EXERCISES FOR CHAPTER II.1

1. Sheaves and sheafification

Exercise 1.1. Exercise II.1.1

Exercise 1.2. Exercise II.1.14. Also, show carefully that the support of a sheaf $\mathcal{F}$ need not be a closed subset of X. (Hint: try constructing a suitable sheaf on the two-pointed space $X = \{p, q\}$ where the closed subsets are $\emptyset, \{p\}, X$.)

Exercise 1.3. A rational function f over $\mathbb{C}$ is a ratio of polynomials. Writing f as a product of linear factors (with positive and negative coefficients), we define the multiplicity of f at a point p to be the exponent of $(z - p)$ in f.

When working with rational functions it is helpful to equip $\mathbb{C}$ with a non-standard topology. Recall that the cofinite topology on $\mathbb{C}$ is the topology whose non-trivial closed sets are exactly the finite subsets of $\mathbb{C}$. Using this topology, we define the sheaf $\mathcal{R}$ of rational functions on $\mathbb{C}$ by assigning to each open set U the abelian group

$$\mathcal{R}(U) := \{ f \mid \mathrm{mult}_p(f) \geq 0, \forall p \in U \} .$$

Given an inclusion of open subsets $V \subset U$ we define $\rho_{U,V} : \mathcal{R}(U) \to \mathcal{R}(V)$ to be the inclusion map.

1. (1) Check carefully that $\mathcal{R}$ defines a sheaf on $\mathbb{C}$ equipped with the cofinite topology. (Suppose we instead tried to equip $\mathbb{C}$ with the Euclidean topology and defined $\mathcal{R}$ in a similar way. Would it still define a sheaf? What goes wrong?)
2. (2) Suppose that $U \subset \mathbb{C}$ is the complement of the points $p_1, \ldots, p_r$. Show that $\mathcal{R}(U)$ is the localization of $\mathbb{C}[x]$ along the multiplicatively closed subset obtained by taking products of the functions $z - p_1, \ldots, z - p_r$.
3. (3) Show that for any point $p \in \mathbb{C}$ the stalk of $\mathcal{R}$ at the point p is the same as the localization of $\mathbb{C}[z]$ along the prime ideal $(z - p)$.

Exercise 1.4. The Riemann sphere is obtained from $\mathbb{C}$ by adding on a single point ∞. Just as in the previous exercise we equip this set with the cofinite topology.

Given a rational function f, we define the multiplicity of f at ∞ in the following way: if we write $f = \frac{p}{q}$, then we set $\mathrm{mult}_\infty(f) = \deg(q) - \deg(p)$. When $\mathrm{mult}_\infty(f) \geq 0$, then we can "evaluate" f at ∞ by setting $f(\infty) = \lim_{p \to \infty} f(p)$. (One readily checks that this limit is well-defined when the multiplicity is non-negative.)

We now define the sheaf $\mathcal{R}_\infty$ on $\mathbb{C}_\infty$ in exactly the same way as before: we set

$$\mathcal{R}_\infty(U) := \{ f \mid \mathrm{mult}_p(f) \geq 0, \forall p \in U \}$$

and define the restriction maps via inclusion.

1. (1) Suppose that $U \subset \mathbb{C}_\infty$ is an open subset containing ∞. Show that $\mathcal{R}_\infty(U)$ is the subset of $\mathcal{R}(U \backslash \infty)$ (which was defined in the previous problem) consisting of rational functions $f = p/q$ such that $\deg(p) \leq \deg(q)$.
2. (2) Compute the stalk of $\mathcal{R}_\infty$ at the point $\infty \in \mathbb{C}_\infty$.

Exercise 1.5. For each of the following examples of a topological space X equipped with a presheaf $\mathcal{F}$, describe the sheafification $\mathcal{F}^+$.

