0$.

Next we fix an intersection profile λ. We define the *minimal degree with respect to* λ by

$$m(\pi, \lambda) :=$$
$$\min\{-K_{\mathcal{X}/B}.C \,|\, [C] \in \mathrm{Sec}(\mathcal{X}/B)(k), \text{ the intersection profile of } C \text{ is } \lambda\}.$$

Next we define the *intersection profile index* by

$$r(\pi)' := \min\{-K_{\mathcal{X}/B}.\alpha \,|\, \alpha \in N_1(\mathcal{X}_\eta) \cap N_1(\mathcal{X})_{\mathbb{Z}}, -K_{\mathcal{X}/B}.\alpha > 0\}.$$

Note $r(\pi)'$ does not depend on λ. For any section $[C] \in \mathrm{Sec}(\mathcal{X}/B)(k)$ with intersection profile λ, we have

$$-K_{\mathcal{X}/B}.C = m(\pi, \lambda) + r(\pi)'d,$$

for some non-negative integer $d \geq 0$. It is clear from the definition that we have $r(\pi)|r(\pi)'$.

We introduce the following definition to specify the exceptional set:

Definition 5.2 ([LST22], [LRT25, Definition 4.7])**.** Let $f : \mathcal{Y}_\eta \to \mathcal{X}_\eta$ be a thin map from a geometrically integral smooth projective variety over $K(B)$. We say f is a *breaking thin map* if either there is a strict inequality

$$(a(\mathcal{X}_\eta, -K_{\mathcal{X}_\eta}), b(K(B), \mathcal{X}_\eta, -K_{\mathcal{X}_\eta})) < (a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}), b(K(B), \mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}))$$

in the lexicographic order or if equality holds and f is an exceptional map.

Inspired by [LST22], we define the *exceptional set* to be

$$\mathsf{Z} := \bigcup_f f(\mathcal{Y}_\eta(K(B))) \subset \mathcal{X}_\eta(K(B)) = \mathrm{Sec}(\mathcal{X}/B)(k)$$

where f runs over all breaking thin maps. The following conjecture has been proved in [LST22, Theorem 5.7] in characteristic 0:

Conjecture 5.3. The exceptional set $\mathsf{Z} \subset \mathcal{X}_\eta(K(B))$ is a thin subset of $\mathcal{X}_\eta(K(B))$.

With these definitions, we can set up the counting function whose asymptotic behavior we are hoping to understand:

Definition 5.4. The standard counting function assigns to any non-negative integer d the quantity

$$N_{\text{stan}}(\pi, d) := \#\{[C] \in (\mathrm{Sec}(\mathcal{X}/B)(k) \setminus \mathsf{Z}) \mid -K_{\mathcal{X}/B}.C \leq m(\pi) + r(\pi)d\}.$$

Now our goal is to describe the prediction of the asymptotic formula for this counting function as $d \to \infty$. First, we introduce the alpha constant and beta constant:

Definition 5.5. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. To the vector space of real 1-cycles $N_1(\mathcal{X}_\eta)$ we assign the Lebesgue measure such that the fundamental domain of the lattice $N_1(\mathcal{X}_\eta) \cap V_{\mathbb{Z}}$ has volume equal to 1. Let $\mathsf{C} \subset \mathrm{Nef}_1(\mathcal{X}_\eta) \subset N_1(\mathcal{X}_\eta)$ be a closed cone. We define the *alpha constant* of C as

$$\alpha(\mathcal{X}_\eta, \mathsf{C}) = (\dim\, N_1(\mathcal{X}_\eta))\,.\mathrm{vol}(\{\alpha \in \mathsf{C} \mid -K_{\mathcal{X}/B}.\alpha \leq 1\}).$$

When $\mathsf{C} = \mathrm{Nef}_1(\mathcal{X}_\eta)$, we write $\alpha(\mathcal{X}_\eta, \mathsf{C}) = \alpha(\mathcal{X}_\eta)$.

Definition 5.6. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. The *beta constant* is

$$\beta(\mathcal{X}_\eta) = \#(\mathrm{Br}(\mathcal{X}_\eta)/\mathrm{Br}(K(B))).$$

A natural question is whether the beta constant is finite under the assumption that $\mathcal{X}_\eta$ is smooth Fano and geometrically globally F-regular. The following proposition is an affirmative answer to this question:

Proposition 5.7. *Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. Then $\beta(\mathcal{X}_\eta)$ is finite.*

Proof. We address the algebraic and transcendental contributions to $\beta(\mathcal{X}_\eta)$ separately. We first claim that $\mathrm{Pic}(\mathcal{X}_{k(\eta)^s})$ is torsion-free. By [CTS21, Corollary 5.1.3], we have $\mathrm{Pic}(\mathcal{X}_{k(\eta)^s}) \cong \mathrm{Pic}(\mathcal{X}_{\overline{\eta}})$ and so it suffices to show that $\mathrm{Pic}(\mathcal{X}_{\overline{\eta}})$ is torsion free. Suppose that D is a divisor on $\mathcal{X}_{\overline{\eta}}$ that is numerically trivial. By Grothendieck-Riemann-Roch and Corollary 2.10 $\chi(\mathcal{O}_{\mathcal{X}_{\overline{\eta}}}(D)) = \chi(\mathcal{O}_{\mathcal{X}_{\overline{\eta}}}) = 1$. By Theorem 2.9 we see that $H^0(\mathcal{X}_{\overline{\eta}}, \mathcal{O}_{\mathcal{X}_{\overline{\eta}}}(D)) = 1$. Thus $D \sim 0$.

Because of the torsion-freeness, [Ser79, Chapter VII, Proposition 4] and [Ser79, Chapter VIII, Corollary 2] show that $H^1(K(B), \mathrm{Pic}(\mathcal{X}_{k(\eta)^s}))$ is finite. Since $\mathcal{X}_\eta$ has a rational point, the Hochschild-Serre spectral sequence

$$0 \to \mathrm{Br}(K(B)) \to \mathrm{Br}_1(\mathcal{X}_\eta) \to H^1(K(B), \mathrm{Pic}(\mathcal{X}_{k(\eta)^s})) \to 0$$

implies the finiteness of $\mathrm{Br}_1(\mathcal{X}_\eta)/\mathrm{Br}(K(B))$.

