FREE CURVES AND FUNDAMENTAL GROUPS

ERIC JOVINELLY, BRIAN LEHMANN, AND ERIC RIEDL,
WITH AN APPENDIX BY AISE JOHAN DE JONG

ABSTRACT. We show that klt Fano varieties and certain lc Fano varieties contain free higher-genus curves in their smooth loci. Our methods also allow us to find free curves on varieties in positive characteristic and on quasiprojective varieties, under a natural positivity condition on the tangent bundle. We then use the existence of free curves to deduce finiteness of the fundamental group of the smooth locus in these settings.

The paper includes an appendix by de Jong that establishes the Künneth formula for tame étale fundamental groups.

1. INTRODUCTION

Rational curves provide an essential tool for studying the geometry of a smooth complex Fano variety X. A paragon is [Mor79] in which Mori proved Hartshorne's Conjecture by studying the existence and deformation properties of rational curves. A rational curve $s : \mathbb{P}^1 \to X$ exhibits the best deformation-theoretic properties when $s^* T_X$ is positive: the map s is said to be free if $s^* T_X$ is nef and very free if $s^* T_X$ is ample. Using the innovative techniques of [Mor79], Kollár showed that every smooth Fano variety carries a free rational curve (see [Mor84, Page 749], [MM86, Page 66]). The existence of very free curves on smooth Fano varieties was subsequently established by [KMM92].

When X is a singular Fano variety, the situation is more subtle. It is still true that X carries many rational curves; for example, [HM07] and [Zha06] prove that a klt Fano pair (X, Δ) admits a rational curve through two general points. However, these rational curves can meet the singular locus of X in which case they no longer exhibit good deformation-theoretic properties. For potential applications it is crucial to find rational curves in the smooth locus of X.

The following well-known conjecture is motivated by work of Keel-McKernan and Miyanishi-Tsunoda.

Conjecture 1.1. Let (X, Δ) be a klt Fano pair over an algebraically closed field of characteristic 0. Then there is a free rational curve in the smooth locus of X.

Building on the earlier works [MT84a, MT84b, Zha88], [KM99, 1.6 Corollary] verified this conjecture in dimension 2: a klt Fano surface always admits a free rational curve in its smooth locus. This result is a central step in Keel and McKernan's groundbreaking work on the existence of $\mathbb{A}^1$-curves and also has a number of important applications in its own right. In the decades since, the question of existence of rational curves in the smooth locus of a Fano variety has remained open, even for terminal Fano threefolds.

1.1. **Main results: characteristic 0.** We take a novel approach: we relax the requirement that our curves be rational and instead search for curves of arbitrary genus. The following definition identifies a natural analogue of freeness for higher genus curves.

Definition 1.2. Let X be a variety and let C be a smooth proper integral curve. Fix a non-negative number r. A morphism $s : C \to X$ is an r-free curve if $s(C)$ is contained in the smooth locus $X^{sm} \subset X$ and every positive rank quotient of $s^* T_X$ has slope at least $2g(C) + r$.

With this change in perspective, we are able to prove the analogue of Conjecture 1.1 for higher genus curves.

Theorem 1.3. *Let (X, Δ) be a dlt Fano pair over an algebraically closed field of characteristic 0. Then there is a curve C such that for every $r \geq 0$ there is an r-free curve $s : C \to X^{sm}$.*

Using this result and [JLR25], we take the next step toward the original Conjecture 1.1 in higher dimensions.

Theorem 1.4. *Let X be a terminal Fano threefold over an algebraically closed field of characteristic 0. Then there is a free rational curve $s : \mathbb{P}^1 \to X^{sm}$.*

Our techniques are applicable to lc Fano pairs (X, Δ). Note that such an X need not contain any free curves at all. (The simplest counterexample is a cone over an elliptic curve.) Instead, we show that the existence of free curves can be precisely accounted for by the positivity of the tangent bundle. Furthermore, this class of lc Fano pairs shares many of the nice geometric properties of klt Fano pairs; see Section 6.2.

Theorem 1.5. *Let (X, Δ) be a $\mathbb{Q}$-factorial lc Fano pair over an algebraically closed field of characteristic 0. Then the following conditions are equivalent:*

(1) X admits a 1-free curve.
(2) There is no positive rank quotient of T_X with numerically trivial first Chern class.
(3) For every $m > 0$ we have $h^0(X^{sm}, \Omega_{X^{sm}}^{\otimes m}) = 0$.

If these equivalent conditions hold, then there is a curve C such that for every $r \geq 0$ there is an r-free curve $s : C \to X$.

Just as with free rational curves, the existence of a 1-free curve of arbitrary genus has a number of important geometric consequences. The most significant is finiteness of the fundamental group. Recently there have been many results concerning fundamental groups associated to klt singularities; see e.g. [Xu14], [GKP16a], [BGO17], [TX17], [CRST18], [BCRG$^+$19], [Bra21]. In combination with Theorem 1.5, the following result extends our understanding to the log canonical case.

Theorem 1.6. *Let X be a projective variety over $\mathbb{C}$. If X admits a 1-free curve, then X^{sm} has finite topological fundamental group.*

In particular, we obtain a new (and short) proof of the finiteness of the topological fundamental group of the smooth locus of a klt Fano pair first proved by [Bra21].

1.2. **Main results: characteristic p.** Much less is known about the existence of free curves in characteristic p. A famous open problem of Kollár asks whether a smooth Fano variety over an algebraically closed field of characteristic p is separably rationally connected. If we allow singularities, the answer is no: counterexamples are given in [Kol95] and [Xu12, Section 5]. These examples use a "negative" quotient of the tangent bundle to obstruct the existence of free rational curves. In particular, they demonstrate that in characteristic p we

must impose stronger assumptions on the possible quotients of the tangent bundle to obtain free curves.

The following theorem provides a new link in the relationship between positivity of the tangent bundle and existence of free curves, in the same direction as earlier work by [Miy87, KSCT07, SCT09, She10, Tia15, Gou16, BM16, CP19] and many others. The positivity assumption is phrased in terms of Mumford-Takemoto slope stability.

Theorem 1.7. *Let X be a normal projective variety of dimension n over an algebraically closed field. Suppose that there is an ample divisor H such that $\mu_{H^{n-1}}^{min}(T_X) > 0$. Then there is a curve C such that for every $r \geq 0$ there is an r-free curve $s : C \to X^{sm}$.*

In characteristic p, this theorem is interesting even when X is smooth; for example, it applies whenever X is a smooth Fano variety with a stable tangent bundle. Just as in characteristic 0, we can use 1-free curves to deduce finiteness of fundamental groups.

Theorem 1.8. *Let X be a projective variety over an algebraically closed field. Suppose that X carries a 1-free curve. Then the étale fundamental group of X^{sm} is finite.*

We emphasize that it is the étale fundamental group and not the tame étale fundamental group that appears in Theorem 1.8.

1.3. **Main results: log setting.** There are several famous conjectures that predict that in the quasiprojective setting curves have similar behavior to the projective setting. According to Iitaka's philosophy, the best way to analyze curves on a smooth quasiprojective variety U is to pass to a SNC compactification (X, Δ) and study log curves. In particular, when (X, Δ) has "positive curvature" we should expect to find log curves on (X, Δ) which are 1-free in the sense of Definition 5.4.

In our setting, it is more natural to start from an lc pair (X, Δ) and to define U to be the open subscheme of X obtained by removing all irreducible components of Δ with coefficient 1. Even when (X, Δ) is an lc Fano pair, it need not admit a 1-free log curve. (The simplest counterexample is $\mathbb{P}^2$ equipped with two lines.) As before, we show that the positivity of the log tangent bundle is the key ingredient for correcting this deficiency.

Theorem 1.9. *Let (X, Δ) be a $\mathbb{Q}$-factorial lc Fano pair over an algebraically closed field of characteristic 0. Then the following conditions are equivalent:*
   (1) (X, Δ) admits a 1-free log curve.
   (2) The log tangent bundle admits no positive rank quotient with numerically trivial first Chern class.
   (3) For every $m > 0$ we have $h^0(V, \Omega_V(\log \lfloor \Delta \rfloor)^{\otimes m}) = 0$ where $V \subset X$ is the SNC locus of the pair $(X, \lfloor \Delta \rfloor)$.
If these equivalent conditions hold, then there is a curve C and a finite subset $D \subset C$ such that for every $r \geq 0$ there is an r-free log curve $s : (C, D) \to (X, \Delta)$.

In dimension 2 [Zhu16] proves a similar but more precise statement. We also prove a version of Theorem 1.9 in positive characteristic; see Theorem 5.7.

Remark 1.10. We expect the conditions of Theorem 1.9 to be equivalent to $\mathbb{A}^1$-connectedness of the open locus. More generally, we expect these conditions to be important in other contexts which rely on "geometric positivity" in the log setting, e.g. the study of integral points for Fano pairs.

In the log setting, we would like to understand the fundamental group of the open locus U. Note that U need not be smooth; our next result bounds the fundamental group of the smooth locus of U.

Theorem 1.11. *Let (X, Δ) be a projective lc pair over an algebraically closed field k. Set $U = X \backslash \lfloor \Delta \rfloor$. Suppose that (X, Δ) carries a 1-free log curve. Then:*

(1) *The curve-tame étale fundamental group of U^{sm} is finite.*
(2) *If $k = \mathbb{C}$, then the topological fundamental group of U^{sm} is finite.*

The proof of Theorem 1.11 relies on Appendix A by de Jong which establishes the behavior of tame étale fundamental groups under products.

2. Preliminaries

Throughout we work with schemes which are separated and whose connected components have finite type over the ground field. A variety is irreducible and reduced.

Over an algebraically closed field of arbitrary characteristic, we let $\overline{\mathcal{M}}_{g,n}(X)$ denote the Kontsevich moduli stack of stable maps and let $\mathcal{M}_{g,n}(X)$ denote the open substack parametrizing morphisms with smooth irreducible domain.

2.1. **Non $\mathbb{Q}$-factorial varieties.** When running the MMP, it is possible to obtain a Fano fibration whose fibers are not $\mathbb{Q}$-factorial. For this reason, we need to briefly review how to work with curves on non-$\mathbb{Q}$-factorial varieties. Note that for such varieties the behavior of Cartier and Weil divisors can be quite different and we must be careful to choose the correct notion.

Suppose that X is a normal projective variety of dimension n over an algebraically closed field. We first define spaces associated to Cartier divisors. We let $N^1(X)_{\mathbb{R}}$ denote the space of $\mathbb{R}$-Cartier divisors up to numerical equivalence and let $N_1(X)_{\mathbb{R}}$ denote the dual space of $\mathbb{R}$-1-cycles up to numerical equivalence. We let $\overline{\mathrm{Eff}}^1(X) \subset N^1(X)_{\mathbb{R}}$ denote the pseudo-effective cone of divisors and let $\overline{\mathrm{Eff}}_1(X) \subset N_1(X)_{\mathbb{R}}$ denote the pseudo-effective cone of curves. Dual to these are the nef cones $\mathrm{Nef}^1(X) \subset N^1(X)_{\mathbb{R}}$ and $\mathrm{Nef}_1(X) \subset N_1(X)_{\mathbb{R}}$.

We next define spaces associated to Weil divisors. We let $N_{n-1}(X)_{\mathbb{R}}$ denote the space of $\mathbb{R}$-Weil divisors up to intersection against homogeneous polynomials of weight $(n-1)$ in Chern classes of vector bundles on X. We let $N^{n-1}(X)_{\mathbb{R}}$ denote the dual space spanned by polynomials in Chern classes. We let $\overline{\mathrm{Eff}}_{n-1}(X) \subset N_{n-1}(X)_{\mathbb{R}}$ denote the closure of the cone generated by effective Weil divisors. Dually we have the nef cone $\mathrm{Nef}^{n-1}(X) \subset N^{n-1}(X)_{\mathbb{R}}$.

Remark 2.1. For any normal projective variety X there is an injective map $N^1(X)_{\mathbb{R}} \xrightarrow{\cap [X]} N_{n-1}(X)_{\mathbb{R}}$ from numerical Cartier classes to numerical Weil classes. Dually, we get a surjection from $\mathrm{Nef}^{n-1}(X)$ to $\mathrm{Nef}_1(X)$. In particular, when we want to find curves representing a nef class the strongest statements are obtained by working with $\mathrm{Nef}^{n-1}(X)$.

