EXERCISES FOR CHAPTER II.2, CONTINUED

1. Morphisms

Exercise 1.1. Consider the map $f : \mathbb{A}^2_{u,v} \to \mathbb{A}^2_{x,y}$ defined by $f^{ring} : k[x,y] \to k[u,v]$ sending $x \mapsto u, y \mapsto uv$. Prove algebraically that the image of f is $D(x) \cup \{(x,y)\}$.

Exercise 1.2. Let $f : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ be a morphism induced by a homomorphism $f^{ring} : S \to R$.

(1) Show that if f^{ring} is surjective then f is injective. What is an example where the converse fails?
(2) Show that if f^{ring} is injective then f has dense image. What is an example where the converse fails?

Exercise 1.3. Consider the parabola X defined by the equation $x^2 - y$ in $\mathbb{A}^2_k$. By projecting onto the y-coordinate we obtain a map $f : X \to \mathbb{A}^1_k$. Show that the fiber of f over a closed point with residue field L will have one of the following three types:

- A disjoint union of two L-points.
- A single point with residue field F where $[F : L] = 2$.
- A single point with residue field L whose ring of functions R satisfies $\dim_L(R) = 2$.

How can you determine which of the three possibilities happens at a given point? (Be careful if the characteristic is 2.)

Exercise 1.4. Set $X = \mathrm{Spec}(k[x,y]/(y^2 - x^3 - x^2))$. Describe all the fibers of the map $f : \mathbb{A}^1 \to X$ defined by $f^{ring} : k[x,y]/(y^2 - x^3 - x^2) \to k[t]$ sending $x \mapsto t^2 - 1$, $y \mapsto t(t^2 - 1)$. What is the geometric interpretation of this map?

2. Schemes

Exercise 2.1. Exercise II.2.3

Exercise 2.2. Exercise II.2.4

Exercise 2.3. Exercise II.2.7

Exercise 2.4. Exercise II.2.9

3. Projective schemes

Exercise 3.1. Exercise II.2.14

Exercise 3.2. Prove that a closed subset $X \subset \mathbb{P}^n$ is irreducible if and only if there is a homogeneous prime ideal $\mathfrak{p} \subset \mathbb{K}[x_0, \ldots, x_n]$ such that $X = V(\mathfrak{p})$.

Exercise 3.3. Consider the distinguished open affine $D_+(x^2 + y^2 + z^2)$ in $\mathbb{P}^2$. Identify explicitly the finitely generated k-algebra S such that $D_+(x^2 + y^2 + z^2) \cong \mathrm{Spec}(S)$.

Exercise 3.4. Describe the quotient morphism $\mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ as a morphism of schemes. (For example, you could define this morphism on affine charts and show that these morphisms are compatible on the overlaps.)