To prove that $\beta(\mathcal{X}_\eta)$ is finite, by [CTS21, Theorem 5.2.5(ii)] it suffices to prove the finiteness of the geometric Brauer group $\mathrm{Br}(\mathcal{X}_{\overline{\eta}})$. Since a smooth Fano variety over an algebraically closed field is rationally chain connected, an argument of Starr as explained in [GJ18, Proposition 4.2] shows that there is some integer N that uniformly bounds the order of any element of $\mathrm{Br}(\mathcal{X}_{\overline{\eta}})$. [CTS21, Corollary 5.2.8] shows that the the prime to p part of $\mathrm{Br}(\mathcal{X}_{\overline{\eta}})$ is finite. For the p-primary part, it follows from [Sko25, Appendix Theorem A.1] that we have an isomorphism

$$\mathrm{Br}(\mathcal{X}_{\overline{\eta}})\{p\} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n \oplus H^3(\mathcal{X}_{\overline{\eta}}, \mathbb{Z}_p(1))\{p\},$$

where $M\{p\}$ denotes the p-primary part of a group M. Since $\mathrm{Br}(\mathcal{X}_{\overline{\eta}})\{p\}$ is annihilated by some power of p, n has to be zero. Then

$$H^3(\mathcal{X}_{\overline{\eta}}, \mathbb{Z}_p(1))\{p\},$$

can be interpreted as the extension of a finite group by the group of $\overline{K(B)}$-points on a connected unipotent group U (see e.g. [Mil76, Cor. 2.7 (a)]). Since the Picard scheme is smooth, [GSY25, Proposition 3.4] shows that the formal Brauer group of $\mathcal{X}_{\overline{\eta}}$ is representable and the dimension of the tangent space is $h^2(\mathcal{X}_{\overline{\eta}}, \mathcal{O}_{\mathcal{X}_{\overline{\eta}}})$. Then [GSY25, Equation (19)] shows that

$$h^2(\mathcal{X}_{\overline{\eta}}, \mathcal{O}_{\mathcal{X}_{\overline{\eta}}}) \geq \dim \mathsf{U}.$$

Since we have $h^2(\mathcal{X}_{\overline{\eta}}, \mathcal{O}_{\mathcal{X}_{\overline{\eta}}}) = 0$, U is 0. Thus our assertion follows. □

The main ingredient of the leading constant of the asymptotic formula is the Tamagawa number which was first introduced by Peyre [Pey95] for the anticanonical divisor and Batyrev–Tschinkel [BT98a] for arbitrary big divisors. A modern account is [CLT10], and we closely follow its exposition. First we need to introduce the local Tamagawa measures:

Definition 5.8. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. Let $b \in |B|$ be a closed point and denote the completion of $K(B)$ with respect to b by $K(B)_b$. Let ω be a top degree rational differential form on $\mathcal{X}_{K(B)_b}$ and $\mathrm{div}(\omega)$ be its corresponding divisor. Let Ω be the flat closure of $\mathrm{div}(\omega)$ in $\pi_b : \mathcal{X}_{\mathfrak{o}_b} \to \mathrm{Spec}\,\mathfrak{o}_b$ where $\mathfrak{o}_b$ is the ring of integers for the local field $K(B)_b$ and π_b is the base change of π. We define the local function

$$\|\omega\| : (\mathcal{X} \setminus \mathrm{Supp}(\Omega))(K(B)_b) \to \mathbb{R}_{>0},$$

that assigns to any $x \in (\mathcal{X} \setminus \mathrm{Supp}(\Omega))(K(B)_b)$ the quantity

$$\|\omega\|(x) = q_b^{-v_b(\sigma^*\Omega)},$$

where $\sigma : \operatorname{Spec} \mathfrak{o}_b \to \mathcal{X}_{\mathfrak{o}_v}$ is the corresponding jet, v_b is the discrete valuation of $\mathfrak{o}_v$ and q_b is the size of the residue field.

Now ω induces a Radon measure $|\omega|$ on $\mathcal{X}(K(B)_b)$ and we define the local Tamagawa measure by

$$\tau_{\mathcal{X},b} = \frac{|\omega|}{\|\omega\|}.$$

Note that this definition does not depend on the choice of ω and it only depends on our model $\pi_b : \mathcal{X}_{\mathfrak{o}_v} \to \operatorname{Spec} \mathfrak{o}_b$.

Next, we introduce the local convergence factors:

Definition 5.9. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. Let $\Sigma \subset |B|$ be a finite set such that π is smooth outside of Σ. We define the *local convergence factor* λ_b for $b \in |B|$ by

$$\lambda_b = \begin{cases} \det(1 - q_b^{-1}\mathrm{Fr}_{\kappa(b)})^{-1} & \text{if } b \in B \backslash \Sigma \\ 1 & \text{if } b \in \Sigma \end{cases},$$

where $\kappa(b)$ is the residue field at b and $\mathrm{Fr}_{\kappa(b)}$ is the geometric Frobenius acting on $\mathrm{Pic}(\mathcal{X}_{\overline{\kappa(b)}}) \otimes \mathbb{Q}$. We also define

$$L_{\Sigma,*}(1, \mathrm{Pic}(\mathcal{X}_{\overline{\eta}}) \otimes \mathbb{Q}) := \lim_{t \to 1-} (1-t)^{\rho(\mathcal{X}_\eta)} \prod_{b \in |B \backslash \Sigma|} \det(1 - q_b^{-1} t^{|b|} \cdot \mathrm{Fr}_{\kappa(b)})^{-1},$$

where $|b| = [\kappa(b) : k]$. This limit exists as a positive real number. We define the *Tamagawa measure* $\tau_{\mathcal{X}}$ on the adelic space

$$\mathcal{X}_\eta(\mathbb{A}_{K(B)}) := \prod_{b \in |B|} \mathcal{X}(K(B)_b),$$

by

$$\tau_{\mathcal{X}} := L_{\Sigma,*}(1, \mathrm{Pic}(\mathcal{X}_{\overline{\eta}}) \otimes \mathbb{Q}) \prod_{b \in |B|} \lambda_b^{-1} \tau_{\mathcal{X},b}.$$

Since we have $h^i(\mathcal{X}_\eta, \mathcal{O}_{\mathcal{X}_\eta}) = 0$ for $i = 1, 2$, using the Weil estimates [CLT10, Section 2] proves that this measure is well-defined as a Borel measure and that $\tau_{\mathcal{X}}(\mathcal{X}(\mathbb{A}_{K(B)}))$ is a finite positive real number as soon as $\mathcal{X}(\mathbb{A}_{K(B)})$ is non-empty.

Finally we define the *Tamagawa number* $\tau_{\mathcal{X}}(-\mathcal{K}_{\mathcal{X}_\eta})$ by

$$\tau_{\mathcal{X}}(-\mathcal{K}_{\mathcal{X}_\eta}) := q^{m(\pi) + n(1 - g(B))} \int_{\mathcal{X}_\eta(\mathbb{A}_{K(B)})^{\mathrm{Br}(\mathcal{X}_\eta)}} \mathrm{d}\tau_{\mathcal{X}},$$

where $n = \dim \mathcal{X}_\eta$. Note that by Proposition 5.7, the set

$$\mathrm{Br}(\mathcal{X}_\eta)/\mathrm{Br}(K(B)),$$

is a finite set, so the Brauer–Manin set

$$\mathcal{X}_\eta(\mathbb{A}_{K(B)})^{\mathrm{Br}(\mathcal{X}_\eta)} \subset \mathcal{X}_\eta(\mathbb{A}_{K(B)}),$$

is an open and closed set on the adelic space. Colliot-Thélène's conjecture predicts that

$$\mathcal{X}_\eta(K(B)) \subset \mathcal{X}_\eta(\mathbb{A}_{K(B)})^{\mathrm{Br}(\mathcal{X}_\eta)},$$

is dense so that the above integral is expected to be an integration over $\overline{\mathcal{X}_\eta(K(B))}$.