(1) $X = \mathbb{R}^n$, $\mathcal{F}$ assigns to each open set U the set of bounded functions on U.
(2) $X = S^1$, $\mathcal{F}$ assigns to each open set U the set of continuous functions f on U which satisfy $f(x) = f(-x)$ for every pair of antipodal points $x, -x$ in U.
   (Note that $\mathcal{F}$ consists of those functions which can be obtained by composing the the quotient map $S^1 \to \mathbb{RP}^1$ with a continuous map $\mathbb{RP}^1 \to \mathbb{R}$. In geometric language, $\mathcal{F}$ is the "pullback" of the sheaf of continuous functions on $\mathbb{RP}^1$.)
(3) $X = \mathbb{C}$, $\mathcal{F}$ assigns to each open set U the set of holomorphic functions on U which admit a square root.

2. Kernels and images

Exercise 2.1. Exercise II.1.2

Exercise 2.2. Exercise II.1.3

Exercise 2.3. Exercise II.1.4

Exercise 2.4. We use notation for the Riemann sphere from an earlier exercise. Fix a point $p \in \mathbb{C}_\infty$ and consider the evaluation morphism $eval : \mathcal{R}_\infty \to \mathbb{C}(p)$. The kernel of this morphism is the sheaf which assigns to each open set U the subset of $\mathcal{R}_\infty(U)$ consisting of rational functions with multiplicity ≥ 1 at p. We will denote this sheaf by $\mathcal{I}_p$.

Consider the morphism $\phi : \mathcal{I}_0 \oplus \mathcal{I}_\infty \to \mathcal{R}_\infty$ obtained by sending $(f, g) \mapsto f + g$.

(1) Show that ϕ is a surjective morphism of sheaves.
(2) Show that $\phi(\mathbb{C}_\infty)$ is not surjective on global sections.

Exercise 2.5. Consider the sheaf $\mathcal{R}_\infty$ on $\mathbb{C}_\infty$ as above. Let A be the stalk of $\mathcal{R}_\infty$ at the point 0. The various stalk restriction maps $\rho_{U,0} : \mathcal{R}_\infty(U) \to A$ combine to give a morphism of sheaves $\phi : \mathcal{R}_\infty \to A(0)$ where $A(0)$ denotes the skyscraper sheaf at the origin with value A.

Show that ϕ is a surjective morphism of sheaves. Show however that there is no open neighborhood U of the origin such that the map $\phi(U) : \mathcal{R}_\infty(U) \to A(0)(U)$ is surjective. (Can you leverage this idea to find a surjective morphism of sheaves such that there is no open set U with $\phi(U)$ surjective?)

Exercise 2.6. Let X be a topological space. Suppose that to each point $x \in X$ we assign a divisible abelian group Q_x. Define the sheaf $\mathcal{Q}$ by assigning to any open set U the product $\prod_{x \in U} Q_x$ and to any inclusion $V \subset U$ the corresponding projection map. Prove that $\mathcal{Q}$ is an injective object in $\mathbf{Sh}(X)$.

Exercise 2.7. Let X be a manifold of dimension ≥ 1. Fix a point $x \in X$. For any open neighborhood $V \subset X$ define the sheaf $\mathbb{Z}_V$ via the prescription:

$$\mathbb{Z}_V(U) = \begin{cases} \mathbb{Z}^{\pi_0(U)} \text{ if } U \subset V \\ 0 \text{ if } U \not\subset V \end{cases}$$

with the obvious restriction maps.

(1) Show that for any open neighborhood V of x there is a surjection $\rho_V : \mathbb{Z}_V \to \mathbb{Z}(x)$ where $\mathbb{Z}(x)$ denotes the skyscraper sheaf at x with value $\mathbb{Z}$.
(2) Use the surjections ρ_V to show that there is no projective object in $\mathbf{Sh}(X)$.

3. Pushforward and pullback

Exercise 3.1. Exercise II.1.17

Exercise 3.2. Exercise II.1.18

Exercise 3.3. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. Prove that:

(1) $(g \circ f)_* = g_* \circ f_*$.
(2) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. (Hint: since these constructions involve a sheafification, it is not easy to directly compare the values of the two constructions on open sets. Instead, you should construct a morphism between the two constructions and show that it induces an isomorphism of stalks.)