Remark 2.2. When X is $\mathbb{Q}$-factorial, the map $N^1(X)_{\mathbb{R}} \xrightarrow{\cap [X]} N_{n-1}(X)_{\mathbb{R}}$ is an isomorphism. Thus we can naturally identify $\mathrm{Nef}^{n-1}(X)$ and $\mathrm{Nef}_1(X)$. We will preferentially use the latter notation when X is $\mathbb{Q}$-factorial.

Remark 2.3. Suppose that C is a smooth curve and $s : C \to X$ is a map whose image lies in the smooth locus of X. Note that s is an lci morphism of codimension $n-1$; indeed, we can write s as the regular embedding $\Gamma_s : C \to C \times X^{sm}$ followed by the smooth projection $\pi_2 : C \times X^{sm} \to X$. Then [Ful84, Example 19.2.3] shows that the Gysin map $s^* : A_{n-1}(X) \to A_0(C)$ descends to numerical groups. In this way, any curve C in the smooth locus of X naturally defines a class $s_*C \in N^{n-1}(X)_{\mathbb{R}}$. If furthermore s deforms in a dominant family then s_*C defines a class in $\mathrm{Nef}^{n-1}(X)$.

The advantage of $\mathrm{Nef}^{n-1}(X)$ is that it is compatible with birational pullbacks, as explained by the following lemma.

Lemma 2.4. *Let $\phi : X' \to X$ denote a birational morphism between normal projective varieties over an algebraically closed field. Then:*
  (1) *The pullback map $\phi^* : \mathrm{Nef}^{n-1}(X) \to \mathrm{Nef}^{n-1}(X')$ is injective.*
  (2) *If ϕ is a small contraction, then ϕ^* takes interior classes for $\mathrm{Nef}^{n-1}(X)$ to interior classes for $\mathrm{Nef}^{n-1}(X')$.*
  (3) *The pushforward map $\phi_* : N_1(X')_{\mathbb{R}} \to N_1(X)_{\mathbb{R}}$ takes $\mathrm{Nef}_1(X')$ surjectively onto $\mathrm{Nef}_1(X)$.*

Proof. (1) follows from the surjectivity of $\overline{\mathrm{Eff}}_{n-1}(X') \xrightarrow{\phi_*} \overline{\mathrm{Eff}}_{n-1}(X)$. Furthermore, [FL16, Theorem 3.21] shows that when ϕ is a small contraction the kernel of $N_{n-1}(X')_{\mathbb{R}} \xrightarrow{\phi_*} N_{n-1}(X)_{\mathbb{R}}$ only intersects $\overline{\mathrm{Eff}}_{n-1}(X')$ at $\{0\}$. Dualizing, we obtain (2). (3) is dual to the fact that the pseudo-effective cone of Cartier divisors injects upon birational pullback. $\qquad\square$

2.2. **Stability of torsion-free sheaves.** We quickly review Harder-Narasimhan filtrations with respect to nef curve classes as developed by [CP11, GKP14, GKP16b]. While these references work in characteristic 0, the arguments used for the construction below work equally well in characteristic p.

Notation 2.5. Let X be a $\mathbb{Q}$-factorial normal projective variety and let $\alpha \in \mathrm{Nef}_1(X)$. For any non-zero torsion-free sheaf $\mathcal{E}$ on X, let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_s = \mathcal{E}.$$

denote the α-Harder-Narasimhan filtration of $\mathcal{E}$ with respect to α. Thus each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is α-semistable and the α-slopes of the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are strictly decreasing in i. When X is a curve, we will always implicitly use the polarization α given by the fundamental class.

We denote by $\mu_\alpha^{max}(\mathcal{E})$ the maximal α-slope of any torsion-free subsheaf of $\mathcal{E}$, that is, $\mu_\alpha^{max}(\mathcal{E}) = \mu_\alpha(\mathcal{F}_1)$. We denote by $\mu_\alpha^{min}(\mathcal{E})$ the minimal α-slope of any torsion-free quotient of $\mathcal{E}$, that is, $\mu_\alpha^{min}(\mathcal{E}) = \mu_\alpha(\mathcal{E}/\mathcal{F}_{s-1})$.

Remark 2.6. In fact, one can develop a theory of slope stability for arbitrary normal projective varieties by working with $\mathrm{Nef}^{n-1}(X)$ instead of $\mathrm{Nef}_1(X)$. While we will not develop such a theory here, we will need one special case: when H is an ample divisor on a normal projective variety, one can define a theory of slope stability with respect to H^{n-1}. Indeed, this is the classical notion of Mumford-Takemoto slope stability; see [Mar81].

2.3. **Positive curves.** The notion of an r-free curve was given in Definition 1.2. Note that for a rational curve, free is the same as 0-free and very free is the same as 1-free; we will use the same shorthand for higher genus curves. (Note also that freeness in our sense implies the related notion of [Kol96, II.3.1 Definition].) We will need the following properties of free curves:

Proposition 2.7. *Let X be a projective variety. Suppose that $s : C \to X$ is an r-free curve for some $r \geq 0$. Then:*

  (1) *For any codimension ≥ 2 closed subset $Z \subset X$ a general deformation of s will be an r-free curve whose image is disjoint from Z. ([LRT24, Lemma 3.8])*
  (2) *If $M \subset \mathrm{Mor}(C, X)$ is the irreducible component containing s then the $(\lfloor r \rfloor + 1)$-fold evaluation map $C^{\times(\lfloor r \rfloor+1)} \times M \to C^{\times(\lfloor r \rfloor+1)} \times X^{\times(\lfloor r \rfloor+1)}$ is dominant. ([LRT24, Lemma 3.6.(4)])*

Proof. The results of [LRT24] are only explicitly stated in characteristic 0 but the arguments also work in characteristic p. $\qquad\square$

The following lemma is frequently useful when working with free curves.

Lemma 2.8. *Let X be a $\mathbb{Q}$-factorial normal projective variety. Suppose that $M \subset \mathrm{Mor}(C, X)$ parametrizes a family of maps $s : C \to X$ with the property that for any codimension 2 subset $Z \subset X$ there is a deformation of s such that $s(C) \cap Z = \emptyset$.*

Let $\mathcal{E}$ be a torsion-free sheaf on X. If for a general map $s : C \to X$ parametrized by M we have $\mu^{min}(s^* \mathcal{E}) > 0$, then $\mu_{s_* C}^{min}(\mathcal{E}) > 0$.

Proof. Consider a non-trivial torsion-free quotient $\mathcal{E} \to \mathcal{Q}$. By assumption, a general deformation $s' : C \to X$ of s will have the property that $s'(C)$ lies in the locus where $\mathcal{E}$ and $\mathcal{Q}$ are locally free. Thus the surjection $s'^* \mathcal{E} \to s'^* \mathcal{Q}$ is also a morphism of locally free sheaves. By assumption we have $0 < \deg(s'^* \mathcal{Q}) = c_1(\mathcal{Q}) \cdot s'_* C$ so that $\mu_{s_* C}(\mathcal{Q}) > 0$. This implies the desired statement. $\qquad\square$

The following weaker positivity condition is also useful.

Definition 2.9. Let X be a projective variety over an algebraically closed field and let r be a non-negative number. We say that a curve $s : C \to X$ is almost r-free if there is a smooth open subset $U \subset X$, a smooth morphism $\pi : U \to Z$, and a fiber F of π such that s has image in F and is an r-free curve in F.

Note that for an almost r-free curve there is a positive integer q such that

$$s^* T_X \cong \mathcal{O}_C^{\oplus q} \oplus \mathcal{E}$$

where $\mathcal{E}$ is a vector bundle satisfying $\mu^{min}(\mathcal{E}) \geq 2g(C) + r$. By [Kol96, I.2.17 Theorem] a general fiber F' of π will contain a deformation of s which is again an r-free curve in F'. Furthermore, if M is an irreducible component of $\mathrm{Mor}(C, X)$ that generically parametrizes almost r-free curves then $\dim M = H^0(C, s^* T_X)$.

2.4. **Pairs and the MMP.** In this subsection we work over an algebraically closed field of characteristic 0. A pair (X, Δ) consists of a normal variety X and an effective $\mathbb{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We will use the standard definitions of Kawamata log terminal (klt), divisorially log terminal (dlt) and log canonical (lc) as in [KM98]. We will frequently use the following results:

(1) If (X, Δ) is dlt, then for any open neighborhood U of $0 \in N^1(X)_{\mathbb{R}}$ there is a divisor Δ' such that (X, Δ') is klt and $\Delta' - \Delta \in U$. (See [KM98, Proposition 2.43].)

(2) Every dlt pair (X, Δ) admits a $\mathbb{Q}$-factorialization, i.e. a small projective birational morphism $\phi : Y \to X$ such that Y is $\mathbb{Q}$-factorial. Furthermore, the strict transform Δ_Y of Δ satisfies that (Y, Δ_Y) is dlt. (See [BCHM10, Corollary 1.4.3], [Loh13, Corollary 4.4].)

(3) If (X, Δ) is a klt Fano pair and $\phi : Y \to X$ is a $\mathbb{Q}$-factorialization, then there is a divisor Δ' on Y such that (Y, Δ') is a klt Fano pair. (See [GOST15, Lemma 3.1].)

Combining these results, we see that for every dlt log Fano pair (X, Δ) there is a small projective birational map $\phi : Y \to X$ and a divisor Δ' on Y such that (Y, Δ') is a $\mathbb{Q}$-factorial klt log Fano pair.

2.5. **Positivity of the tangent bundle.** We start by recalling an important theorem of [Ou23].

Theorem 2.10 ([Ou23, Theorem 1.4])**.** *Let (X, Δ) be a projective $\mathbb{Q}$-factorial lc pair over an algebraically closed field of characteristic 0. Suppose that $-(K_X + \Delta)$ is nef. Then for every surjection $T_X \to \mathcal{Q}$ to a torsion-free sheaf $\mathcal{Q}$ we have $c_1(\mathcal{Q})$ is pseudo-effective.*

Equivalently, we have $\mu_\alpha^{min}(T_X) \geq 0$ for every nef curve class α on X.

Proof. While [Ou23, Theorem 1.4] only states the result when $\Delta = 0$, the argument works in this more general context. The only point where the proof of [Ou23, Theorem 1.4] needs to be modified for $\Delta \neq 0$ is when we apply [Ou23, Theorem 1.10]. Note however that [Ou23, Theorem 1.10] is stated for pairs (X, Δ) and yields the chain of inequalities

$$K_{\mathcal{F}} \cdot \alpha \geq (K_X + \Delta_{ver}) \cdot \alpha \geq K_X \cdot \alpha$$

for $\alpha \in \mathrm{Nef}_1(X)$ which is all that is needed to finish the proof. $\qquad\square$

We will be interested in lc Fano pairs which satisfy a slightly stronger property.

Definition 2.11. Let X be a normal projective variety over an algebraically closed field of arbitrary characteristic. Suppose $\mathcal{E}$ is a torsion-free sheaf on X. We say that $\mathcal{E}$ has the positive quotient property (PQP) if for every surjection $\mathcal{E} \to \mathcal{Q}$ onto a positive rank torsion-free sheaf, the first Chern class $c_1(\mathcal{Q})$ lies in $\overline{\mathrm{Eff}}_{n-1}(X)$ and is not numerically trivial.

The following result from [CP19] shows that if T_X fails the PQP, then there is a rational map associated to the largest quotient of T_X with numerically trivial first Chern class.

Lemma 2.12. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial lc pair over an algebraically closed field of characteristic 0. Suppose that $-(K_X + \Delta)$ is nef. Amongst all surjections $T_X \to \mathcal{Q}$ to torsion-free sheaves such that $c_1(\mathcal{Q})$ is numerically trivial there is a unique "maximal quotient", i.e. a quotient $T_X \to \mathcal{Q}_{max}$ through which all other such morphisms factor. The kernel $\mathcal{F}$ of this surjection defines an algebraically integrable foliation with rationally connected leaves.*

Proof. If there are no quotients of slope 0, $\mathcal{Q}_{max} = 0$ and the foliation is trivial. Otherwise, choose a curve class α in the interior of $\mathrm{Nef}_1(X)$ and consider the α-Harder-Narasimhan filtration of T_X:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_s = T_X.$$

Theorem 2.10 shows that every successive quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ has non-negative α-slope. Thus every quotient of slope 0 must be a quotient of the final term $\mathcal{F}_s/\mathcal{F}_{s-1}$. Due to the semistability of $\mathcal{F}_s/\mathcal{F}_{s-1}$ we see that it also has slope zero so that $\mathcal{Q}_{max} = \mathcal{F}_s/\mathcal{F}_{s-1}$.