Finally we state a provisional version of Manin's conjecture over global function fields:

Provisional Conjecture 5.10 (Standard Manin's conjecture over global function fields)**.** Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. Suppose that $\mathcal{X}_\eta(K(B))$ is not thin. Then

$$N_{\mathrm{stan}}(\pi, d) \sim (1-q^{-r(\pi)})^{-1}\alpha(\mathcal{X}_\eta)\beta(\mathcal{X}_\eta)r(\pi)\tau_{\mathcal{X}}(-\mathcal{K}_{\mathcal{X}_\eta})q^{r(\pi)d}(dr(\pi))^{\rho(\mathcal{X}_\eta)-1},$$

as $d \to \infty$ with $d \in \mathbb{Z}$.

Remark 5.11. We expect that when

$$r(\pi) = r(\pi)', \tag{5.1}$$

is true, Provisional Conjecture 5.10 holds. Note that (5.1) is satisfied when every fiber of π is integral. However, when (5.1) fails, the asymptotic formula may have some periodicity observed by [LL25, Example 1.1.4] in the context of Malle's conjecture. [San25] suggests that we might need some averaging over degrees. This is the main reason why Conjecture 5.10 is provisional.

5.2. **All height approach.** It is also natural to consider the following modification of our counting problem. For each algebraic curve class $\alpha \in \mathsf{S}_{\mathcal{X}/B}$ consider the quantity

$$\frac{\#M_\alpha(k)}{q^{-K_{\mathcal{X}/B}.\alpha+n(1-g(B))}}$$

where M_α denotes a Manin component associated to α. We would like to have an asymptotic formula describing the limit of this quantity as $-K_{\mathcal{X}/B}.\alpha$ goes to ∞. As before, it is important to exclude some loci of M_α since such loci can affect the leading constant. To this end, we would like to propose the following definition:

Definition 5.12. A thin set $Z \subset \mathcal{X}_\eta(K(B))$ is properly constructible if there are finitely many finite thin B-maps $f_i : \mathcal{Y}_i \to \mathcal{X}$ from geometrically integral projective B-varieties $\mathcal{Y}_i \to B$ such that

$$Z = \bigcup_i f_i(\mathcal{Y}_{i,\eta}(K(B))).$$

We conjecture the following:

Conjecture 5.13. The exceptional set $\mathsf{Z} \subset \mathcal{X}_\eta(K(B))$ is a properly constructible thin set.

We should note that this conjecture is still open even in characteristic 0. Using this property, one can construct an open subscheme $M_\alpha^\circ \subset M_\alpha$ by removing the exceptional set:

Proposition 5.14. *Let $f_i : \mathcal{Y}_i \to \mathcal{X}$ be finitely many finite thin B-maps. For any extension k'/k, let $Z_{k'} \subset \mathcal{X}_\eta(K(B)\otimes_k k') = \mathrm{Sec}(\mathcal{X}/B)(k')$ be the properly constructible thin set defined by $f_{i,k'}$'s. Then for any irreducible component $M \subset \mathrm{Sec}(\mathcal{X}/B)$, there exists a unique open subscheme $M^\circ \subset M$ such that for any extension k'/k we have*

$$M^\circ(k') = M(k') \setminus Z_{k'}.$$

Proof. We may assume that k is algebraically closed. Then our claim follows from the fact that for any finite B-morphism $f : \mathcal{Y} \to \mathcal{X}$, the induced map

$$f_* : \overline{M}_{g(B),0}(\mathcal{Y}) \to \overline{M}_{g(B),0}(\mathcal{X})$$

is proper. Note that since f is finite, for any stable map that is numerically equivalent to a section its composition with f is still a stable map. This means f_* maps a stable map with reducible domain to a stable map with reducible domain. Thus our assertion follows. ☐

We next set up the counting problem. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. Let

$$\mathsf{Z} \subset \mathrm{Sec}(\mathcal{X}/B)(k),$$

be the exceptional set. By Conjecture 5.13, Z is defined by finitely many finite thin B-maps $\{f_i\}$. Let λ be an intersection profile. Recall that S_λ denotes the set of real nef classes of section type which are compatible with the intersection profile λ. We fix a rational max norm on $N_1(\mathcal{X})$.

Definition 5.15. We let $\ell_\lambda : \mathsf{S}_\lambda \to \mathbb{R}_{\geq 0}$ denote the rational piecewise linear function that measures the distance to the relative boundary of S_λ.

Let α be an algebraic class of a section such that its numerical class is contained in S_λ. We assume $\ell_\lambda(\alpha)$ is sufficiently large so that by Conjecture 4.14, there exists a unique Manin component M_α. Let $M_\alpha^\circ \subset M_\alpha$ be the open subscheme induced by f_i's. With these setups, we have the following conjecture. There are very similar conjectures proposed by David Bourqui (see [Bou11a, Question 1.10] and [Bou13, Definition 2.2]) and Emmanuel Peyre (see [Pey21, Question 6.27]):

Conjecture 5.16 (All height approach version of Manin's conjecture)**.** There exists constants $c(\mathcal{X},\lambda) \geq 0$ and $\delta > 0$ such that we have

$$\#M_\alpha^\circ(k) = c(\mathcal{X},\lambda)q^{-K_{\mathcal{X}/B}.\alpha+n(1-g(B))} + O(q^{-K_{\mathcal{X}/B}.\alpha-\delta\ell_\lambda(\alpha)}),$$

as $\ell_\lambda(\alpha) \to \infty$. Here $n = \dim \mathcal{X}_\eta$.

