EXERCISES FOR CHAPTER II.5

1. Quasicoherent sheaves

Exercise 1.1. Exercise II.5.2.

Exercise 1.2. Suppose $X = \mathrm{Spec}(R)$ is affine. Prove carefully that an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for every distinguished open affine $D(g) \subset X$ we have $\mathcal{F}(D(g)) \cong \mathcal{F}(X)_g$ compatibly with restriction maps.

Exercise 1.3. Find an affine scheme X and a SES of $\mathcal{O}_X$-modules such that after taking the global sections functor we only get a left-exact sequence.

Exercise 1.4. Let X be the disjoint union of countably many copies of $\mathrm{Spec}(\mathbb{Z})$ and consider the map $f : X \to \mathrm{Spec}(\mathbb{Z})$ that is the identity map on each copy. Prove that $f_*\mathcal{O}_X$ is not quasicoherent.

2. Coherent sheaves

Exercise 2.1. Exercise II.5.6.

Exercise 2.2. Suppose that X is a finite-type k-scheme and $\mathcal{F}$ is a coherent sheaf on X. It is not necessarily true that $\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module for every open set U.

Consider $\mathbb{P}^2$ equipped with the sheaf $\mathcal{F} = i_*\mathcal{O}_{\mathbb{P}^1}$ where i is the inclusion of a line. Find an open set $U \subset \mathbb{P}^2$ where this finite generation property fails.

Exercise 2.3. Let k be a field. Suppose that $X \subset \mathbb{P}^{n+1}_k$ is a hypersurface and that $x \in \mathbb{P}^{n+1}_k$ is a k-point that is not contained in X. Let $f : X \to \mathbb{P}^n_k$ be the restriction of projection away from x. Prove that f is finite.

Exercise 2.4. Let $f : X \to Y$ be a finite morphism of Noetherian schemes such that $\deg_y(f) \leq 1$ for every point $y \in Y$. Prove that f is a closed embedding.

3. Locally free sheaves

Exercise 3.1. Exercise II.5.1 (if you didn't do it earlier)

Exercise 3.2. Exercise II.5.18. (Note Hartshorne uses the dual convention for vector bundles.)

Exercise 3.3.
(1) Suppose $f : X \to Y$ is a morphism of schemes. Show that if $\mathcal{G}$ is a locally free sheaf on Y then $f^*\mathcal{G}$ is locally free.
(2) Find an example of a morphism $f : X \to Y$ and a locally free sheaf $\mathcal{F}$ on X such that $f_*\mathcal{F}$ is not locally free.

Exercise 3.4. Let X be a scheme and let $\mathcal{F}$ be a finitely generated locally free sheaf of rank r on X. Suppose that $\{U_i\}$ is an open affine cover that trivializes $\mathcal{F}$ and let $\psi_{ij} : \mathcal{O}_{U_{i \cap j}}^{\oplus r} \to \mathcal{O}_{U_{i \cap j}}^{\oplus r}$ be the associated transition functions.

(1) Prove that $\{U_i\}$ also trivializes $\mathcal{F}^\vee$ with the transition functions ψ_{ij}^{-1}.
(2) Suppose that $\mathcal{G}$ is another finitely generated locally free sheaf trivialized by $\{U_i\}$ with transition functions μ_{ij}. Show that the transition functions for $\mathcal{F} \otimes \mathcal{G}$ are $\psi_{ij} \otimes \mu_{ij}$.