[Pan15, Proposition 1.3.32] shows that $\mathcal{F}_{s-1}$ is a foliation. Consider a smooth birational model $\phi : X' \to X$ and let $\mathcal{F}'$ denote the foliation on X' induced by $\mathcal{F}_{s-1}$. Using the compatibility of slopes with pull-back of nef curve classes as established by [GKP16b, Proposition 2.8], we see that $\mu_{\phi^*\alpha}^{min}(\mathcal{F}') > 0$. Thus $\mathcal{F}'$ (and equivalently $\mathcal{F}_{s-1}$) is algebraically integrable with rationally connected leaves by [CP19, Theorem 1.1]. $\square$

The arguments of [Ou23] show that the tangent bundle of a $\mathbb{Q}$-factorial dlt Fano variety always has the PQP.

Proposition 2.13. *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt Fano pair over an algebraically closed field of characteristic 0. Then for every surjection $T_X \to \mathcal{Q}$ to a positive rank torsion-free sheaf $\mathcal{Q}$ we have $c_1(\mathcal{Q})$ is pseudo-effective and is not numerically trivial.*

Equivalently, we have $\mu_\alpha^{min}(T_X) > 0$ for every nef curve class in the interior of $\mathrm{Nef}_1(X)$.

Proof. By perturbing Δ we can find an effective divisor Δ' such that (X, Δ') is a $\mathbb{Q}$-factorial klt Fano pair. Theorem 2.10 shows that every positive rank torsion-free quotient of T_X has pseudo-effective first Chern class. Suppose for a contradiction that there exists a quotient whose first Chern class is numerically trivial. Let $\phi : X \dashrightarrow Z$ denote the rational map induced by the maximal quotient $T_X \to \mathcal{Q}_{max}$ as in Lemma 2.12. Let D denote an effective Weil divisor on X defined by pulling back a very ample divisor H on Z and taking its closure. Choose $\epsilon > 0$ such that $(X, \Delta' + \epsilon D)$ is klt and $-(K_X + \Delta' + \epsilon D)$ is still ample. Then [Ou23, Theorem 1.10] shows that $c_1(\mathcal{Q}_{max}) - \epsilon D$ is pseudo-effective. Thus $c_1(\mathcal{Q}_{max})$ is not numerically trivial, a contradiction. $\square$

3. Existence of free curves

This section is devoted to proving the existence of free curves in the smooth locus of a mildly singular Fano variety. In the first section, we use characteristic p techniques to demonstrate that a curve $s : C \to X^{sm}$ such that $\mu^{min}(s^*T_X) > 0$ can be modified to obtain a free curve. In the later sections, we show to how verify the existence of curves $s : C \to X^{sm}$ such that $\mu^{min}(s^*T_X) > 0$ and then use these constructions to prove Theorem 1.3, Theorem 1.5, Theorem 1.7, and several variants.

3.1. **Finding free curves.** We start with a well-known lemma (see e.g. [Gou16, Section 2]) about Frobenius pullbacks of ample vector bundles on curves. One can also give an effective version of this lemma using [Lan04, Corollary 6.2]. Given a field k of positive characteristic, we let F denote the relative Frobenius map over $\mathrm{Spec}(k)$.

Lemma 3.1. *Let k be a field of characteristic p. Suppose that C is a smooth projective curve over k and $\mathcal{E}$ is an ample vector bundle on C. For any positive number r, there is an iterated Frobenius pullback $F^k : C \to C$ such that $\mu^{min}(F^{k*}\mathcal{E}) > r$.*

Proof. The proof is by induction on the rank. When $\mathcal{E}$ is a line bundle, then $\mu^{min}(F^{k*}\mathcal{E}) = p^k\mu(\mathcal{E}) \geq p^k$ and it suffices to take $k > \log_p(r)$.

In general, first suppose that $F^{k*}\mathcal{E}$ is semistable for every $k \geq 0$. Then $\mu^{min}(F^{k*}\mathcal{E}) = p^k\mu(\mathcal{E}) \geq p^k\frac{1}{\mathrm{rk}(\mathcal{E})}$ so it suffices to choose $k > \log_p(r\,\mathrm{rk}(\mathcal{E}))$. Next suppose that $F^{j*}\mathcal{E}$ is not semistable for some $j \geq 0$. Since $F^{j*}\mathcal{E}$ is ample, every successive quotient in the Harder-Narasimhan filtration is ample and we conclude by induction on the rank. $\square$

The following theorem shows that if we can identify one "positive" curve in the smooth locus of X then we can modify it to obtain a free curve in the smooth locus of X. The argument is similar to Mori's proof of the Bend-and-Break theorem: we spread out to characteristic p and use the Frobenius morphism to improve the positivity of the curve. There are related statements in [KSCT07, Section 4] and [Gou16, Theorem 5.5].

Theorem 3.2. *Let X be a projective variety over an algebraically closed field of arbitrary characteristic. Let U be an open subset of the smooth locus of X. Suppose that C is a smooth projective curve and $s : C \to X$ is a morphism such that $s(C) \subset U$ and $\mu^{min}(s^*T_X) > 0$.*

Then for any positive number r, there is an r-free curve $s' : C \to X$ whose image is contained in U and whose numerical class in $N^{n-1}(X)_{\mathbb{R}}$ is proportional to s_*C.

The content of this theorem is that we can improve the bound $\mu^{min}(s^*T_X) > 0$ to the bound $\mu^{min}(s'^*T_X) \geq 2g(C) + r$ required for r-freeness.

Proof. When k has characteristic p, we find the desired curve by precomposing s with a high iterate of the Frobenius and applying Lemma 3.1.

When k has characteristic 0, we start by spreading out to characteristic p. Set $Z = X\backslash U$ and choose a resolution of singularities $\phi : Y \to X$ that is an isomorphism over U. Choose a finitely generated extension $K/\mathbb{Q}$ such that X, Z, C, Y, and the morphisms s, ϕ are defined over K. Choose an integral domain R that is finitely generated over $\mathbb{Z}$ whose fraction field is equal to K. We will repeatedly replace $\mathrm{Spec}(R)$ by a smaller open affine subset, which by an abuse of notation we will continue to call $\mathrm{Spec}(R)$.

After perhaps replacing $\mathrm{Spec}(R)$ by an open affine subset, we can spread X, Z, and C out to schemes $\mathcal{X}$, $\mathcal{Z}$ which are flat and projective over $\mathrm{Spec}(R)$ and a scheme $\mathcal{C}$ which is smooth projective over $\mathrm{Spec}(R)$. We set $\mathcal{U} = \mathcal{X}\backslash\mathcal{Z}$; we may assume $\mathcal{U}$ is smooth over $\mathrm{Spec}(R)$ by shrinking $\mathrm{Spec}(R)$ if necessary. Consider the relative hom scheme $\mathrm{Mor}_R(\mathcal{C}, \mathcal{X})$. After perhaps shrinking $\mathrm{Spec}(R)$ further, the morphism $s : C \to X$ spreads out to a morphism $\tilde{s} : \mathcal{C} \to \mathcal{X}$ whose image is contained in $\mathcal{U}$. Finally, we can also ensure that ϕ spreads out to give a family of resolutions $\phi_R : \mathcal{Y} \to \mathcal{X}$ over $\mathrm{Spec}(R)$ that are isomorphisms over $\mathcal{U}$.

[Nit11, Theorem 5] shows that after replacing $\mathrm{Spec}(R)$ by an open affine subset, we may ensure that as we vary the prime $\mathfrak{p} \in \mathrm{Spec}(R)$ the slopes in the Harder-Narasimhan filtration of $\tilde{s}_\mathfrak{p}^*T_{\mathcal{X}_\mathfrak{p}}$ remain constant. In particular, there exists a prime ideal $\mathfrak{p}$ such that $\mu^{min}(\tilde{s}_\mathfrak{p}^*T_{\mathcal{X}_\mathfrak{p}}) > 0$. By Lemma 3.1, there is an iterated geometric Frobenius morphism $F^k : \mathcal{C}_\mathfrak{p} \to \mathcal{C}_\mathfrak{p}$ such that $\mu^{min}(F^{k*}\tilde{s}_\mathfrak{p}^*T_{\mathcal{X}_\mathfrak{p}}) \geq 2g(C) + r$. Thus the composition $s'_\mathfrak{p} := s_\mathfrak{p} \circ F^k : \mathcal{C}_\mathfrak{p} \to \mathcal{X}_\mathfrak{p}$ is an r-free curve. It is clear that the numerical class of $s'_{\mathfrak{p}*}\mathcal{C}_\mathfrak{p}$ is proportional to the class of $s_{\mathfrak{p}*}\mathcal{C}_\mathfrak{p}$.

Finally, we deform back to the general fiber using the freeness of $s'_\mathfrak{p}$. Since the obstruction group $H^1(\mathcal{C}_\mathfrak{p}, s'^*_\mathfrak{p}T_{\mathcal{X}_\mathfrak{p}})$ vanishes, [Kol96, I.2.17 Theorem] shows that every irreducible component M of $\mathrm{Mor}_R(\mathcal{C}, \mathcal{X})$ that contains $s'_\mathfrak{p}$ has dimension at least $H^0(\mathcal{C}_\mathfrak{p}, s'^*_\mathfrak{p}T_{\mathcal{X}_\mathfrak{p}}) + \dim_\mathfrak{p}(\mathrm{Spec}(R))$. Furthermore, since $s'_\mathfrak{p}$ is free its deformations in $\mathcal{X}_\mathfrak{p}$ have the expected dimension. By upper

semicontinuity of fiber dimensions, we conclude that the map from M to $\mathrm{Spec}(R)$ is dominant, and in particular, that the image contains the generic point. Using [Nit11, Theorem 5] we see that a general map parametrized by M over the generic point will be an r-free curve $s' : C \to X$. Since a general deformation of $s'_{\mathfrak{p}}$ remains in $\mathcal{U}$, the image of s' is in U.

It only remains to verify the proportionality of the numerical classes. Consider the strict transform over $\phi_R : \mathcal{Y} \to \mathcal{X}$ of the constructed family of deformations of $s_{\mathfrak{p}} : \mathcal{C}_{\mathfrak{p}} \to \mathcal{X}_{\mathfrak{p}}$. Letting $\pi : \mathcal{Y} \to \mathrm{Spec}(R)$ denote the structure map, [MP12, Proposition 3.6] constructs an injective morphism on numerical spaces of Cartier divisors from the geometric fiber of π over the generic point of $\mathrm{Spec}(R)$ to the geometric fiber of π over $\mathfrak{p}$. Thus on Y we conclude that the numerical class of $s'_* C$ is proportional to the class of $s_* C$ as an element of $N_1(Y)_{\mathbb{R}}$. Using the isomorphism $N_1(Y)_{\mathbb{R}} \cong N^{n-1}(Y)_{\mathbb{R}}$ and the injectivity of $\phi^* : N^{n-1}(X)_{\mathbb{R}} \to N^{n-1}(Y)_{\mathbb{R}}$ we obtain the desired proportionality on X. $\qquad\square$

3.2. **Finding starting curves.** In order to apply Theorem 3.2, we must find "positive" curves in the smooth locus of Fano varieties.

Proposition 3.3. *Let (X, Δ) be a normal projective variety of dimension n over an algebraically closed field of arbitrary characteristic. Suppose there is an ample Cartier divisor H such that $\mu^{min}_{H^{n-1}}(T_X) > 0$. Then there is a curve $s : C \to X$ whose image is contained in the smooth locus of X that satisfies $\mu^{min}(s^*T_X) > 0$.*

Proof. By repeatedly applying [Fle84, 1.2 Theorem] (in characteristic 0) or [Lan24, 0.1 Theorem] (in characteristic p) we see that a general complete intersection curve C constructed by intersecting divisors linearly equivalent to a sufficiently positive multiple of H will have the desired property. $\qquad\square$

3.3. **Proving main results.** We are now prepared to prove some of the results from the introduction.

Theorem 3.4. *Let (X, Δ) be a $\mathbb{Q}$-factorial projective lc pair over an algebraically closed field of characteristic 0. Suppose $-(K_X + \Delta)$ is nef. Then the following conditions are equivalent:*

(1) There is no positive rank quotient of T_X with numerically trivial first Chern class.
(2) X admits a 1-free curve.
(3) For every $m > 0$ we have $h^0(X^{sm}, \Omega^{\otimes m}_{X^{sm}}) = 0$.

If these conditions hold, then there is a smooth projective curve C such that for every $r \geq 0$ there is an r-free morphism $s : C \to X$.