Moreover, one can provide a precise prediction for the value of $c(\mathcal{X},\lambda)$. Let $\pi : \mathcal{X} \to B$ be a good Fano fibration. Let $\Sigma_\pi \subset |B|$ be the set of closed points $b \in |B|$ such that the fiber $\mathcal{X}_b \to \operatorname{Spec} \kappa(b)$ is not smooth. We denote the smooth locus of this fiber by $\mathcal{X}_b^{\mathrm{sm}}$. Since for any B-morphism $\sigma : \operatorname{Spec} K(B)_b \to \mathcal{X}$, its specialization $\sigma(b)$ lands in $\mathcal{X}_b^{\mathrm{sm}}(\kappa(b))$, we have the following continuous map:

$$\Phi : \mathcal{X}_\eta(\mathbb{A}_{K(B)})^{\mathrm{Br}(\mathcal{X}_\eta)} \subset \prod_{b\in|B|} \mathcal{X}_\eta(K(B)_b) \to \prod_{b\in\Sigma_\pi} \mathcal{X}_b^{\mathrm{sm}}(\kappa(b)).$$

For an intersection profile λ and $b \in \Sigma_\pi$, let $\mathcal{X}_{b,\lambda}^{\mathrm{sm}}$ be the component of $\mathcal{X}_b^{\mathrm{sm}}$ specified by the intersection profile λ. Then we define

$$\mathcal{X}_\eta(\mathbb{A}_{K(B)})_\lambda^{\mathrm{Br}(\mathcal{X}_\eta)} := \Phi^{-1}\left(\prod_{b\in\Sigma_\pi} \mathcal{X}_{b,\lambda}^{\mathrm{sm}}(\kappa(b))\right),$$

which is an open and closed subset of $\mathcal{X}_\eta(\mathbb{A}_{K(B)})$ because of Proposition 5.7. Here is a conjectural description of $c(\mathcal{X},\lambda)$:

Conjecture 5.17. In Conjecture 5.16, we have

$$c(\mathcal{X},\lambda) = \frac{\beta(\mathcal{X}_\eta)}{\mathsf{B}} \int_{\mathcal{X}_\eta(\mathbb{A}_{K(B)})_\lambda^{\mathrm{Br}(\mathcal{X}_\eta)}} \mathrm{d}\tau_{\mathcal{X}},$$

where B is the number of algebraic equivalence classes representing one single numerical class. It is natural to wonder whether B is equal to $\#\mathrm{Br}(\mathcal{X})$.

5.3. **Relationship between conjectures.** Finally we give a heuristic argument explaining how the all height approach (when combined with earlier conjectures) implies a version of Provisional Conjecture 5.10.

Assumption 5.18. *In this subsection we assume the validity of Conjectures 4.9, 4.13, 4.14, 5.16, and 5.17. We also assume that (5.1) holds to avoid some lattice issues.*

Consider the following counting function:

$$N_{\text{Manin}}(\pi, d) := \sum_{\lambda} \sum_{\substack{\alpha \in \mathsf{S}_{\lambda,\mathbb{Z}} \\ -K_{\mathcal{X}/B}.\alpha \leq m(\pi)+dr(\pi)}} \sum_{M_\alpha} \#M^\circ_\alpha(k),$$

where λ runs over all intersection profiles and M_α runs over all Manin components associated to a numerical class α. Note that $N_{\text{Manin}}(\pi, d) \leq N_{\text{stan}}(\pi, d)$, but they need not agree when there are exceptional components that do not come from breaking thin maps. We expect that the difference between $N_{\text{stan}}(\pi, d)$ and $N_{\text{Manin}}(\pi, d)$ is asymptotically negligible, but do not have a rigorous proof. Here we focus on $N_{\text{Manin}}(\pi, d)$.

Let $\epsilon > 0$ be a sufficiently small rational number. Then we define the following shrunken set:

$$\mathsf{S}_{\lambda,\epsilon} := \{\alpha \in \mathsf{S}_\lambda \mid \ell_\lambda(\alpha) > -\epsilon K_{\mathcal{X}/B}.\alpha\}.$$

In the view of Conjecture 4.9, this is a rational polyhedral convex set with the recession cone $\mathrm{Nef}_1(\mathcal{X}_\eta)_\epsilon \subset \mathrm{Nef}_1(\mathcal{X}_\eta)$. By Conjecture 4.14, there exists $\alpha_0 \in \mathsf{S}_{\lambda,\mathbb{Z}}$, such that for any $\alpha \in \alpha_0 + \mathrm{Nef}_{\text{vert},\mathbb{Z}}$, there is a unique Manin component for any algebraic class whose numerical class is α. We denote those Manin components by $M_{\alpha,i}$ for $i = 1, \cdots, \mathsf{B}$. Now note that when $-K_{\mathcal{X}/B}.\alpha$ is sufficiently large, any class in $\mathsf{S}_{\lambda,\epsilon}$ is contained in $\alpha_0 + \mathrm{Nef}_{\text{vert},\mathbb{Z}}$. To compute the asymptotic behavior, we may assume that every class α in $\mathsf{S}_{\lambda,\epsilon,\mathbb{Z}}$ is represented by B Manin components.

Let us consider the counting function for classes in $\mathsf{S}_{\lambda,\epsilon,\mathbb{Z}}$:

$$\sum_{\substack{\alpha \in \mathsf{S}_{\lambda,\epsilon,\mathbb{Z}} \\ -K_{\mathcal{X}/B}.\alpha \leq m(\pi)+dr(\pi)}} \sum_{i=1}^{\mathsf{B}} \#M^\circ_{\alpha,i}(k).$$

By Conjectures 5.16 and 5.17, this is equal to

$$\sum_{\substack{\alpha \in \mathsf{S}_{\lambda,\epsilon,\mathbb{Z}} \\ -K_{\mathcal{X}/B}.\alpha \leq m(\pi)+dr(\pi)}} \left(\beta(\mathcal{X}_\eta)\tau_{\mathcal{X}}(\mathcal{X}_\eta(\mathbb{A}_{K(B)})_\lambda^{\mathrm{Br}(\mathcal{X}_\eta)})q^{-K_{\mathcal{X}/B}.\alpha+n(1-g(B))} + O(q^{-(1-\delta\epsilon)K_{\mathcal{X}/B}.\alpha})\right).$$

Then using the counting arguments of [LRT25, Proposition 4.3] (combined with Conjecture 4.9 and the argument of [LRT25, Theorem 5.7]), the asymptotic formula for the sum is given by

$$(1-q^{-r(\pi)})^{-1}\alpha(\mathcal{X}_\eta, \mathrm{Nef}_1(\mathcal{X}_\eta)_\epsilon)\beta(\mathcal{X}_\eta)r(\pi)\tau_{\mathcal{X}}(\mathcal{X}_\eta(\mathbb{A}_{K(B)})_\lambda^{\mathrm{Br}(\mathcal{X}_\eta)})q^{r(\pi)d}(dr(\pi))^{\rho(\mathcal{X}_\eta)-1},$$

as $d \to \infty$ with $d \in \mathbb{Z}$. Hence after taking the summation over λ, we obtain

$$(1-q^{-r(\pi)})^{-1}\alpha(\mathcal{X}_\eta, \mathrm{Nef}_1(\mathcal{X}_\eta)_\epsilon)\beta(\mathcal{X}_\eta)r(\pi)\tau_{\mathcal{X}}(-\mathcal{K}_{\mathcal{X}_\eta})q^{r(\pi)d}(dr(\pi))^{\rho(\mathcal{X}_\eta)-1}.$$

To finish the argument, we introduce one more conjecture:

Conjecture 5.19. There exists uniform constants B' and C such that for any $\alpha \in \mathsf{S}_{\lambda,\mathbb{Z}}$, the number of Manin components of the class α is bounded by B' and for such a Manin component M_α, we have

$$\# M_\alpha^\circ(k) \leq \mathsf{C} q^{-K_{\mathcal{X}/B}.\alpha+n(1-g(B))}.$$

Since $\mathsf{S}_{\lambda,\epsilon,\mathbb{Z}} \subset \mathsf{S}_{\lambda,\mathbb{Z}}$ and the difference is controlled by the region $\mathrm{Nef}_1(\mathcal{X}_\eta)\backslash\mathrm{Nef}_1(\mathcal{X}_\eta)_\epsilon$, we have

$$\begin{aligned}
&(1-q^{-r(\pi)})^{-1}\alpha(\mathcal{X}_\eta, \mathrm{Nef}_1(\mathcal{X}_\eta)_\epsilon)\beta(\mathcal{X}_\eta)r(\pi)\tau_{\mathcal{X}}(-\mathcal{K}_{\mathcal{X}_\eta}) \\
&\leq \liminf_{d\to\infty} \frac{N_{\mathrm{Manin}}(\pi,d)}{q^{r(\pi)d}(dr(\pi))^{\rho(\mathcal{X}_\eta)-1}} \\
&\leq \limsup_{d\to\infty} \frac{N_{\mathrm{Manin}}(\pi,d)}{q^{r(\pi)d}(dr(\pi))^{\rho(\mathcal{X}_\eta)-1}} \\
&\leq (1-q^{-r(\pi)})^{-1}\alpha(\mathcal{X}_\eta, \mathrm{Nef}_1(\mathcal{X}_\eta)_\epsilon)\beta(\mathcal{X}_\eta)r(\pi)\tau_{\mathcal{X}}(-\mathcal{K}_{\mathcal{X}_\eta}) \\
&+ (1-q^{-r(\pi)})^{-1}\mathsf{C}\mathsf{B}'\alpha(\mathcal{X}_\eta, \mathrm{Nef}_1(\mathcal{X}_\eta)\setminus \mathrm{Nef}_1(\mathcal{X}_\eta)_\epsilon))r(\pi).
\end{aligned}$$

As $\epsilon \to 0$, we conclude

$$\lim_{d\to\infty} \frac{N_{\mathrm{Manin}}(\pi,d)}{q^{r(\pi)d}(dr(\pi))^{\rho(\mathcal{X}_\eta)-1}} = (1-q^{-r(\pi)})^{-1}\alpha(\mathcal{X}_\eta)\beta(\mathcal{X}_\eta)r(\pi)\tau_{\mathcal{X}}(-\mathcal{K}_{\mathcal{X}_\eta}).$$

6. Examples

Finally, we discuss a few examples where (weaker forms of) the conjectures of the previous sections have been verified. We do not attempt to give a complete account of the literature, instead focusing on a few notable examples.

6.1. **Toric varieties.** Manin's conjecture for smooth projective toric varieties over $\mathbb{F}_q(t)$ has been settled by David Bourqui in [Bou03] and [Bou11b]. The first paper used the method of universal torsors developed by Salberger [Sal98] over number fields and the second paper used harmonic analysis on tori following ideas of Batyrev–Tschinkel in [BT96] and [BT98b]. Indeed, using these methods, Bourqui obtained an analytic continuation of the height zeta function. Then Conjecture 5.10 follows from the Tauberian theorem as in [CLT10, Corollary A.13]. Bourqui also proves the all height version of Manin's Conjecture in [Bou11a, Section 2.9].

Bourqui's pioneering work has been quite influential. In fact [Bou09] proved a motivic version of Manin's Conjecture for toric varieties; his work has been revisited in [BDH22] and [Fai25].

6.2. **Low degree hypersurfaces.** Manin's conjecture over global function fields has been proved for low degree hypersurfaces in $\mathbb{P}^n$ with p being greater than the degree using the Hardy–Littlewood circle method which is a technique from analytic number theory. See [Lee11], [BV17], [BS23], [Saw24], and [HL25] for more details.

6.3. **Homological sieve method.** Let $k = \mathbb{F}_q$ be a finite field and S be a split smooth del Pezzo surface of degree $d \leq 7$ defined over k. Here split means that we have $\rho(S) = \rho(S_{\overline{k}})$. [DLTT25] and [Tan25b] studied Manin's conjecture for the trivial family $\pi : S \times \mathbb{P}^1 \to \mathbb{P}^1$. This is based on the homological sieve method developed in [DLTT25]. Let us state the main result of [DLTT25]: we assume that $d = 4$. Let ℓ be non-negative, rational, homogeneous, continuous, and piecewise linear function on the nef cone $\mathrm{Nef}_1(S)$. Let $\epsilon > 0$ be a sufficiently small rational number. Then we define the shrunken nef cone by

$$\mathrm{Nef}_1(S)_\epsilon := \{\alpha \in \mathrm{Nef}_1(S) \,|\, \ell(\alpha) \geq -\epsilon K_S.\alpha\}.$$

We define the counting function by

$$N_{\mathrm{stan},\epsilon}(\pi, d) := \sum_{\alpha \in \mathrm{Nef}_1(S)_{\epsilon,\mathbb{Z}}, -K_S.\alpha \leq d} \#\mathrm{Mor}(\mathbb{P}^1, S, \alpha)(k),$$

where $\mathrm{Mor}(\mathbb{P}^1, S, \alpha)$ is the morphism scheme parametrizing morphisms $s : \mathbb{P}^1 \to S$ such that $s_*[\mathbb{P}^1] = \alpha$. Here is the main theorem of [DLTT25]:

Theorem 6.1 ([DLTT25, Theorem 1.1])**.** *Fix a sufficiently small rational number $\epsilon > 0$. Let q be a power of a prime number such that $q^\epsilon > 2^{32}$. Let S be a split smooth quartic del Pezzo surface defined over $\mathbb{F}_q$. Then there exists a non-negative, rational, homogeneous, continuous, and piecewise linear function ℓ on $\mathrm{Nef}_1(S)$ which does not depend on ϵ and q and takes positive values on a dense open cone $U \subset \mathrm{Nef}_1(S)$ such that*

$$N_{\mathrm{stan},\epsilon}(\pi, d) \sim (1 - q^{-1})^{-1}\alpha(S, \mathrm{Nef}_1(S)_\epsilon)\tau_{S\times\mathbb{P}^1}(-\mathcal{K}_{\mathcal{X}_\eta})q^d d^5,$$

as $d \to \infty$.

The method of the proof, developed by Das, Tosteson, and the authors, is called the homological sieve method. Its ingredients are the following:

- algebraic geometry (birational geometry of moduli spaces of rational curves on smooth quartic del Pezzo surfaces);
- arithmetic geometry (simplicial schemes, their homotopy theory and Grothendieck–Lefschetz trace formula);
- algebraic topology (the inclusion–exclusion principle and the Vassiliev type method of bar complexes), and;
- elementary analytic number theory.

