EXERCISES FOR CHAPTER II.2

1. Spec of a ring

Exercise 1.1. In contrast to traditional geometry, every open set in the Zariski topology is quite "large." For example, suppose that R is a domain.

(1) Show that any two non-empty open subsets of $\mathrm{Spec}(R)$ will have non-empty intersection.
(2) Show that every non-empty open subset of $\mathrm{Spec}(R)$ is dense.

Exercise 1.2.
(1) Consider $\mathbb{A}^2_{\mathbb{R}}$. Suppose that $\mathfrak{m}$ is a non-traditional closed point. Show that the ideal $\mathfrak{m} \in \mathbb{C}[x, y]$ defines a conjugate pair of points in $\mathbb{C}$. Deduce that $\mathfrak{m}$ must contain a (real) linear function and so has the form $\mathfrak{m} = (\ell, q)$ for some linear ℓ and quadratic q.
(2) Consider the ring $R = \mathbb{R}[x, y]/(y - x^2)$. Identify the traditional and non-traditional points of R. Check that $\mathrm{Spec}(R)$ is homeomorphic to $\mathbb{A}^1_{\mathbb{R}}$ (via the identification $x \leftrightarrow t, y \leftrightarrow t^2$).

Exercise 1.3. Suppose that $\mathbb{K}$ is an infinite field. Prove that if $X \subset \mathbb{A}^n_{\mathbb{K}}$ is a closed subset that contains every traditional point then $X = \mathbb{A}^n_{\mathbb{K}}$. In contrast, for the finite field $\mathbb{F}_q$ identify a proper closed subset of $\mathbb{A}^n_{\mathbb{F}_q}$ that contains every traditional point.

Exercise 1.4. Let R be a finitely generated k-algebra. Show that $\mathrm{Spec}(R)$ is a finite set if and only if R is an Artinian ring. (Depending on how much you are willing to assume from the theory of Artinian rings this exercise may be trivial.)

Exercise 1.5. Exercise II.2.11

2. Zariski topology

Exercise 2.1. Show that the following conditions are equivalent:

(1) $\mathrm{Spec}(R)$ is disconnected.
(2) There exist non-zero idempotents $e_1, e_2 \in R$ such that $e_1e_2 = 0$ and $e_1 + e_2 = 1$.

We can think of the idempotents e_1, e_2 as indicator functions for unions of connected components of X.

Exercise 2.2. Some of the properties of the Zariski topology we discussed in this section will hold more generally for any Noetherian topological space.

(1) Verify that every Noetherian topological space is compact.
(2) Verify every closed subset can be written as a union of irreducible closed subsets in an essentially unique way. (Hint: let $\mathcal{W}$ denote the set of closed sets which cannot be written as a finite union of irreducible closed sets. Show that if $\mathcal{W}$ is non-empty then it has a minimal element. Use this minimal element to derive a contradiction.)

Exercise 2.3. Let X be an irreducible topological space. Prove that:

- Every non-empty open subset of X is dense.
- X is connected.
- X is Hausdorff if and only if X is a single point.

3. Sheaf of functions

Exercise 3.1. Suppose that U is an open subset of $\mathrm{Spec}(R)$. Let $I \subset \mathcal{O}_{\mathrm{Spec}(R)}(U)$ be an ideal. The vanishing locus $V(I)$ is defined to be set of points $x \in U$ such that the stalk map $\rho_{U,x} : \mathcal{O}_{\mathrm{Spec}(R)}(U) \to \mathcal{O}_{\mathrm{Spec}(R),x}$ satisfies the property that $\rho_{U,x}(I) \subset \mathfrak{m}_{\mathrm{Spec}(R),x}$.

(1) Double check that when $U = \mathrm{Spec}(R)$ this agrees with our usual notion of vanishing locus.
(2) Show that $V(I)$ is a closed subset of U.

Exercise 3.2. Let $U \subset \mathbb{A}^2$ be the complement of the origin. Show that $\mathcal{O}_{\mathbb{A}^2}(U) = k[x,y]$ in two different ways:

- By appealing to the description of the structure sheaf for a domain.
- By choosing a cover of distinguished open affines and using the sheaf property.

Exercise 3.3. Consider the ring $R = k[w,x,y,z]/(wy, wz, xy, xz)$. Geometrically this is two copies of $\mathbb{A}^2$ in $\mathbb{A}^4$ which meet at the origin.

Let $U \subset \mathrm{Spec}(R)$ be the complement of the origin. Prove that $O_{\mathrm{Spec}(R)}(U)$ is not the same as the localization of R along all the functions which vanish along the origin. (Hint: consider the function which is identically 1 on one component of U and identically 0 on the other.)

Exercise 3.4. Suppose that R is a finitely generated k-algebra. For every non-empty open subset $U \subset R$ it is true that $\mathcal{O}_{\mathrm{Spec}(R)}(U)$ will be a k-algebra, but it might fail to be finitely generated!

Let $R = k[s,t,u]/(tu)$ and set $W = \mathrm{Spec}(R)$. Geometrically W is a union of two coordinate hyperplanes in $\mathbb{A}^3$. Let $U \subset W$ be the open subset obtained by removing the line $V(s,t)$ in W. Show that $\mathcal{O}_W(U)$ is not finitely generated. Precisely, we have

$$\mathcal{O}_W(U) \cong R[z_1, z_2, z_3, \ldots]/(tz_n, s^n z_n - u, s^{m+n} z_m - z_n)$$