Proof. (1) $\implies$ (2): suppose that (X, Δ) is a $\mathbb{Q}$-factorial lc pair with $-(K_X + \Delta)$ nef such that T_X has no positive rank quotient with numerically trivial first Chern class. By Theorem 2.10 T_X satisfies the PQP. Then Proposition 3.3 and Theorem 3.2 combine to prove the existence of r-free curves for every $r \geq 0$.

(2) $\implies$ (3): suppose that X carries a 1-free curve. Then through any general point $x \in X^{sm}$ there is a curve $s : C \to X^{sm}$ such that $s^*T_{X^{sm}}$ is ample. Thus every section of the locally free sheaf $\Omega^{\otimes m}_{X^{sm}}$ must vanish at x. Since this is true for a dense set of points $x \in X^{sm}$, we conclude that the space of global sections is 0.

(3) $\implies$ (1): Let $\phi : Y \to X$ be a resolution of singularities. Then we have $h^0(Y, \Omega^{\otimes m}_Y) = 0$ for every $m > 0$ as well. In particular $\mathrm{Pic}^0_{Y/\mathbb{C}} = 0$, so every numerically trivial Cartier divisor

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on Y is torsion. By appealing to $\mathbb{Q}$-factoriality and pulling back divisors from X to Y, we see the same is true of numerically trivial Weil divisors on X.

Suppose we had a surjection $T_X \to \mathcal{Q}$ where $c_1(\mathcal{Q})$ was numerically trivial. Since a torsion sheaf always has pseudo-effective first Chern class, we may suppose $\mathcal{Q}$ is torsion-free. Then for any $m > 0$ we have a chain of surjections on the smooth locus X^{sm}

$$T_{X^{sm}}^{\otimes m\,\mathrm{rk}(\mathcal{Q})} \to \mathcal{Q}|_{X^{sm}}^{\otimes m\,\mathrm{rk}(\mathcal{Q})} \to \left(\bigwedge\nolimits^{\mathrm{rk}(\mathcal{Q})} \mathcal{Q}|_{X^{sm}} \right)^{\otimes m}.$$

In particular, when m satisfies $mc_1(\mathcal{Q}) \sim 0$ we obtain a surjection $T_{X^{sm}}^{\otimes m\,\mathrm{rk}(\mathcal{Q})} \to \mathcal{O}_{X^{sm}}$. Dually, we get a non-vanishing tensor field on X^{sm}, a contradiction. $\square$

Proof of Theorem 1.5: Follows immediately from Theorem 3.4. $\square$

Proof of Theorem 1.7: Follows immediately from Proposition 3.3 and Theorem 3.2. $\square$

Proof of Theorem 1.3: As discussed in Section 2.4, there is a $\mathbb{Q}$-factorialization $\phi : X' \to X$ and a divisor Δ' on X' such that (X', Δ') is $\mathbb{Q}$-factorial klt log Fano. Theorem 3.4 and Proposition 2.13 show that there is a curve C such that for every non-negative r there is an r-free morphism $s' : C \to X'$. Since a general deformation of an r-free curve $s' : C \to X'$ will miss the codimension ≥ 2 sublocus of X' where ϕ is not an isomorphism, by taking images on X we obtain the desired curves. $\square$

3.4. **Representing nef classes by free curves.**
In this section we prove two variants of Theorem 1.3 which give more details about which numerical classes are represented by free curves. We first discuss the situation for pairs which are not necessarily Fano.

Proposition 3.5. *Let (X, Δ) be a dlt pair over an algebraically closed field of characteristic 0. Suppose that $\mathcal{F}$ is a $(K_X + \Delta)$-negative extremal face of the cone $\mathrm{Nef}_1(X) + \overline{\mathrm{Eff}}_1(X)_{K_X + \Delta \geq 0}$. Then there is some curve C such that for any $r \geq 0$ there is an almost r-free curve $s : C \to X^{sm}$ whose class is in $\mathcal{F}$.*

In particular, if (X, Δ) is a dlt pair such that $K_X + \Delta$ fails to be pseudo-effective then there is an almost r-free curve $s : C \to X^{sm}$.

Proof. Let $\phi : X' \to X$ be a $\mathbb{Q}$-factorialization of (X, Δ) and let Δ' be the strict transform of Δ. There is an extremal face $\mathcal{F}'$ of the analogous cone for (X', Δ') such that $\phi_* \mathcal{F}' = \mathcal{F}$. Since a general deformation of an almost free curve will avoid the codimension 2 locus where ϕ is not an isomorphism, the desired statement for X follows from the statement for X'. Thus we may assume (X, Δ) is $\mathbb{Q}$-factorial. After perturbing Δ we may also suppose that (X, Δ) is klt.

Let D be a pseudo-effective $\mathbb{Q}$-divisor such that the nef curve classes with vanishing intersection against D are exactly the classes in the face $\mathcal{F}$. [Leh12, Lemma 4.3] shows that (after rescaling) we may write $D = K_X + \Delta + A$ for some ample $\mathbb{Q}$-divisor A such that $(X, \Delta + A)$ is klt. By [BCHM10, Theorem C] we can run a D-MMP with scaling of A to obtain a birational contraction $\psi : X \dashrightarrow X'$ such that $\psi_* D$ is nef. We can write $\psi_* D \sim_{\mathbb{Q}} K_{X'} + \psi_* \Delta + (1 - \epsilon)\psi_* A + \epsilon E + H$ for some choice of $\epsilon > 0$, effective $\mathbb{Q}$-divisor E on X', and ample $\mathbb{Q}$-divisor H on X' such that $(X', \psi_* \Delta + (1 - \epsilon)\psi_* A + \epsilon E)$ is klt. Thus we may apply the basepoint free theorem (see [KM98, Theorem 3.3]) to find a fiber space $\pi : X' \to Z$ which contracts precisely the curves with vanishing intersection against $\psi_* D$.

Note that for a general fiber F of π the pair $(F, (\psi_*\Delta + (1-\epsilon)\psi_*A + \epsilon E)|_F)$ will be klt Fano. In particular, by Theorem 1.3 there is a curve C such that for any $r \geq 0$ we have an r-free curve $s : C \to F$. By freeness, a general deformation of s will avoid the indeterminacy locus of ψ^{-1}. Thus the strict transform of these maps on X will be almost free curves. Since they have vanishing intersection against D, their numerical classes lie in $\mathcal{F}$. $\square$

Our next result addresses the homological classes of free curves. Let X be a projective variety and consider the set of cycles on X with a fixed codimension > 1. It is frequently a delicate question whether or not a given homological class is represented by an effective cycle, and even harder to determine whether it is represented by a "positive" cycle. We prove a strong result in this direction for curves on dlt Fano pairs:

Theorem 3.6. *Let (X, Δ) be a dlt Fano pair over an algebraically closed field of characteristic 0. Every rational ray in the interior of $\mathrm{Nef}^{n-1}(X)$ is represented by a 1-free curve.*

Before proving Theorem 3.6, we need a more precise version of Proposition 3.3 for dlt varieties.

Proposition 3.7. *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt log Fano pair over an algebraically closed field of characteristic 0. There is a dense set of rays in $\mathrm{Nef}_1(X)$ which are represented by curves $s : C \to X$ whose images are contained in the smooth locus of X and which satisfy $\mu^{min}(s^*T_X) > 0$.*

Proof. [BCHM10, Corollary 1.3.1] proves that X is a Mori Dream Space. Suppose that $\phi : X \dashrightarrow X'$ is a small $\mathbb{Q}$-factorial modification. Let A be a very ample divisor on X'. [Fle84, 1.2 Theorem] implies that for some sufficiently large m we can find a curve C' which is a general complete intersection of elements in $|mA|$ and satisfies

$$\mu^{min}(T_{X'}|_{C'}) = \mu^{min}_{C'}(T_{X'}).$$

Since C' is a general complete intersection, we may ensure that it is contained in the locus where ϕ^{-1} is an isomorphism. Let C denote the strict transform curve on X. By [GKP14, Corollary A.22] we see that $\mu^{min}_{C'}(T_{X'}) = \mu^{min}_C(T_X)$. Combining the equalities above with Proposition 2.13, we have

$$\mu^{min}(T_X|_C) = \mu^{min}(T_{X'}|_{C'}) = \mu^{min}_{C'}(T_{X'}) = \mu^{min}_C(T_X) > 0$$

showing that C has the desired properties.

It only remains to show that the rays spanned by classes of the above form are dense in $\mathrm{Nef}_1(X)$. Given any small $\mathbb{Q}$-factorial modification $\phi : X \dashrightarrow X'$, let N_ϕ denote the strict transform of the classes of the form A^{n-1} on X' where A is an ample $\mathbb{Q}$-Cartier divisor. [LX19, Example 5.4] shows that $\mathrm{Nef}_1(X) = \cup_\phi \overline{N_\phi}$ and this concludes the proof. $\square$

We now can apply this to representing rays of the nef cone by free curves.

Proof of Theorem 3.6: As discussed in Section 2.4, there is a $\mathbb{Q}$-factorialization $\phi : X' \to X$ and a divisor Δ' on X' such that (X', Δ') is $\mathbb{Q}$-factorial klt log Fano. By Lemma 2.4 the induced injective map $\phi^* : \mathrm{Nef}^{n-1}(X) \to \mathrm{Nef}^{n-1}(X')$ preserves the property of being a boundary or interior class. Thus it suffices to prove the statement when (X, Δ) is a $\mathbb{Q}$-factorial klt log Fano pair.

By combining Proposition 3.7 and Theorem 3.2, we see there is a dense set of rays $\{R_i\}$ in the interior of $\mathrm{Nef}_1(X)$ which are represented by classes of 1-free curves. For any rational

ray R in the interior of the nef cone, we can choose a simplicial cone in the interior of $\text{Nef}_1(X)$ containing it. By choosing sufficiently small open neighborhoods of the generators of this simplicial cone and appealing to density, we can find a finite set of 1-free curves $\{C_j\}_{j=1}^r$ such that the cone generated by their numerical classes contains R in its interior. In particular, there is a sum of the C_j with positive integer coefficients which has numerical class lying on R. By choosing appropriate deformations of the C_j, we can glue them to obtain a stable map $s' : C' \to X$ such that the restriction of s' to each component is 1-free. Since $H^1(C', s'^*T_X) = 0$, this curve can be smoothed to obtain a 1-free curve. $\qquad\square$

Remark 3.8. With a little additional effort along the lines of the proof of Proposition 3.5, one can show that every rational ray on the boundary of the nef cone is represented by an almost free curve.

4. RATIONAL CURVES ON TERMINAL FANO THREEFOLDS

In this section we prove the existence of free rational curves in the smooth locus of a terminal Fano threefold. Throughout we work over an algebraically closed field of characteristic 0. Starting with a family of free (higher genus) curves in the smooth locus of X, we carefully study how such curves can break. This allows us to construct a dominant family of rational curves which meet the singular locus no more than twice. Finally, an application of [KM99, Corollary 5.9] yields a free rational curve.

Remark 4.1. When X is a terminal Fano variety with LCI singularities, there is a well-known direct proof of the existence of free rational curves in the smooth locus. Suppose $M \subset \text{Mor}(\mathbb{P}^1, X)$ is an irreducible component parametrizing a dominant family of rational curves $s : \mathbb{P}^1 \to X$. Let $\phi : Y \to X$ be a resolution of singularities and let $s' : \mathbb{P}^1 \to Y$ denote the strict transform of a general s. By [Kol96, II.1.3 Theorem]

$$-K_X \cdot s_*\mathbb{P}^1 + \dim X \leq \dim M = -K_Y \cdot s'_*\mathbb{P}^1 + \dim Y.$$

Thus $K_{Y/X} \cdot C' \leq 0$. Since $K_{Y/X}$ is effective and C' is general in a dominant family, we conclude that equality holds, i.e. $\phi(C')$ avoids the singular locus of X.

We start with a couple helpful lemmas. The first is a basic observation about families of curves which are generically disjoint from an effective divisor E.

Lemma 4.2. *Let X' be a smooth projective variety and let E be an effective divisor on X'. Suppose B is a one-parameter family of stable maps in $\overline{\mathcal{M}}_{g,n}(X', \beta)$ whose general member is disjoint from E.*
Let $s : C \to X'$ be a morphism parametrized by B and suppose that there is an irreducible component $T \subset C$ and a point $q \in T$ such that $s(q) \in \text{Supp}(E)$ but $s(T) \not\subset \text{Supp}(E)$. Then C must have an irreducible component R that meets T at q such that $s(R) \subset \text{Supp}(E)$.