To our knowledge, this is the first time that such a geometric and topological method was used to establish Manin's conjecture over global function fields for highly non-trivial examples. Moreover, the second author applied this method to study split del Pezzo surfaces of degree ≥ 5 in [Tan25b]. Readers interested in this method should consult [DLTT25].

Remark 6.2. Over $\mathbb{C}$, the Cohen–Jones–Segal conjecture predicts that the homology of the irreducible components $M_\alpha \subset \mathrm{Mor}(B, X)$ stabilizes to the homology of the space of continuous maps $\mathrm{Top}(B, X)_\alpha$ as α increases. [DLTT25] also proved a version of the Cohen–Jones–Segal conjecture for quartic del Pezzo surfaces over $\mathbb{C}$ using similar techniques. This is based on the method of the bar complexes developed by Das–Tosteson in [DT24] which settles the Cohen–Jones–Segal conjecture for quintic del Pezzo surfaces. An upcoming paper of Das, Tosteson, and the authors gives a general description of the relationship between Manin's conjecture and the Cohen–Jones–Segal conjecture.

References

[Bat88] V. V. Batyrev. Distribution of rational points of bounded height. a lecture at Math. Inst. Berlin, Thu 21th Jul, 1988.

[BCHM10] C. Birkar, P. Cascini, Chr. D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[BD25] Christian Bernert and Ulrich Derenthal. Points of bounded height on quintic del Pezzo surfaces over number fields. *Adv. Math.*, 482:Paper No. 110561, 41, 2025.

[BDH22] Margaret Bilu, Ronno Das, and Sean Howe. Zeta statistics and Hadamard functions. *Adv. Math.*, 407:Paper No. 108556, 68, 2022.

[BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Paun, and Thomas Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Algebraic Geom.*, 22(2):201–248, 2013.

[BHB17] Tim Browning and Roger Heath-Brown. Forms in many variables and differing degrees. *J. Eur. Math. Soc. (JEMS)*, 19(2):357–394, 2017.

[Bil23] Margaret Bilu. Motivic Euler products and motivic height zeta functions. *Mem. Amer. Math. Soc.*, 282(1396):v+185, 2023.

[Bir19] Caucher Birkar. Anti-pluricanonical systems on Fano varieties. *Ann. of Math. (2)*, 190(2):345–463, 2019.

[Bir21] Caucher Birkar. Singularities of linear systems and boundedness of Fano varieties. *Ann. of Math. (2)*, 193(2):347–405, 2021.

[Bir62] B. J. Birch. Forms in many variables. *Proc. Roy. Soc. London Ser. A*, 265:245–263, 1961/62.

[BJ22] Andrew Burke and Eric Jovinelly. Geometric Manin's Conjecture for Fano 3-folds, 2022.

[BK13] Roya Beheshti and N. Mohan Kumar. Spaces of rational curves on complete intersections. *Compos. Math.*, 149(6):1041–1060, 2013.

[BLRT22] Roya Beheshti, Brian Lehmann, Eric Riedl, and Sho Tanimoto. Moduli spaces of rational curves on Fano threefolds. *Adv. Math.*, 408:Paper No. 108557, 60, 2022.

[BLRT23] R. Beheshti, B. Lehmann, E. Riedl, and S. Tanimoto. Rational curves on del Pezzo surfaces in positive characteristic. *Trans. Amer. Math. Soc. Ser. B*, 10:407–451, 2023.

[BM90] V. V. Batyrev and Yu. I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.*, 286(1-3):27–43, 1990.

[Bou03] David Bourqui. Fonction zêta des hauteurs des variétés toriques déployées dans le cas fonctionnel. *J. Reine Angew. Math.*, 562:171–199, 2003.

[Bou09] D. Bourqui. Produit eulérien motivique et courbes rationnelles sur les variétés toriques. *Compos. Math.*, 145(6):1360–1400, 2009.

[Bou10] David Bourqui. Fonctions L d'Artin et nombre de Tamagawa motiviques. *New York J. Math.*, 16:179–233, 2010.

[Bou11a] David Bourqui. Asymptotic behaviour of rational curves, 2011.

[Bou11b] David Bourqui. Fonction zêta des hauteurs des variétés toriques non déployées. *Mem. Amer. Math. Soc.*, 211(994):viii+151, 2011.

[Bou12] D. Bourqui. Moduli spaces of curves and Cox rings. *Michigan Math. J.*, 61(3):593–613, 2012.

[Bou13] David Bourqui. Exemples de comptages de courbes sur les surfaces. *Math. Ann.*, 357(4):1291–1327, 2013.

[Bou16] D. Bourqui. Algebraic points, non-anticanonical heights and the Severi problem on toric varieties. *Proc. Lond. Math. Soc. (3)*, 113(4):474–514, 2016.

[BS23] T. Browning and W. Sawin. Free rational curves on low degree hypersurfaces and the circle method. *Algebra Number Theory*, 17(3):719–748, 2023.

[BT96] V. Batyrev and Yu. Tschinkel. Height zeta functions of toric varieties. volume 82, pages 3220–3239. 1996. Algebraic geometry, 5.

[BT98a] V. V. Batyrev and Y. Tschinkel. Tamagawa numbers of polarized algebraic varieties. *Astérisque*, (251):299–340, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).

[BT98b] Victor V. Batyrev and Yuri Tschinkel. Manin's conjecture for toric varieties. *J. Algebraic Geom.*, 7(1):15–53, 1998.

[BV17] T. D. Browning and P. Vishe. Rational curves on smooth hypersurfaces of low degree. *Algebra Number Theory*, 11(7):1657–1675, 2017.

[Che25] Raymond Cheng. Free curves in Fano hypersurfaces must have high degree. *Proc. Amer. Math. Soc.*, 153(7):2841–2846, 2025.

[CLL16] Antoine Chambert-Loir and Fran¸cois Loeser. Motivic height zeta functions. *Amer. J. Math.*, 138(1):1–59, 2016.

[CLT02] Antoine Chambert-Loir and Yuri Tschinkel. On the distribution of points of bounded height on equivariant compactifications of vector groups. *Invent. Math.*, 148(2):421–452, 2002.

[CLT10] Antoine Chambert-Loir and Yuri Tschinkel. Igusa integrals and volume asymptotics in analytic and adelic geometry. *Confluentes Math.*, 2(3):351–429, 2010.

[CS09] I. Coskun and J. Starr. Rational curves on smooth cubic hypersurfaces. *Int. Math. Res. Not. IMRN*, (24):4626–4641, 2009.

[CS22] Izzet Coskun and Geoffrey Smith. Very free rational curves in Fano varieties. *J. Algebra*, 611:246–264, 2022.

[CTS21] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov. *The Brauer-Grothendieck group*, volume 71 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, [2021] ©2021.