Exercise 3.4. Consider the inclusion $i : \mathbb{C} \to \mathbb{C}_\infty$ as an open set. Compute the stalks of $i_*\mathcal{R}$ (the sheaf defined in an earlier exercise) at every point $x \in \mathbb{C}_\infty$.

Exercise 3.5. Let X be a topological space and let $\mathcal{B} = \{V_i\}$ be a base for the topology. A $\mathcal{B}$-sheaf $\widetilde{\mathcal{F}}$ assigns to every open set $V_i \in \mathcal{B}$ an abelian group $\widetilde{\mathcal{F}}(V_i)$ and to each inclusion $V_i \subset V_j$ of open sets in $\mathcal{B}$ a restriction map $\widetilde{\rho}_{V_j,V_i}$ such that the following properties hold:

(1) $\widetilde{\mathcal{F}}(\emptyset) = 0$.
(2) The assignments $\widetilde{\mathcal{F}}, \widetilde{\rho}$ define a contravariant functor from the category of open subsets of X contained in $\mathcal{B}$ (with morphisms = inclusions) to the category of abelian groups.
(3) For any open set $V_i \in \mathcal{B}$ and any open cover of V_i by elements in $\mathcal{B}$ the identity and gluing axioms hold.

Verify that a $\mathcal{B}$-sheaf $\widetilde{\mathcal{F}}$ extends to a sheaf $\mathcal{F}$ on X in a unique way. Hint: define $\mathcal{F}(U)$ as a subset of the product $\prod_{\substack{V_i \subset U \\ V_i \in \mathcal{B}}} \widetilde{\mathcal{F}}(V_i)$:

$$\mathcal{F}(U) := \left\{ (f_i \in \widetilde{\mathcal{F}}(V_i))_{V_i \subset U} \;\middle|\; \widetilde{\rho}_{V_{i_1}, V_{i_1} \cap V_{i_2}}(f_{i_1}) = \widetilde{\rho}_{V_{i_2}, V_{i_1} \cap V_{i_2}}(f_{i_2}) \;\; \forall i_1, i_2 \right\}. \tag{3.1}$$

Exercise 3.6. Let X be a topological space and let $\mathcal{B}$ be a base for the topology. Suppose that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are two $\mathcal{B}$-sheaves. A morphism $\widetilde{\phi}$ of $\mathcal{B}$-sheaves assigns to each open set $V_i \in \mathcal{B}$ a homomorphism $\widetilde{\phi}_{V_i} : \widetilde{\mathcal{F}}(V_i) \to \widetilde{\mathcal{G}}(V_i)$ in such a way that the various $\widetilde{\phi}_{V_i}$ commute with restriction.

Prove that a morphism of $\mathcal{B}$-sheaves $\widetilde{\phi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ induces a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of the sheaves $\mathcal{F}, \mathcal{G}$ constructed by the previous exercise such that for every $V_i \in \mathcal{B}$ we have $\widetilde{\phi}_{V_i} = \phi_{V_i}$. Show that ϕ is uniquely determined by this condition.

Exercise 3.7. Let X be a topological space equipped with an open cover $\{U_i\}$. Suppose that for each index i we have a sheaf $\mathcal{F}_i$ on U_i. Suppose furthermore that for every pair of indices i, j we have an isomorphism

$$\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$$

and that ϕ_{ii} is the identity map, $\phi_{ij} = \phi_{ji}^{-1}$ and $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ (as isomorphisms of sheaves on $U_i \cap U_j \cap U_k$). Show that there is a sheaf $\mathcal{F}$ on X (unique up to isomorphism) such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{F}_i$.

(Remark: the conditions on the ϕ_{ij} are known as the "cocycle condition." It is worth comparing these conditions with the similar problem of constructing a manifold by gluing together open subsets of $\mathbb{R}^n$; the requirements are exactly the same.)