Proof. After perhaps replacing B by a cover, we let $\varphi : \mathcal{C} \to B$ denote the universal family equipped with the evaluation map $\mathfrak{s} : \mathcal{C} \to X'$. Note that every irreducible component of the divisor $\mathfrak{s}^*E$ is φ-vertical. Since T meets $\mathfrak{s}^*E$ at q but is not contained in it, there must be an irreducible component R of $\mathfrak{s}^*E$ meeting T at q. Since every irreducible component of $\mathfrak{s}^*E$ is φ-vertical, R must be an irreducible component of C. $\qquad\square$

The next lemma will allow us to compare intersection numbers on our terminal threefold X and on a resolution of singularities $\phi : X' \to X$. In practice, we will apply it when T is a free rational curve on X'.

Lemma 4.3. *Let $\phi : X' \to X$ be a resolution of singularities of a Fano variety with canonical singularities. Suppose B is a one-parameter family of stable maps in $\overline{\mathcal{M}}_{g,n}(X', \beta)$ whose general member is disjoint from the ϕ-exceptional locus. Let $s : C_0 \to X'$ be any stable map parametrized by B. Suppose $S \subset C_0$ is a connected subcurve such that*

(1) s does not contract S to a point in X', and

(2) for any irreducible component $T \subset \overline{C_0 \setminus S}$ meeting S, $K_{X'} \cdot s_* T \leq 0$ and $s(T)$ is not contained in the exceptional locus of ϕ.

Then $(\phi^* K_X - K_{X'}) \cdot s_* S \geq 0$. Moreover, $-K_{X'} \cdot s_* S \geq 1$ if $\phi \circ s$ does not contract S or if $s(S \cap \overline{C_0 \setminus S})$ meets a ϕ-exceptional divisor which has positive discrepancy.

Proof. We denote the ϕ-exceptional divisors by $\{E_i\}_{i=1}^q$. Let $C_1 = \overline{C_0 \setminus S}$. After perhaps replacing B by a cover, there is a family of curves $\varphi : \mathcal{C} \to B$ with a morphism $\mathfrak{s} : \mathcal{C} \to X'$ realizing our family of stable maps where $\mathcal{C}$ is a surface with $\mathbb{Q}$-factorial singularities and every fiber of φ is reduced.

We claim $E_i \cdot s_* S \leq 0$ for each i. Indeed, since the general curve is disjoint from E_i the pullback $\mathfrak{s}^* E_i$ is φ-vertical. Suppose $\mathfrak{s}^* E_i|_{C_0}$ contains an irreducible component R of C_0. Then $R \cdot C_0 = 0$ because C_0 is the class of a φ-fiber. In particular, if $\overline{R} \subset C_0$ is the union of R with all components in C_0 adjacent to it, then $R \cdot \overline{R} = 0$. If S meets an irreducible component $R \subset \mathfrak{s}^* E_i|_{C_0}$, then $R \subset S$ by assumption (2). Hence, $R \cdot S \leq R \cdot \overline{R} = 0$, with equality if and only if S contains $\overline{R}$. As this holds for all $R \subset \mathfrak{s}^* E_i|_{C_0}$, we conclude $\mathfrak{s}^* E_i \cdot S \leq 0$. Since X has canonical singularities, the inequalities $E_i \cdot s_* S \leq 0$ for all i together imply the first statement.

To see the last statement, first note that since $K_{X'}$ and $s_* S$ are integral classes their intersection must be an integer. Thus it suffices to show that either $K_X \cdot (\phi \circ s)_* S < 0$ or $E_i \cdot s_* S < 0$ for some exceptional divisor E_i on X' with positive discrepancy. By assumption (1), the image $s(S)$ has dimension one in X'. Because X is Fano, if $\phi \circ s$ does not contract S, $K_X \cdot (\phi \circ s)_* S < 0$. So we may assume each component of $s(S)$ is contracted by ϕ. By hypothesis, there is an exceptional divisor E_i with positive discrepancy and a point $p \in S \cap C_1$ such that $s(p) \in E_i$. By Lemma 4.2, there must be a curve $R \subset S \cap \mathfrak{s}^{-1} E_i$ that contains p. For this component, $\overline{R} \not\subset S$, proving $\mathfrak{s}^* E_i \cdot S \leq R \cdot S < 0$. $\qquad \square$

Given a projective variety X and a rational curve $s : \mathbb{P}^1 \to X$, we say that s meets the singular locus k times if the complement of $s^{-1}(X^{sm})$ is a union of k closed points of $\mathbb{P}^1$.

Theorem 4.4. *Let X be a terminal Fano variety. There is a dominant family of rational curves on X such that the general curve meets the singular locus of X at most $\dim X - 1$ times.*

Proof. Set $n = \dim X$. We may suppose $n \geq 3$, as otherwise X is smooth. Suppose for a contradiction that every rational curve that deforms in a dominant family meets the singular locus at least n times.

Let $\phi : X' \to X$ be a resolution of singularities of X which is an isomorphism away from the singular locus of X. Let $\{a_i\}_{i=1}^t$ be the discrepancies of the ϕ-exceptional divisors $\{E_i\}_{i=1}^t$ on X' and set $a_{\min} = \min\{a_1, \ldots, a_t, 1\}$.

By Theorem 1.3, there is a curve C of genus g such that for any $r \geq 0$ the variety X admits an r-free curve $s : C \to X$. By possibly replacing C by a finite cover we may suppose that $g \geq 2$. Choose a positive integer r so that $a_{\min} r > ng + 1$.

According to the previous paragraph, we may select an irreducible component M of $\mathrm{Mor}(C, X)$ that generically parametrizes $2r$-free curves. Let M' be the irreducible component of $\mathrm{Mor}(C, X')$ parametrizing the strict transforms $s' : C \to X'$ of general curves parametrized by M. Let $Z \subset X'$ denote the union of the ϕ-exceptional divisors and the locus in X' swept out by all families of non-free rational curves whose degree against a fixed ample divisor is at most the degree of $s'_* C$.

We let d denote the $-K_X$-degree of the curves parametrized by M. Note that d is also the $-K_{X'}$-degree of the curves parametrized by M'. Since M has the expected dimension we have

$$\dim M = d + (1 - g)n.$$

The image of a general curve $s' : C \to X'$ parametrized by M' will not be contained in Z. By applying [JLR25, Proposition 2.2] to M', we see that for $k = \lfloor \frac{d - n(g-1)}{n+1} \rfloor$ the closure of the image of M' in $\overline{\mathcal{M}}_{g,0}(X)$ parametrizes a stable map $s' : C' \to X'$ where C' contains:

- an irreducible component C_0 isomorphic to C, and
- k trees of rational curves T_i whose image in X' is not contained in Z.

Because C is $2r$-free, $d \geq (2r + 2g)n$ and thus k is at least r. From the definition of k we also see that $(n + 1)k > d - n(g - 1) - (n + 1) = d - ng - 1$.

For each i, we can choose a component $R_i \subset T_i$ whose image in X' is not contained in Z. By construction R_i deforms in a dominant family on X' and thus $-K_{X'} \cdot R_i \geq 2$. By assumption, each R_i meets the ϕ-exceptional locus in at least n points. By Lemma 4.2 there must be at least $n - 1$ connected components of $T_i \setminus R_i$ not adjacent to C_0. By Lemma 4.3, the $-K_{X'}$-degree of each of these connected components must be at least 1. Moreover, either there are n such connected components, or R_i meets the ϕ-exceptional locus along a point which is not incident to any of these components. Altogether we see that the $-K_X$-degree of $\phi_* s_* T_i$ is at least $2 + (n - 1) + a_{\min}$. It follows that d is at least

$$(n + 1 + a_{\min})k = (n + 1)k + a_{\min}k$$
$$> d - ng - 1 + a_{\min}r$$
$$> d - ng - 1 + ng + 1 = d$$

This is impossible. $\qquad\square$

Proof of Theorem 1.4: When X is a terminal Fano threefold, Theorem 4.4 shows that there is a dominant family of rational curves on X which meet the singular locus at most 2 times. Recall that every terminal Fano threefold has LCIQ singularities (see [KM98, Corollary 5.39]). Thus the existence of a free rational curve follows from [KM99, 5.9 Corollary] which shows that any terminal Fano variety that has LCIQ singularities and admits a dominant family of rational curves meeting the singular locus at most twice will also have a free rational curve in its smooth locus. $\qquad\square$

5. Log free curves

In this section we extend previous results to the log setting. Given a $\mathbb{Q}$-Weil divisor Δ, we define the Weil divisor $\lfloor \Delta \rfloor$ by rounding down the coefficients of the irreducible divisors in the formal sum defining Δ.

5.1. **Log tangent bundle and positivity.** Let (X, Δ) be a projective lc pair. There is a smooth open subset $V \subset X$ such that $X \backslash V$ has codimension ≥ 2 and $\lfloor \Delta \rfloor|_V$ is a SNC divisor. We call the largest such open set V the SNC locus of (X, Δ). The log tangent bundle $T_V(-\log \lfloor \Delta \rfloor)$ is the kernel of the map $T_V \to \oplus_i N_{\Delta_i \cap V}$, where the Δ_i are the irreducible components of $\lfloor \Delta \rfloor$.

Definition 5.1. Let (X, Δ) be a projective lc pair with SNC locus V. We define the log tangent bundle $T^{log}_{(X,\Delta)}$ to be the unique saturated subsheaf of T_X that extends $T_V(-\log \lfloor \Delta \rfloor) \subset T_X|_V$.

The log tangent bundle satisfies a natural positivity property.

Theorem 5.2 ([Ou23]). *Let (X, Δ) be a $\mathbb{Q}$-factorial projective lc pair over an algebraically closed field of characteristic 0. Suppose that $-(K_X + \Delta)$ is nef. For every torsion-free quotient $T^{log}_{(X,\Delta)} \to \mathcal{Q}$ the divisor class $c_1(\mathcal{Q})$ is pseudo-effective.*

[Ou23] proves this statement for the orbifold tangent bundle as defined by [CP19]. One could prove Theorem 5.2 by comparing the orbifold tangent bundle and $T^{log}_{(X,\Delta)}$ using [Cla17, Définition 2.10]. We will instead prove the desired positivity simply by repeating the arguments of [Ou23].

Proof. Assume for a contradiction that the statement fails. Then there is a nef curve class α such that $\mu^{min}_\alpha(T^{log}_{(X,\Delta)}) < 0$. On the other hand, since $-K_X - \Delta$ is nef the divisor $-K_X - \lfloor \Delta \rfloor$ is pseudo-effective and thus $\mu_\alpha(T^{log}_{(X,\Delta)}) \geq 0$. We conclude that the α-Harder-Narasimhan filtration of $T^{log}_{(X,\Delta)}$ includes a term with positive slope. By taking the last such term, we find $\mathcal{G} \subset T^{log}_{(X,\Delta)}$ such that $\mu^{min}_\alpha(\mathcal{G}) > 0$, $\mu^{max}_\alpha(T^{log}_{(X,\Delta)}/\mathcal{G}) \leq 0$, and $\mu_\alpha(T^{log}_{(X,\Delta)}/\mathcal{G}) < 0$.

Let $\phi : Y \to X$ be a log resolution of (X, Δ) which is an isomorphism over the SNC locus and let Δ_Y denote the sum of the strict transform of $\lfloor \Delta \rfloor$ with the reduced ϕ-exceptional divisor. Then (Y, Δ_Y) is a smooth SNC pair. By [GKP14, Corollary A.22] we see that the terms in the $\phi^*\alpha$-Harder-Narasimhan filtration of $T_Y(-\log \Delta_Y)$ will coincide with the terms of the α-Harder-Narasimhan filtration of $T^{log}_{(X,\Delta)}$ when restricted to the locus where ϕ is an isomorphism. In particular, there is a subsheaf $\mathcal{G}_Y \subset T_Y(-\log \Delta_Y)$ that is isomorphic to $\mathcal{G}$ over the SNC locus of X and satisfies the same slope inequalities as $\mathcal{G}$ on X. By [CP19, Theorem 1.4] the saturation $\mathcal{F}_Y$ of $\mathcal{G}_Y$ in T_Y defines an algebraically integrable foliation corresponding to a rational map $\pi_Y : Y \dashrightarrow Z$. By [Cla17, Proposition 2.17] we have that $c_1(\mathcal{G}_Y) = c_1(\mathcal{F}_Y) - \Delta_{Y,hor}$ where $\Delta_{Y,hor}$ denotes the horizontal part of Δ_Y for the rational map π_Y.