[CZ14] Qile Chen and Yi Zhu. Very free curves on Fano complete intersections. *Algebr. Geom.*, 1(5):558–572, 2014.

[Das20] Omprokash Das. Finiteness of log minimal models and nef curves on 3-folds in characteristic $p > 5$. *Nagoya Math. J.*, 239:76–109, 2020.

[Der14] Ulrich Derenthal. Singular del Pezzo surfaces whose universal torsors are hypersurfaces. *Proc. Lond. Math. Soc. (3)*, 108(3):638–681, 2014.

[dlB02] Régis de la Bretèche. Nombre de points de hauteur bornée sur les surfaces de del Pezzo de degré 5. *Duke Math. J.*, 113(3):421–464, 2002.

[DlBB11] R. De la Bretèche and T. D. Browning. Manin's conjecture for quartic del Pezzo surfaces with a conic fibration. *Duke Math. J.*, 160(1):1–69, 2011.

[dlBBD07] Régis de la Bretèche, Tim D. Browning, and Ulrich Derenthal. On Manin's conjecture for a certain singular cubic surface. *Ann. Sci. École Norm. Sup. (4)*, 40(1):1–50, 2007.

[dlBBP12] Régis de la Bretèche, Tim Browning, and Emmanuel Peyre. On Manin's conjecture for a family of Châtelet surfaces. *Ann. of Math. (2)*, 175(1):297–343, 2012.

[DLTT25] Ronno Das, Brian Lehmann, Sho Tanimoto, and Philip Tosteson. Homological stability and manin's conjecture for rational curves on quartic del pezzo surfaces, 2025.

[DP19] Ulrich Derenthal and Marta Pieropan. Cox rings over nonclosed fields. *J. Lond. Math. Soc. (2)*, 99(2):447–476, 2019.

[DP20] Ulrich Derenthal and Marta Pieropan. The split torsor method for Manin's conjecture. *Trans. Amer. Math. Soc.*, 373(12):8485–8524, 2020.

[DT24] Ronno Das and Philip Tosteson. Homology of spaces of curves on blowups, 2024.

[Fai25] Loïs Faisant. Motivic distribution of rational curves and twisted products of toric varieties. *Algebra Number Theory*, 19(5):883–965, 2025.

[FM17] Christopher Frei and Manfred Madritsch. Forms of differing degrees over number fields. *Mathematika*, 63(1):92–123, 2017.

[FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel. Rational points of bounded height on Fano varieties. *Invent. Math.*, 95(2):421–435, 1989.

[Gao23] Runxuan Gao. A Zariski dense exceptional set in Manin's conjecture: dimension 2. *Res. Number Theory*, 9(2):Paper No. 42, 15, 2023.

[GJ18] Frank Gounelas and Ariyan Javanpeykar. Invariants of Fano varieties in families. *Mosc. Math. J.*, 18(2):305–319, 2018.

[Gla25] Jakob Glas. Linear growth and moduli spaces of rational curves, 2025.

[GLP$^+$15] Yoshinori Gongyo, Zhiyuan Li, Zsolt Patakfalvi, Karl Schwede, Hiromu Tanaka, and Runhong Zong. On rational connectedness of globally F-regular threefolds. *Adv. Math.*, 280:47–78, 2015.

[GNT19] Yoshinori Gongyo, Yusuke Nakamura, and Hiromu Tanaka. Rational points on log Fano threefolds over a finite field. *J. Eur. Math. Soc. (JEMS)*, 21(12):3759–3795, 2019.

[GSY25] Livia Grammatica, Alexei Skorobogatov, and Yuan Yang. Brauer groups of abelian varieties over fields of finite characteristic. 2025.

[HL25] Matthew Hase-Liu. A higher genus circle method and an application to geometric Manin's conjecture. 2025.

[HRS04] J. Harris, M. Roth, and J. Starr. Rational curves on hypersurfaces of low degree. *J. Reine Angew. Math.*, 571:73–106, 2004.

[HTT15] B. Hassett, S. Tanimoto, and Y. Tschinkel. Balanced line bundles and equivariant compactifications of homogeneous spaces. *Int. Math. Res. Not. IMRN*, (15):6375–6410, 2015.

[IP99] V. A. Iskovskikh and Y. G. Prokhorov. *Algebraic geometry. V*, volume 47 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1999. Fano varieties, A translation of ıt Algebraic geometry. 5 (Russian), Ross. Akad. Nauk, Vseross. Inst. Nauchn. i Tekhn. Inform., Moscow, Translation edited by A. N. Parshin and I. R. Shafarevich.

[Isk77] V. A. Iskovskih. Fano threefolds. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(3):516–562, 717, 1977.

[Isk79] V. A. Iskovskih. Anticanonical models of three-dimensional algebraic varieties. In *Current problems in mathematics, Vol. 12 (Russian)*, pages 59–157, 239 (loose errata). VINITI, Moscow, 1979.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.*, 36(3):765–779, 1992.

[KN22] Tatsuro Kawakami and Masaru Nagaoka. Pathologies and liftability of Du Val del Pezzo surfaces in positive characteristic. *Math. Z.*, 301(3):2975–3017, 2022.

[Kol95] János Kollár. Nonrational hypersurfaces. *J. Amer. Math. Soc.*, 8(1):241–249, 1995.

[KP01] B. Kim and R. Pandharipande. The connectedness of the moduli space of maps to homogeneous spaces. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 187–201. World Sci. Publ., River Edge, NJ, 2001.

[KT25] János Kollár and Zhiyu Tian. Stable maps of curves and algebraic equivalence of 1-cycles. *Duke Math. J.*, 174(5):911–947, 2025.

[Lee11] Siu-lun Alan Lee. Birch's theorem in function fields. 2011.

[LL25] Aaron Landesman and Ishan Levy. Homological stability for Hurwitz spaces and applications. 2025.

[LRT25] Brian Lehmann, Eric Riedl, and Sho Tanimoto. Non-free sections of Fano fibrations. *Mem. Amer. Math. Soc.*, 2025. to appear, arXiv:2301.01695 [math.AG].

[LS24] D. Loughran and T. Santens. Malle's conjecture and Brauer groups of stacks. arXiv:2412.04196, 2024.

[LST22] Brian Lehmann, Akash Kumar Sengupta, and Sho Tanimoto. Geometric consistency of Manin's conjecture. *Compos. Math.*, 158(6):1375–1427, 2022.

[LT17] B. Lehmann and S. Tanimoto. On the geometry of thin exceptional sets in Manin's conjecture. *Duke Math. J.*, 166(15):2815–2869, 2017.

[LT19a] B. Lehmann and S. Tanimoto. Geometric Manin's conjecture and rational curves. *Compos. Math.*, 155(5):833–862, 2019.

[LT19b] B. Lehmann and S. Tanimoto. On exceptional sets in Manin's conjecture. *Res. Math. Sci.*, 6(1):Paper No. 12, 41, 2019.

