EXERCISES FOR CHAPTER II.3

1. Properties of morphisms

Exercise 1.1. Let $f : X \to Y$ be a morphism of schemes. Suppose there is an open cover $\{V_i\}$ of Y such that $f^{-1}V_i$ is affine.

(1) Let $V \subset Y$ be any affine. Prove that V admits an open cover by distinguished open affines that are simultaneously distinguished opens in some V_i.
(2) Set $U = f^{-1}V$. Prove that U is qcqs.
(3) Use the affineness criterion to verify that U is affine.

Exercise 1.2. Let $f : X \to Y$ be a morphism of schemes.

(1) Prove that f is quasicompact if and only if the preimage of every open affine in Y is quasicompact.
(2) Prove that quasicompactness is local on the target.
(3) Prove that quasicompactness is "reasonable".

2. Separatedness

Exercise 2.1. Exercise II.4.2.

Exercise 2.2. Exercise II.4.3. (Hint: show that $U \cap V$ is the same as $\Delta_{X/Y} \cap (U \times V)$.

Exercise 2.3. Suppose that $f : X \to Y$ is a monomorphism in the category of schemes. Prove that the diagonal morphism is an isomorphism. In particular, show that monomorphisms are separated.

Exercise 2.4. Over a field k, the Segre embedding is the map

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$$
$$(x_0, \dots, x_n) \times (y_0, \dots, y_m) \to (x_0y_0, \dots, x_ny_m).$$

(1) For any ring R, define an analogous morphism of schemes $\mathbb{P}^n_R \times_{\mathrm{Spec}(R)} \mathbb{P}^n_R \to \mathbb{P}^{nm+n+m}_R$ and prove that it is a closed embedding.
(2) In the special case $n = m$, show that the diagonal is the same as the intersection of the image of the Segre embedding with a linear subspace of $\mathbb{P}^{nm+n+m}_R$. Use this to give a "geometric" proof that the map $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ is separated.

3. Properness

Exercise 3.1. Exercise II.4.4.

Exercise 3.2. Suppose that X is a projective variety over a field k. Prove that the image of any morphism $f : X \to \mathbb{A}^1_k$ must be a finite union of closed points of $\mathbb{A}^1$.

Exercise 3.3. (1) Suppose that X is an integral projective variety over an algebraically closed field k. Prove that $\mathcal{O}_X(X) \cong k$.

(2) Find examples of non-integral projective varieties which still have the property that $\mathcal{O}_X(X) \cong k$.
(3) Show that (1) is false if we remove the algebraically closed condition.

Exercise 3.4. Let $H \subset \mathbb{P}^n_k$ be a hypersurface, i.e. the vanishing locus of a single homogeneous equation $H = V(f)$. Suppose that $X \subset \mathbb{P}^n_k$ is a closed subscheme that is not a finite union of points. Prove that $H \cap X \neq \emptyset$.