Passing back down to X, we see that the saturation of $\mathcal{G}$ inside T_X, which we denote by $\mathcal{F}$, is the foliation induced by a rational map $\pi : X \dashrightarrow Z$. We define Δ_{hor} and Δ_{ver} to be the horizontal and vertical parts of Δ with respect to π. Furthermore we know that $c_1(\mathcal{G}) = c_1(\mathcal{F}) - \lfloor \Delta_{hor} \rfloor$. On the other hand, [Ou23, Theorem 1.10] shows that $-c_1(\mathcal{F}) - K_X - \Delta_{ver}$

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is pseudo-effective. Thus the same is true of $-c_1(\mathcal{F}) - K_X - \lfloor \Delta_{ver} \rfloor$. Since we can identify

$$c_1(T^{log}_{(X,\Delta)}/\mathcal{G}) = -K_X - \lfloor \Delta \rfloor - (c_1(\mathcal{F}) - \lfloor \Delta_{hor} \rfloor) = -c_1(\mathcal{F}) - K_X - \lfloor \Delta_{ver} \rfloor$$

this contradicts the fact that $\mu_\alpha(T^{log}_{(X,\Delta)}/\mathcal{G}) < 0$. $\qquad\square$

Just as with the usual tangent bundle, one can characterize the failure of the PQP for the log tangent bundle via a maximal quotient. However, we will not need this statement in this paper.

5.2. **Log free curves.** Let (X, Δ) be a projective lc pair. We define the space of log curves associated to (X, Δ) as follows:

Definition 5.3. Let (X, Δ) be a projective lc pair and let C be a smooth projective curve equipped with a reduced divisor D. Suppose that $\{\Delta_i\}_{i=1}^k$ are the irreducible components of $\lfloor \Delta \rfloor$. For each index i choose an effective divisor $D_i \subset C$ whose support is contained in D. We define the space $\mathrm{Mor}^{log}_{\{D_i\}}(C, X)$ to be the locally closed subscheme of $\mathrm{Mor}(C, X)$ consisting of maps $s : C \to X$ such that

    (1) the image $s(C)$ is contained in the SNC locus of $(X, \lfloor \Delta \rfloor)$ but not in $\mathrm{Supp}(\lfloor \Delta \rfloor)$, and
    (2) $s^*\Delta_i = D_i$ for every index i.

We will say that a map $s : (C, D) \to (X, \Delta)$ is a log curve when we want to think of it as an element of the appropriate space $\mathrm{Mor}^{log}_{\{D_i\}}(C, X)$. We write $\mathrm{Mor}^{log}(C, X)$ for the union of the moduli spaces as we vary over all $\{D_i\}$.

Let $V \subset X$ denote the open locus where the pair $(X, \lfloor \Delta \rfloor)$ is SNC. We can equip $C \times V$ with the log structure defined by the SNC divisor $C \times \lfloor \Delta \rfloor|_V$. By identifying a morphism $s : C \to V$ with its graph, one can identify $\mathrm{Mor}^{log}(C, X)$ as an open substack of the moduli space of log sections $\mathrm{Sec}_{log}(C \times V/C)$ as constructed in [CLT25, Section 3.1]. According to [CLT25, Discussion after Equation (3.5)], the log deformation theory of s is controlled by the cohomology groups of the pullback of $T_V(-\log\lfloor \Delta \rfloor)$, or equivalently, by the cohomology groups of $s^*T^{log}_{(X,\Delta)}$.

Definition 5.4. Let (X, Δ) be a projective lc pair and let C be a smooth projective curve of genus g. Fix an $r \geq 0$. A log curve $s : (C, D) \to (X, \Delta)$ is r-free if $\mu^{min}(s^*T^{log}_{(X,\Delta)}) \geq 2g(C) + r$.

We say that $s : (C, D) \to (X, \Delta)$ is an unmixed r-free log curve if additionally for every pair of distinct irreducible components Δ_i, Δ_j of $\lfloor \Delta \rfloor$ the preimages $s^*\Delta_i$ and $s^*\Delta_j$ are disjoint. (In other words, $s(C)$ is only allowed to meet the boundary $\lfloor \Delta \rfloor$ along codimension 1 strata.)

Lemma 5.5. *Let (X, Δ) be a projective lc pair over an algebraically closed field of characteristic 0. Suppose that $s : (C, D) \to (X, \Delta)$ is an r-free log curve. Then there is a log resolution $\phi : Y \to X$ of $(X, \lfloor \Delta \rfloor)$ and an effective divisor Δ_Y on Y such that $\phi : (Y, \Delta_Y) \to (X, \Delta)$ is log étale on a neighborhood of $s(C)$ and the strict transform of s on Y is an unmixed r-free curve with respect to (Y, Δ_Y).*

In characteristic p, one can accomplish the same goal simply by repeatedly blowing up strata of the boundary, but we will not need such a result.

Proof. For any log resolution $\phi : Y \to X$ of $(X, \lfloor \Delta \rfloor)$ we define Δ_Y to be the sum of the strict transform of $\lfloor \Delta \rfloor$ with the reduced ϕ-exceptional divisor.

Let $\phi : Y \to X$ be any log resolution of $(X, \lfloor \Delta \rfloor)$ that is an isomorphism over the SNC locus. Thus the strict transform s_Y of s is r-free. If s_Y is not unmixed, then we let $\psi : \widetilde{Y} \to Y$ denote the blow-up of any minimal dimension strata of $\lfloor \Delta_Y \rfloor$ that meets every log deformation of s_Y. Since this blow-up is log étale, the strict transform $s_{\widetilde{Y}}$ is again r-free. After repeating this construction finitely many times we obtain the desired birational model. $\square$

The following lemma verifies the basic properties of free log curves.

Lemma 5.6. *Let (X, Δ) be a projective lc pair. Suppose that $s : (C, D) \to (X, \Delta)$ is an r-free log curve with $r \geq 0$.*

(1) *If s is unmixed, then for any codimension ≥ 2 closed subset $Z \subset X$ a general log deformation of s will be disjoint from Z.*
(2) *If $M \subset \operatorname{Mor}^{log}(C, X)$ is the irreducible component containing s then the $(\lfloor r \rfloor + 1)$-fold evaluation map $\psi : C^{\times (\lfloor r \rfloor + 1)} \times M \to C^{\times (\lfloor r \rfloor + 1)} \times X^{\times (\lfloor r \rfloor + 1)}$ is dominant.*

Proof. We first explain (1). Observe that unmixed log free curves only meet the smooth locus of $\lfloor \Delta \rfloor$. Let M° be the open locus in M parametrizing r-free log curves. For $x \in X$, consider the set $U_{p,x} \subset M^\circ \times C$ of pairs (s, p) with $s(p) = x$. For $x \in X^{sing} \cup \lfloor \Delta \rfloor^{sing}$, $U_{p,x}$ is empty. For $x \in X \setminus \lfloor \Delta \rfloor$, $U_{p,x}$ has dimension at most $\dim M - \dim X$ because the map $U_{p,x} \to X$ is smooth due to the r-free assumption. For $x \in \lfloor \Delta \rfloor^{sm}$, $U_{p,x}$ has dimension at most $\dim M - \dim X + 1$ because the map $T^{log}_{(X,\Delta)}|_p \to T_X|_p$ has image of codimension 1; however, $U_{p,x}$ will be empty if $p \notin D$. Taking the union over all $p \in C$, we get a set U_x of dimension at most $\dim M - \dim X + 1$. Taking the union over all $x \in Z$, we get a set U_Z of dimension at most $\dim M - 1$. Since U_Z does not dominate M, the result follows.

To see (2), it suffices to find a single point $(s, p_1, \ldots, p_{\lfloor r \rfloor + 1})$ where the map $d\psi$ is surjective. In other words, it suffices to find $(s, p_1, \ldots, p_{\lfloor r \rfloor + 1})$ with

$$h^1(C, s^* T^{log}_{(X,\Delta)}(-p_1 - \cdots - p_{\lfloor r \rfloor + 1})) = 0.$$

The existence of such an s follows immediately from the hypothesis that s is r-free: Serre duality guarantees the vanishing of h^1 for any choice of $\lfloor r \rfloor + 1$ points on C. $\square$

5.3. **Existence of log free curves.** The next result is the analogue of Theorem 1.7 in the log setting.

Theorem 5.7. *Let (X, Δ) be a projective lc pair of dimension n over an algebraically closed field. Suppose that H is an ample divisor such that $\mu^{min}_{H^{n-1}}(T^{log}_{(X,\Delta)}) > 0$. Then there is a smooth projective curve C and a finite set $D \subset C$ such that for every $r \geq 0$ there is an r-free log curve $s : (C, D) \to (X, \Delta)$.*

Proof. Let $i : C \to X$ denote a general curve obtained as a complete intersection of elements of $|mH|$ for m sufficiently large. We may ensure that $i(C)$ will lie in the SNC locus of (X, Δ). By a repeated application of [Fle84, 1.2 Theorem] (in characteristic 0) or [Lan24, 0.1 Theorem] (in characteristic p) we may also ensure that $\mu^{min}(i^* T^{log}_{(X,\Delta)}) > 0$.

In characteristic p, Lemma 3.1 shows that we can precompose i by an iterated Frobenius to construct an r-free log curve s. Since the Frobenius morphism is a homeomorphism, the support of the pullback of $\lfloor \Delta \rfloor$ does not depend on r and thus we may choose $D = i^{-1} \lfloor \Delta \rfloor$.

In characteristic 0, as in the proof of Theorem 3.2 we spread out to characteristic p by constructing

- a family $\mathcal{X} \to \mathrm{Spec}(R)$ equipped with a boundary Δ_R and an open locus $\mathcal{U}$, and
- a smooth projective curve $\mathcal{C}$ over $\mathrm{Spec}(R)$ equipped with a morphism $i : \mathcal{C} \to \mathcal{X}$ and with a reduced divisor $\mathcal{D} \subset \mathcal{C}$ such that $i^{-1}(\lfloor \Delta_R \rfloor) \subset \mathcal{D}$.

By [Nit11, Theorem 5], for a general prime $\mathfrak{p} \in \mathrm{Spec}(R)$ we will still have $\mu^{min}(i_{\mathfrak{p}}^* T^{log}_{(\mathcal{X}_{\mathfrak{p}}, \Delta_{\mathfrak{p}})}) > 0$. Lemma 3.1 shows that by precomposing by an iterated Frobenius we obtain an r-free log curve $s_{\mathfrak{p}} : \mathcal{C}_{\mathfrak{p}} \to \mathcal{X}_{\mathfrak{p}}$. Since the Frobenius is a homeomorphism we see that $s_{\mathfrak{p}}^{-1}(\lfloor \Delta_{R,\mathfrak{p}} \rfloor)$ will continue to lie in $\mathcal{D}_{\mathfrak{p}}$.

Arguing as in [Kol96, p95], one may construct a space of relative log maps $\mathrm{Mor}_R^{log}(\mathcal{C}, \mathcal{X})$ that is cut out from the relative morphism scheme $\mathrm{Mor}_R(\mathcal{C}, \mathcal{X})$ by $\deg s_{\mathfrak{p}}^* \lfloor \Delta \rfloor$ equations in a neighborhood of $s_{\mathfrak{p}}$. Thus any irreducible component M of $\mathrm{Mor}_R^{log}(\mathcal{C}, \mathcal{X})$ that contains $s_{\mathfrak{p}}$ has dimension at least $H^0(\mathcal{C}_{\mathfrak{p}}, s_{\mathfrak{p}}'^* T^{log}_{(\mathcal{X}_{\mathfrak{p}}, \lfloor \Delta_{R,\mathfrak{p}} \rfloor)}) + \dim_{\mathfrak{p}}(\mathrm{Spec}(R))$ as in [Kol96, I.2.17 Theorem]. Because $s_{\mathfrak{p}}$ is an r-free log curve, it follows that $\mathrm{Mor}^{log}(\mathcal{C}_{\mathfrak{p}}, \mathcal{X}_{\mathfrak{p}})$ must have the expected dimension at $s_{\mathfrak{p}}$, and hence M must dominate $\mathrm{Spec}\,R$. A general log deformation of $s_{\mathfrak{p}}$ in the generic fiber of M yields an r-free log curve $s : (C, D) \to (X, \Delta)$ by [Nit11, Theorem 5]. $\square$

In characteristic 0, we prove the analogue of Theorem 3.4 in the log setting.