[LT21] B. Lehmann and S. Tanimoto. Rational curves on prime Fano threefolds of Picard rank 1. *J. Algebraic Geom.*, 30(1):151–188, 2021.

[LT22] Brian Lehmann and Sho Tanimoto. Classifying sections of del Pezzo fibrations, II. *Geom. Topol.*, 26(6):2565–2647, 2022.

[LT24] Brian Lehmann and Sho Tanimoto. Classifying sections of del Pezzo fibrations, I. *J. Eur. Math. Soc. (JEMS)*, 26(1):289–354, 2024.

[LTT18] B. Lehmann, S. Tanimoto, and Y. Tschinkel. Balanced line bundles on Fano varieties. *J. Reine Angew. Math.*, 743:91–131, 2018.

[Mad16] Zachary Maddock. Regular del Pezzo surfaces with irregularity. *J. Algebraic Geom.*, 25(3):401–429, 2016.

[Mil76] J. S. Milne. Duality in the flat cohomology of a surface. *Ann. Sci. École Norm. Sup. (4)*, 9(2):171–201, 1976.

[MM83] Shigefumi Mori and Shigeru Mukai. On Fano 3-folds with $B_2 \geq 2$. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 101–129. North-Holland, Amsterdam, 1983.

[MM03] Shigefumi Mori and Shigeru Mukai. Erratum: "Classification of Fano 3-folds with $B_2 \geq 2$" [Manuscripta Math. **36** (1981/82), no. 2, 147–162; MR0641971 (83f:14032)]. *Manuscripta Math.*, 110(3):407, 2003.

[MM82] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.*, 36(2):147–162, 1981/82.

[Mor79] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Ann. of Math. (2)*, 110(3):593–606, 1979.

[Mor82] Shigefumi Mori. Threefolds whose canonical bundles are not numerically effective. *Ann. of Math. (2)*, 116(1):133–176, 1982.

[MTiB25] Yusuf Mustopa and Montserrat Teixidor i Bigas. Rational curves on moduli spaces of vector bundles. *Internat. J. Math.*, 36(3):Paper No. 2450080, 20, 2025.

[Oka24] Fumiya Okamura. The irreducibility of the spaces of rational curves on del Pezzo manifolds. *Int. Math. Res. Not. IMRN*, (12):9893–9909, 2024.

[Oka25] F. Okamura. Moduli spaces of rational curves on Artin-Mumford double solids. arXiv:2501.09269, 2025.

[Pey95] E. Peyre. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.*, 79(1):101–218, 1995.

[Pey03] E. Peyre. Points de hauteur bornée, topologie adélique et mesures de Tamagawa. *J. Théor. Nombres Bordeaux*, 15(1):319–349, 2003.

[Pey12] Emmanuel Peyre. Points de hauteur bornée sur les variétés de drapeaux en caractéristique finie. *Acta Arith.*, 152(2):185–216, 2012.

[Pey17] Emmanuel Peyre. Liberté et accumulation. *Doc. Math.*, 22:1615–1659, 2017.

[Pey21] Emmanuel Peyre. Chapter V: Beyond heights: slopes and distribution of rational points. In *Arakelov geometry and Diophantine applications*, volume 2276 of *Lecture Notes in Math.*, pages 215–279. Springer, Cham, [2021] ©2021.

[Ray78] M. Raynaud. Contre-exemple au "vanishing theorem" en caractéristique $p > 0$. In *C. P. Ramanujam—a tribute*, volume 8 of *Tata Inst. Fundam. Res. Stud. Math.*, pages 273–278. Springer, Berlin-New York, 1978.

[RY19] E. Riedl and D. Yang. Kontsevich spaces of rational curves on Fano hypersurfaces. *J. Reine Agnew. Math.*, 748:207–225, 2019.

[Sal98] Per Salberger. Tamagawa measures on universal torsors and points of bounded height on Fano varieties. Number 251, pages 91–258. 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).

[San25] Tim Santens. The leading constant in Malle's conjecture over function fields. 2025.

[Saw24] W. Sawin. The asymptotic in Waring's problem over function fields via singular sets in the circle method. arXiv:2412.14053, 2024.

[Seg79] G. Segal. The topology of spaces of rational functions. *Acta Math.*, 143(1-2):39–72, 1979.

[Sen21] Akash Kumar Sengupta. Manin's conjecture and the Fujita invariant of finite covers. *Algebra Number Theory*, 15(8):2071–2087, 2021.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[Ser08] Jean-Pierre Serre. *Topics in Galois theory*, volume 1 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, second edition, 2008. With notes by Henri Darmon.

[Sko25] Alexei Skorobogatov. Boundedness of the p-primary torsion of the Brauer group of products of varieties. *Forum Math. Sigma*, 13:Paper No. e134, 16, 2025. With an appendix by Alexander Petrov.

[SS10] Karl Schwede and Karen E. Smith. Globally F-regular and log Fano varieties. *Adv. Math.*, 224(3):863–894, 2010.

[ST16] Joseph Shalika and Yuri Tschinkel. Height zeta functions of equivariant compactifications of unipotent groups. *Comm. Pure Appl. Math.*, 69(4):693–733, 2016.

[ST19] Jason Michael Starr and Zhiyu Tian. Separable rational connectedness and weak approximation in positive characteristic, 2019.

[ST25] Kenta Sato and Shunsuke Takagi. Arithmetic and geometric deformations of F-pure and F-regular singularities. *Amer. J. Math.*, 147(2):561–596, 2025.

[STBT07] Joseph Shalika, Ramin Takloo-Bighash, and Yuri Tschinkel. Rational points on compactifications of semi-simple groups. *J. Amer. Math. Soc.*, 20(4):1135–1186, 2007.

[Tan25a] S. Tanimoto. *Birational Geometry and Manin's Conjecture*. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2025. to appear.

[Tan25b] Sho Tanimoto. The spaces of rational curves on del Pezzo surfaces via conic bundles, 2025.

[Tes09] D. Testa. The irreducibility of the spaces of rational curves on del Pezzo surfaces. *J. Algebraic Geom.*, 18(1):37–61, 2009.

[Tho98] J. F. Thomsen. Irreducibility of $\overline{M}_{0,n}(G/P,\beta)$. *Internat. J. Math.*, 9(3):367–376, 1998.

[Tia25] Zhiyu Tian. Local-global principle and integral Tate conjecture for certain varieties. *J. Amer. Math. Soc.*, 38(3):703–782, 2025.

[Voi06] C. Voisin. On integral Hodge classes on uniruled or Calabi-Yau threefolds. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 43–73. Math. Soc. Japan, Tokyo, 2006.

[Xu12] Chenyang Xu. Strong rational connectedness of surfaces. *J. Reine Angew. Math.*, 665:189–205, 2012.

[Zhu24] Yi Zhu. Fano hypersurfaces in positive characteristic. *C. R. Math. Acad. Sci. Paris*, 362:107–115, 2024.

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