EXERCISES FOR CHAPTER 2

0.1. **Geometric properties.**

Exercise 0.1. RPoV Exercise 2.1

Exercise 0.2. Suppose that X and Y are two varieties over a field k.

1. Show that if X and Y are geometrically irreducible then $X \times Y$ is geometrically irreducible. Find a counterexample to this statement if we remove the word "geometrically."
2. Show that if X and Y are geometrically reduced then $X \times Y$ is geometrically reduced. Find a counterexample to this statement if we remove the word "geometrically."

0.2. **Closed points.**

Exercise 0.3. RPoV Exercise 2.11

Exercise 0.4. RPoV Exercise 2.13 and 2.14

Exercise 0.5. Describe the closed points of $\mathbb{P}^1_{\mathbb{F}_q(t)}$. (This is not a well-defined problem; part of the problem is figuring out how to think about these points. One option is to show that these closed points are parametrized by a countable union of $\mathbb{F}_q$-schemes.)

0.3. **Scheme-valued points.**

Exercise 0.6. RPoV Exercise 2.4

Exercise 0.7. RPoV Exercise 2.5

Exercise 0.8. RPoV Exercise 2.7

0.4. **Curves.**

Exercise 0.9. RPoV Exercise 2.12

Exercise 0.10. Let X denote a regular, projective, geometrically integral curve over a field k with genus 0. Prove that the following are equivalent:

1. X is isomorphic to $\mathbb{P}^1$.
2. X has a rational point.
3. X carries a line bundle of degree 1.

Exercise 0.11. Let X denote a regular, projective, geometrically integral curve over a field k. Suppose $\mathcal{L}$ is a line bundle of degree 0 on X. Prove that either $\mathcal{L} \cong \mathcal{O}_X$ or we have $\dim H^0(X, \mathcal{L}) = 0$ and $\dim H^1(X, \mathcal{L}) = g - 1$. (Hint: if $\dim H^0(X, \mathcal{L}) > 0$ then the global section defines a morphism $\mathcal{O}_X \to \mathcal{L}$. What can the kernel and cokernel look like?)

Exercise 0.12. Let X denote the projective curve over $\mathbb{F}_3(t)$ defined by the equation $zy^2 = x^3 + tz^3$ in $\mathbb{P}^2$.

1. Prove that X is regular and geometrically integral.

(2) Prove that X has genus 1.
(3) Let $L = \mathbb{F}_3(t^{1/3})$. Show that X_L is not regular and that its normalization has genus 0.

Exercise 0.13. Let X be the curve $y^2 = -x^6 + x^2 - 1$ over the field $\mathbb{F}_3$. Let $\overline{X}$ be the normal projective model of X.

(1) Show that X does not admit any rational points. (I think $\overline{X}$ also does not but I did not check carefully.)
(2) Prove that $\overline{X}$ admits a line bundle of degree 1.