Theorem 5.8. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial lc pair over an algebraically closed field of characteristic 0. Suppose that $-(K_X + \Delta)$ is nef. Then the following conditions are equivalent:*

(1) *The log tangent bundle admits no positive rank quotient with numerically trivial first Chern class.*
(2) *(X, Δ) admits a 1-free log curve.*
(3) *For every $m > 0$ we have $h^0(V, \Omega_V(\log \lfloor \Delta \rfloor)^{\otimes m}) = 0$ where $V \subset X$ is the SNC locus of the pair $(X, \lfloor \Delta \rfloor)$.*

If these equivalent conditions hold, then there is a smooth projective curve C and a finite set $D \subset C$ such that for every $r \geq 0$ there is an r-free log curve $s : (C, D) \to (X, \Delta)$.

Proof. (1) $\implies$ (2): Under assumption (1), $T^{log}_{(X,\Delta)}$ satisfies the PQP by Theorem 5.2. Thus for any ample divisor H we have $\mu^{min}_{H^{n-1}}(T^{log}_{(X,\Delta)}) > 0$. We conclude the existence of r-free curves by Theorem 5.7.

(2) $\implies$ (3): Under assumption (2), for any general point $x \in V$ there is a curve $s : C \to V$ such that $s^* T^{log}_{(X,\Delta)} = s^* \Omega_V(\log \lfloor \Delta \rfloor)^\vee$ is ample. Thus any section of $\Omega_V(\log \lfloor \Delta \rfloor)^{\otimes m}$ must vanish at x. Since this is true for a dense set of points $x \in V$, we conclude that the space of global sections is 0.

(3) $\implies$ (1): Let $\phi : Y \to X$ be a log resolution of $(X, \lfloor \Delta \rfloor)$ that is an isomorphism over V and let Δ_Y be the sum of the strict transform of $\lfloor \Delta \rfloor$ with the reduced ϕ-exceptional divisor. Then we have $h^0(Y, \Omega_Y(\log \Delta_Y)^{\otimes m}) = 0$ for every $m > 0$ as well. Since Ω_Y is a subsheaf of $\Omega_Y(\log \Delta_Y)$, we conclude that $\mathrm{Pic}^0_{Y/\mathbb{C}} = 0$, so every numerically trivial Cartier

divisor on Y is torsion. By appealing to $\mathbb{Q}$-factoriality and pulling back divisors from X to Y, we see the same is true of numerically trivial Weil divisors on X.

Suppose we had a surjection $s^*T^{log}_{(X,\Delta)} \to \mathcal{Q}$ where $c_1(\mathcal{Q})$ was numerically trivial. Since a torsion sheaf always has pseudo-effective first Chern class, we may suppose $\mathcal{Q}$ is torsion-free. Then for any $m > 0$ we have a chain of surjections on V

$$T_V(-\log\lfloor\Delta\rfloor)^{\otimes m\operatorname{rk}(\mathcal{Q})} \to \mathcal{Q}|_V^{\otimes m\operatorname{rk}(\mathcal{Q})} \to \left(\bigwedge^{\operatorname{rk}(\mathcal{Q})} \mathcal{Q}|_{X^{sm}}\right)^{\otimes m}.$$

In particular, when m satisfies $mc_1(\mathcal{Q}) \sim 0$ we obtain a surjection $T_V(-\log\lfloor\Delta\rfloor)^{\otimes m\operatorname{rk}(\mathcal{Q})} \to \mathcal{O}_V$. Dually, we get a non-vanishing tensor field on V, a contradiction. $\square$

Proof of Theorem 1.9: Follows immediately from Theorem 5.8. $\square$

6. FINITENESS OF FUNDAMENTAL GROUPS

We next address finiteness of the fundamental group. For the smooth locus of a klt Fano variety over $\mathbb{C}$, this famous question has been addressed in various settings – the étale fundamental group, the topological fundamental group, and the regional fundamental group of a klt singularity – by [Xu14], [GKP16a], [BGO17], [TX17], [CRST18], [BCRG$^+$19], [Bra21].

In this section we use the existence of 1-free curves to prove finiteness of fundamental groups. (It was known previously that the existence of a 1-free rational curve in X^{sm} could be used to prove this statement – see e.g. [KM99, 7.5 Lemma].) The proof relies on four properties of (topological or étale) fundamental groups over an algebraically closed field k.

(A) Suppose X, Y are smooth varieties over k with X proper. Given geometric points $x \in X$, $y \in Y$, the natural homomorphism $\pi_1(X \times Y, (x,y)) \to \pi_1(X,x) \times \pi_1(Y,y)$ is an isomorphism. Furthermore the projection map is split by the inclusion of $\{x\} \times Y$.

(B) Suppose $f : X \to Y$ is a dominant morphism of normal varieties over k. For any k-point $x \in X$ the image $f_*\pi_1(X,x)$ has finite index in $\pi_1(Y, f(x))$.

(C) We have $\pi_1(\operatorname{Spec}(k), \operatorname{Spec}(k)) = 0$.

(D) Let X be a variety over k. If x, x' are two k-points of X then there is an isomorphism between $\pi_1(X,x)$ and $\pi_1(X,x')$ that is defined up to conjugacy.

For topological fundamental groups, (A) is established by [Hat02, Proposition 1.12] and (B) is established by [Kol93, 2.9 Proposition]. For étale fundamental groups, (A) is established by [Gro63, Exposé X Corollaire 1.7]. The argument for (B) seems to be well-known: one can write f as a composition of an open embedding, a morphism with geometrically connected fibers, and a generically étale map. The first two maps induce surjections on étale fundamental groups and the last induces a map whose image has finite index. Properties (C) and (D) are well-known in both settings.

Theorem 6.1. *Let U be a smooth variety over an algebraically closed field k and let $x \in U$ be a k-point. Suppose that C is a smooth projective k-curve, M is a k-variety, and $e : C \times M \to U$ is a k-morphism such that the induced map*

$$r = (p_1, p_2, e \circ p_{13}, e \circ p_{23}) : C \times C \times M \to C \times C \times U \times U$$

is dominant, where p_i and p_{ij} denote projections onto the corresponding factors. Then:

(1) The étale fundamental group $\pi_1^{et}(U,x)$ is finite.

(2) If $k = \mathbb{C}$, the topological fundamental group $\pi_1(U, x)$ is finite.

Proof. We start with (1). Fix a general k-point $x \in U$ and a general k-point $c \in C$. Let $S \subset M$ be the set of curves sending c to x, i.e., $S = p_3(r^{-1}(C \times \{c\} \times U \times \{x\}))$. By assumption the map $e : C \times S \to U$ is dominant. We may replace S by a smooth open subset of a suitably chosen irreducible component while still keeping the dominance of $e : C \times S \to U$.

Let $s : C \to U$ be a general curve parametrized by S. We claim that the subgroup $s_* \pi_1^{et}(C, c) \subset \pi_1^{et}(U, x)$ has finite index. Indeed, (B) above shows that the subgroup $e_* \pi_1^{et}(C \times S, c \times s) \subset \pi_1^{et}(U, x)$ has finite index. Note however that e contracts the section $\{c\} \times S$ of $p_2 : C \times S \to S$. Using the isomorphism $\pi_1^{et}(C \times S, c \times s) \cong \pi_1^{et}(C, c) \times \pi_1^{et}(S, s)$ from (A) above and the fact that the projection map is split by the inclusion of $\{c\} \times S$, (C) implies that $e_* \pi_1^{et}(C \times S, c \times s) = s_* \pi_1^{et}(C, c)$, proving the claim.

Thus it suffices to show that $s_* \pi_1^{et}(C, c)$ is finite. Fix general points $c_1, c_2 \in C$ and a general point $y \in U$. Define $R = p_3(r^{-1}(\{c_1\} \times \{c_2\} \times \{y\} \times s(C)))$. By generality of s, the map $e : C \times R \to U$ sends $\{c_1\} \times R$ to y and maps $\{c_2\} \times R$ dominantly to $s(C)$. Let R' be the fiber product of $e|_{\{c_2\} \times R}$ and s over $s(C)$. The induced map $e' : C \times R' \to U$ has the additional property that $e'|_{\{c_2\} \times R'}$ factors through s. Let $f : \{c_2\} \times R' \to C$ be such that $e'|_{\{c_2\} \times R'} = s \circ f$.

After shrinking R', we may assume R' is smooth and irreducible and that the image of f is a dense subset of C. Appealing to generality of $c \in C$, we may assume it lies in the image of f. Fix a point $r \in R'$ such that $f(c_2 \times r) = c$. By (B) above $e'_* \pi_1^{et}(\{c_2\} \times R', c_2 \times r)$ has finite index in $s_* \pi_1(C, c)$. However, e' contracts the section $\{c_1\} \times R'$ of $C \times R' \to R'$. By (D) there is an isomorphism $\pi_1^{et}(C, c_2) \cong \pi_1^{et}(C, c_1)$ which is determined up to an inner automorphism; this induces an isomorphism

$$\pi_1^{et}(C \times R', c_2 \times r) \cong \left(\pi_1^{et}(C, c_2) \times \pi_1^{et}(R', r) \right) \cong \left(\pi_1^{et}(C, c_1) \times \pi_1^{et}(R', r) \right) \cong \pi_1^{et}(C \times R', c_1 \times r)$$

that is determined up to conjugacy on the first factor of the middle terms. We conclude that the image of $\pi_1^{et}(\{c_2\} \times R', c_2 \times r)$ is conjugate to the image of $\pi_1^{et}(\{c_1\} \times R', c_1 \times r)$, which is the trivial group by (B). This concludes the proof.

The proof of (2) is the same as the proof of (1). $\qquad\square$

Proof of Theorem 1.6: Theorem 3.4 proves the existence of a 1-free curve $s : C \to X^{sm}$. Proposition 2.7.(2) shows that deformations of s go through two general points of X^{sm}. We conclude the desired statement by Theorem 6.1.(2). $\qquad\square$

Proof of Theorem 1.8: Follows from Proposition 2.7.(2) and Theorem 6.1.(1). $\qquad\square$

6.1. **Log setting.** We next prove a finiteness statement for fundamental groups in the log setting. Note that in characteristic p even the simplest example $U = \mathbb{A}^1 = \mathbb{P}^1 \backslash \{\infty\}$ will have infinite étale fundamental group. We must instead work with the curve-tame étale fundamental group as defined in [Wie08], [KS10]. We write $\pi_1^t(X, x) := \pi_1^t(X/\operatorname{Spec}(k), x)$ for this group.

The following statement is almost identical to Theorem 6.1. The key difference is that the curve C is only quasiprojective, since in the log setting we can no longer hope to find projective curves in the open locus U. The proof relies on results of Appendix A.

Theorem 6.2. *Let U be a smooth variety over an algebraically closed field k and let $x \in U$ be a k-point. Suppose that C is a smooth irreducible quasi-projective k-curve, M is a k-variety,*

and $e : C \times M \to U$ is a k-morphism such that the induced map

$$r = (p_1, p_2, e \circ p_{13}, e \circ p_{23}) : C \times C \times M \to C \times C \times U \times U$$

is dominant, where p_i and p_{ij} denote projections onto the corresponding factors.

(1) The tame fundamental group $\pi_1^t(U, x)$ is finite.
(2) If $k = \mathbb{C}$, then the topological fundamental group $\pi_1(U, x)$ is finite.

Proof. (1) We have the following properties of tame fundamental groups. The only difference from the corresponding properties for étale fundamental groups is that no properness assumption is necessary in (A).

(A) Suppose X, Y are smooth varieties over k. Given geometric points $x \in X$, $y \in Y$, Theorem A.1 shows that the natural homomorphism $\pi_1^t(X \times Y, (x, y)) \to \pi_1^t(X, x) \times \pi_1^t(Y, y)$ is an isomorphism. Furthermore the projection map is split by the inclusion of $\{x\} \times Y$.

(B) Suppose $f : X \to Y$ is a dominant morphism of normal varieties over k. Then for any k-point $x \in X$, $f_* \pi_1^t(X, x)$ has finite index in $\pi_1^t(Y, f(x))$. This follows from the corresponding fact for étale fundamental groups via the existence of a functorial surjection $\pi_1^{et}(X, x) \to \pi_1^t(X, x)$.

(C) We have $\pi_1^t(\mathrm{Spec}(k), \mathrm{Spec}(k)) = 0$.

(D) Let X be a variety over k. If x, x' are two k-points of X then there is an isomorphism between $\pi_1^t(X, x)$ and $\pi_1^t(X, x')$ that is defined up to conjugacy.

Using these properties, the proof is identical to the proof of Theorem 6.1.(1).

