EXERCISES FOR CHAPTER II.6

1. Weil divisors and class groups

Exercise 1.1. Exercise II.6.1.

Exercise 1.2. Exercise II.6.3.(ab).

Exercise 1.3. Let X be a normal separated integral scheme. For any Weil divisor D we can define a sheaf $\mathcal{O}_X(D)$ in the analogous way:

$$\mathcal{O}_X(D)(V) := \{f \in K(X)^\times \,|\, (\mathrm{div}(f) + D)|_V \geq 0\}.$$

(1) Compute $\mathcal{O}_X(D)$ when X is the quadric cone and D is a line through the origin. Show that $\mathcal{O}_X(D)$ does not need to be locally free.
(2) Prove that if $D = \mathrm{div}(L)$ for a Cartier divisor L then $\mathcal{O}_X(D) = \mathcal{O}_X(L)$ so there is no conflict in notation.
(3) Show that D is a Cartier divisor (where as usual we identify $\mathrm{CDiv} \subset \mathrm{WDiv}$) if and only if $\mathcal{O}_X(D)$ is locally free.

Exercise 1.4. Let X be the cone over a smooth quadric surface, e.g. $X = \mathrm{Spec}(k[w,x,y,z]/(wy-xz))$. Consider the prime divisor $D = (w,x)$. Prove that the complement of D in X is not affine. (Hint: if it were affine, its intersection with the plane $y = z = 0$ would also be affine.) Conclude that no multiple of D is a Cartier divisor (by the exercise below).

2. Cartier divisors and Picard groups

Exercise 2.1. Exercise II.6.9.

Exercise 2.2. Let X be an integral affine scheme. Suppose that L is an effective Cartier divisor on X. The support of L is defined to be the underlying set of $\mathrm{div}(L)$. Prove that the complement U of the support of L is an affine scheme. (Hint: show that the inclusion $U \to X$ is an affine morphism by arguing locally.)

Exercise 2.3. Let X be an integral scheme. Suppose that L is an effective Cartier divisor on X. Show that "multiplication by L" defines an injective map $\mathcal{O}_X(-L) \to \mathcal{O}_X$ whose cokernel is $\mathcal{O}_{\mathrm{div}(L)}$. Conclude that $\mathcal{O}_X(-L)$ is isomorphic to the ideal sheaf of $\mathrm{div}(L)$.

Exercise 2.4. Compute the Picard group of the complement of a degree d hypersurface in $\mathbb{P}^n_{\mathbb{C}}$.

Exercise 2.5. Let X be the affine line with doubled origin. Prove that $\mathrm{Pic}(X) \cong \mathbb{Z}$.