# EXERCISES FOR CHAPTER II.3

## 1. Properties of schemes

**Exercise 1.1.** A generic point $\xi$ of a scheme $X$ is a point such that $\overline{\xi} = X$.

(1) Prove that if $X$ is an irreducible scheme then $X$ has a unique generic point.
(2) Show that if $X$ is an integral scheme, then show that for any open affine $\mathrm{Spec}(R) \subset X$ the fraction field $\mathrm{Frac}(R)$ coincides with the residue field at the generic point of $X$.

The term "generic point" evokes the fact that there are many features of schemes which, if they hold at $\xi$, will then hold for some non-empty open subset of $\overline{\xi}$.

**Exercise 1.2.** Exercise II.3.8

**Exercise 1.3.** Let $X$ be a (non-empty) locally Noetherian scheme. Show that $X$ is integral if and only if $X$ is connected and every stalk $\mathcal{O}_{X,p}$ is an integral domain. (Thus in some sense the property of being an integral domain is "almost", but not quite, a stalk local property.)

**Exercise 1.4.**
(1) Prove that for a locally Noetherian scheme $X$ reducedness is an open property: if a point $p \in X$ is reduced, then $p$ admits an open neighborhood that is reduced. (Hint: think about the primary decomposition of the nilradical of a Noetherian ring.)
(2) Unfortunately reducedness is not an open property in general. Prove this by analyzing the spec of the ring

$$\mathbb{C}[x, y_1, y_2, y_3, \ldots]/(y_1^2, (x-1)y_1, y_2^2, (x-2)y_2, y_3^2, (x-3)y_3, \ldots).$$

## 2. Finiteness properties, open and closed embeddings

**Exercise 2.1.** Prove carefully that:

(1) A scheme $X$ is quasicompact if and only if it admits a finite cover by open affines.
(2) A scheme $X$ is quasiseparated if and only if there exists a cover by open affines $\{U_i\}$ and furthermore each $U_i \cap U_j$ admits a finite cover by open affines.
(3) A scheme $X$ is qcqs if and only if it admits a finite cover by open affines $\{U_i\}$ and furthermore each $U_i \cap U_j$ admits a finite cover by open affines.

**Exercise 2.2.** Suppose that $S$ is a $\mathbb{Z}_{\geq 0}$-graded ring that is finitely generated over $S_0$. Prove that $\mathrm{Proj}(R)$ is qcqs.

**Exercise 2.3.** Exercise II.2.16 (basically: prove the QCQS lemma)

**Exercise 2.4.** Let $i : U \to X$ be an open embedding of schemes. Suppose $f : Z \to X$ is any morphism of schemes such that $f(Z) \subset U$ set-theoretically. Prove that $f$ admits a unique factoring through $i$.

## 3. Products

**Exercise 3.1.** Let $X, Y, Z$ be schemes with morphisms $f : X \to Z$, $g : Y \to Z$. Suppose that $W \subset Z$ is an open subset. Suppose $U \subset f^{-1}(W)$ and $V \subset g^{-1}(W)$ are open subshemes. Show that if $U \times_W V$ exists in the category of schemes, then $U \times_Z V$ does as well and is isomorphic to it. Use this fact to finish the construction of the product.

**Exercise 3.2.** Let $f : X \to Z$ be a morphism. Suppose that $i : Y \to Z$ is an open embedding. Prove that the projection map $X \times_Z Y \to X$ is an open embedding with image homeomorphic to $f^{-1}(Y)$.

**Exercise 3.3.** Let $f : X \to Z$ be a morphism. Suppose that $i : Y \to Z$ is a closed embedding. Prove that the projection map $X \times_Z Y \to X$ is a closed embedding with image homeomorphic to $f^{-1}(Y)$.

**Exercise 3.4.** Let $f : X \to Z$ be a morphism of affine schemes. Prove carefully that the diagonal $\Delta_{X/Z} : X \to X \times_Z X$ is a closed embedding.

**Exercise 3.5.** Exercise II.3.9