(2) Follows from the same argument as Theorem 6.1.(2). $\qquad\square$

Proof of Theorem 1.11: Let $U = X \backslash \lfloor \Delta \rfloor$. Let $M \subset \mathrm{Mor}_{\{D_i\}}^{log}(C, X)$ denote an irreducible component containing a 1-free log curve $s : (C, D) \to (X, \Delta)$ and let $e : C \times M \to X$ denote the evaluation map. Lemma 5.6.(2) shows that the induced map $C^{\times 2} \times M \to C^{\times 2} \times X^{\times 2}$ is dominant.

We let $C^\circ \subset C$ be the complement of D so that we have a map $e : C^\circ \times M \to U^{sm}$. It is clear that $C^{\circ \times 2} \times M \to C^{\circ \times 2} \times (U^{sm})^{\times 2}$ is still dominant. We conclude by applying Theorem 6.2. $\qquad\square$

6.2. Dichotomy.

Theorem 3.4 and Lemma 2.12 establish a fundamental geometric dichotomy for $\mathbb{Q}$-factorial lc Fano pairs (X, Δ) over $\mathbb{C}$. When T_X satisfies the PQP, X also satisfies other desirable geometric properties:

(1) X carries a 1-free curve by Theorem 3.4,
(2) X^{sm} has no non-vanishing tensor fields by Theorem 3.4,
(3) X is rationally connected by [Gou16, Theorem 5.2], and
(4) the smooth locus of X has finite fundamental group by Theorem 1.6.

On the other hand, when T_X does not satisfy the PQP then Lemma 2.12 shows that there is a canonical quotient $\pi : T_X \to \mathcal{Q}$ witnessing this failure where $c_1(\mathcal{Q})$ is numerically trivial. The kernel of π is an algebraically integrable foliation and every almost free curve on X is contained in a fiber of the corresponding rational map. Indeed, as an almost free curve $s : C \to T_X$ can be moved off of any codimension 2 subset, we may assume $s(C)$ is contained in the smooth locus of the map induced by $T_X \to \mathcal{Q}$. Only the trivial factors in $s^* T_X$ can admit non-trivial maps to $s^* \mathcal{Q}$, showing that $s(C)$ must be contained in a fiber.

The following example clarifies that the dichotomy discussed above is not controlled by the rationally connected property or by the fundamental group.

Example 6.3. We give an example of an lc Fano variety X such that X is rationally connected, X^{sm} has finite fundamental group, and T_X fails the PQP. (In other words, the PQP for T_X implies the first two properties but is not necessary for them to hold.)

Let B be a smooth sextic in $\mathbb{P}^2$. Starting from $\mathbb{P}^2 \times \mathbb{P}^1$, we choose a section $S = \mathbb{P}^2 \times \{0\}$ and define $F = B \times \mathbb{P}^1$. We then perform the following blow-ups:

- blow up $S \cap F$ to obtain an exceptional divisor $E_1 \cong B \times \mathbb{P}^1$;
- blow up the intersection of E_1 with the strict transform of F to get an exceptional divisor E_2.

The resulting variety $\widetilde{X}$ is equipped with a morphism $\widetilde{\pi} : \widetilde{X} \to \mathbb{P}^2$ obtained by composing the blow-downs and the projection map. We denote by $\widetilde{S}, \widetilde{F}, \widetilde{E}_1$ the strict transforms on $\widetilde{X}$ of S, F, E_1. We also denote by H_1 and H_2 the pullbacks to $\widetilde{X}$ of $\mathcal{O}(1)$ from $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively. Every fiber of π over B consists of three components: a reduced component $\ell_{\widetilde{F}}$ lying in $\widetilde{F}$, a reduced component $\ell_{\widetilde{E}_1}$ lying in $\widetilde{E}_1$ and a multiplicity two component ℓ_{E_2} lying in E_2.

We next construct a variety X by contracting certain divisors on $\widetilde{X}$. Define the divisor $D = 6H_1 + 2H_2 - \widetilde{E}_1 - 2E_2$. Note that D can also be expressed as $D = \widetilde{F} + 2H_2$ and as $D = 2\widetilde{S} + \widetilde{E}_1 + 6H_1$. From these expressions we see that D is big and nef and that the base locus of $|D|$ is empty. Since D has vanishing intersection against $\ell_{\widetilde{F}}$, $\ell_{\widetilde{E}_1}$, and every curve in $\widetilde{S}$, the linear series $|D|$ defines a birational morphism $\phi : \widetilde{X} \to X$ that contracts the divisors $\widetilde{F}, \widetilde{E}_1, \widetilde{S}$. In particular X has Picard rank 1. Since $\phi^* K_X = K_{\widetilde{X}} + \widetilde{S} + \frac{1}{2}\widetilde{E}_1$ we see that X is a log Fano variety with log canonical singularities.

Note that $\widetilde{\pi} : \widetilde{X} \to \mathbb{P}^2$ induces a rational map $\pi : X \dashrightarrow \mathbb{P}^2$ which is a morphism on X^{sm}. We claim that this map yields a quotient $T_X \to \mathcal{Q}$ such that $c_1(\mathcal{Q})$ is numerically trivial. Indeed, consider the corresponding map $\widetilde{\pi} : \widetilde{X} \to \mathbb{P}^2$. Since the fibers over B have multiplicity 2 along E_2, the image of the map $T_{\widetilde{X}} \to \widetilde{\pi}^* T_{\mathbb{P}^2}$ has first Chern class $3H_1 - E_2$. Since this divisor is $\mathbb{Q}$-linearly equivalent to $\frac{1}{2}(\widetilde{F} + \widetilde{E}_1)$, by pushing forward to X we see that $\mathcal{Q}$ has numerically trivial first Chern class.

The previous paragraph showed that T_X does not have the PQP. It is clear that X is rationally connected. It only remains to verify that X^{sm} has finite fundamental group. The double cover of $\mathbb{P}^2$ ramified over B is a K3 surface Z. Let U denote the normalization of the base change $Z \times_{\mathbb{P}^2} X^{sm}$. Since every fiber of X^{sm} over B has multiplicity 2, $U \to X^{sm}$ is an étale double cover. Furthermore, U is an open subset of an $\mathbb{A}^1$-bundle Y over Z where the complement of U in Y has codimension ≥ 2. This implies that $\pi_1(U)$, and hence also $\pi_1(X^{sm})$, is finite.

APPENDIX A. TAME FUNDAMENTAL GROUP OF A PRODUCT
(WRITTEN BY AISE JOHAN DE JONG)

Setup: we take an algebraically closed field k and we set $S = \operatorname{Spec}(k)$. When X is a smooth variety over k we set

$$\pi_1^t(X) = \pi_1^t(X/S, \overline{x})$$

for some choice of geometric point $\overline{x}$ of X as defined in [KS10, Section 7]. Correspondingly we will say a finite étale cover of X is *tame* if it is curve-tame in the sense of [KS10].

If Y is a second smooth variety over k, then we denote by $X \times Y$ the product variety. Choose closed points $x \in X$, $y \in Y$. We will use x, y, and (x,y) as the base points of our tame fundamental groups. Functoriality of the tame fundamental group (discussed in [KS10]) gives homomorphisms

$$\pi_1^t(X) \to \pi_1^t(X \times Y) \quad \text{and} \quad \pi_1^t(Y) \to \pi_1^t(X \times Y)$$

It turns out that we also have a canonical surjective map

$$(*) \qquad\qquad \pi_1^t(X \times Y) \to \pi_1^t(X) \times \pi_1^t(Y)$$

corresponding to the fully faithful functor

$$F\acute{E}t_X \times F\acute{E}t_Y \to F\acute{E}t_{X \times Y}, (U, V) \mapsto U \times V$$

of categories of (tame) finite étale coverings. This looks strange at first sight; in terms of Galois groups, it corresponds to the fact that the absolute Galois group of the function field $k(X \times Y)$ maps onto the product of the absolute Galois groups of $k(X)$ and $k(Y)$.

Theorem A.1. *The map (*) is an isomorphism.*

[Hos09, Proposition 3] gives a related but slightly weaker statement in a different setting.

Proof. We will reduce this theorem to Lemma A.2.

We claim that the images of $\pi_1^t(X)$ and $\pi_1^t(Y)$ in $\pi_1^t(X \times Y)$ commute. By Lemma A.2 this is true for elements in the image of $\pi_1^t(C) \to \pi_1^t(X)$ and $\pi_1^t(D) \to \pi_1^t(Y)$ for any pair of morphisms $f : C \to X$ and $g : D \to Y$ where C and D are smooth curves and $x \in f(C)$ and $y \in g(D)$. By Lemma A.3 such elements form dense subgroups of $\pi_1^t(X)$ and $\pi_1^t(Y)$ and this proves the claim. Thus we have an induced map $\pi_1^t(X) \times \pi_1^t(Y) \to \pi_1^t(X \times Y)$. It is clear that this map splits the surjection $\pi_1^t(X \times Y) \to \pi_1^t(X) \times \pi_1^t(Y)$ discussed earlier.

It suffices to show that the map $\pi_1^t(X) \times \pi_1^t(Y) \to \pi_1^t(X \times Y)$ is surjective. Suppose that $Z \to X \times Y$ is a connected tame Galois cover. By Lemma A.3 we can choose a morphism $g : C \to X \times Y$ where C is a smooth curve with $(x, y) \in g(C)$ such that $Z \times_{X \times Y} C$ is connected. Let C_1, resp. C_2 be the normalization of the image of C in X, resp. Y. Since g factors through $C_1 \times C_2$, the covering $Z \times_{X \times Y} (C_1 \times C_2)$ is also connected. By Lemma A.2. we see that Z defines a connected $\pi_1^t(C_1) \times \pi_1^t(C_2)$ set. As the action of these groups on Z factors through $\pi_1^t(X) \times \pi_1^t(Y)$, we obtain the claim. $\qquad\square$

Lemma A.2. *If X and Y are curves, then (*) is an isomorphism.*

Proof. Choose smooth projective compactifications $X \subset \bar{X}$ and $Y \subset \bar{Y}$. Let $f : Z \to X \times Y$ be a tame finite étale Galois covering with group G (we do not require connectedness in this proof). The covering $X' = f^{-1}(X \times \{y\}) \to X$ is a tame finite étale Galois covering with

group G. Similarly for $Y' = f^{-1}(x \times Y) \to Y$. The ramification index of $X' \to X$ at a point $x_1 \in \overline{X} \setminus X$ is the same as the ramification index of $Z \to X \times Y$ along the divisor $\{x_1\} \times \bar{Y} \subset \bar{X} \times \bar{Y}$; here we use the Galois structure to know that all the ramification indices above a given point are the same. Similarly for Y'. Thus Abhyankar's lemma shows that the pull back Z' of Z to $X' \times Y'$ does not ramify at all along the generic points of the boundary of $X' \times Y'$ in $\bar{X}' \times \bar{Y}'$. Here $\bar{X}'$ and $\bar{Y}'$ are the smooth projective compactifications of X' and Y'; of course $\bar{X}' \times \bar{Y}'$ is the normalization of $\bar{X} \times \bar{Y}$ in the function field of $X' \times Y'$. By purity we see that Z' extends to a finite étale covering of $\bar{X}' \times \bar{Y}'$. Thus Z' is dominated by a product $X'' \times Y''$ with X'' and Y'' tame finite étale over X and Y. This proves that the map (*) is injective, whence an isomorphism. $\qquad\square$

Lemma A.3 ([EK16b]). *Let X be a smooth variety over k and $x \in X$ be a closed point. Let $Z \to X$ be finite tame étale with Z connected. Then there exists a morphism $f : C \to X$ where C is a smooth curve with $x \in f(C)$ such that $Z \times_X C$ is connected.*

Proof. Since X and thus Z are smooth, Z remains connected after passing to a non-empty open subset. Thus we may assume that X is quasiprojective. Furthermore if X' denotes the blow-up of X at x then the pull back of Z to X' will remain connected. Let $\bar{X}'$ denote a normal projective closure of X' and let $C \subset \bar{X}'$ be a smooth complete intersection curve passing through x that is transverse to $D := \bar{X}' \setminus X'$ and disjoint from the singular locus of D and $\bar{X}'$. By [EK16b, Proposition 6.2] (see [EK16a]), the pull back of Z to C is connected. $\quad\square$

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Department of Mathematics, Brown University, Box 1917, 151 Thayer Street, Providence, RI, 02912

Email address: `eric_jovinelly@brown.edu`

Department of Mathematics, Boston College, Chestnut Hill, MA 02467

Email address: `lehmannb@bc.edu`

Department of Mathematics, University of Notre Dame, 255 Hurley Hall, Notre Dame, IN 46556

Email address: `eriedl@nd.edu`

Department of Mathematics, Columbia University, New York, NY 10027

Email address: `dejong@math.columbia.edu`