## EXERCISES FOR CHAPTER 1

### 0.1. **Global and local fields.**

**Exercise 0.1.** Consider $\mathbb{Q}(i)$ with its ring of integers $\mathbb{Z}[i]$. Recall that $\mathbb{Z}[i]$ is a Euclidean domain and thus admits prime factorizations for elements. A prime $p$ in $\mathbb{Z}$ will remain prime in $\mathbb{Z}[i]$ if $p \equiv 3 \bmod 4$. If $p$ is congruent to 1 mod 4, we can write $p = a^2 + b^2$ and so we have prime elements $(a + bi), (a - bi)$ whose product is $p$. Finally $2 = (1 + i)^2$.

(1) For which primes $\mathfrak{p} \subset \mathbb{Z}[i]$ does the $\mathfrak{p}$-adic absolute value $|x|_{\mathfrak{p}} = \kappa(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)}$ extend the $p$-adic absolute value on $\mathbb{Q}$?
(2) Verify the product formula $\prod_{\nu} |x|_{\nu} = 1$. (Recall that for any real place $|x|_{\nu}$ is normalized to be the usual absolute value. For any complex place, we define $|a+bi|_{\nu} = a^2 + b^2$ – technically this is not an absolute value, but it is equivalent to the usual absolute value and this definition is necessary to make the product formula work.)

**Exercise 0.2.** Set $K = \mathbb{F}_5(t)$ and consider the extension $L = K[y]/(y^2 - t)$. For every place $\nu$ of $K$, determine the splitting and ramification behavior of $\nu$ in $L$: does it correspond to two different places, a single place which ramifies, or a single place with larger residue field?

### 0.2. **Galois theory.**

**Exercise 0.3.** Let $G$ be a profinite group equipped with the profinite topology.

(1) Show that a subgroup of $G$ is open if and only if it is closed and has finite index. (Hint: for the forward implication, first show profinite groups are compact by appealing to Tychonoff's Theorem.)
(2) Find an example of a profinite group $G$ and a closed subgroup of $G$ that is not open.

It is a result of Nikolov and Segal that if $G$ is topologically finitely generated then every subgroup of finite index is open.

**Exercise 0.4.** Suppose that $L/k$ is a finite Galois extension of fields with Galois group $G$. Let $V$ be a $k$-vector space. Prove that the natural inclusion $V \to (V \times_k L)^G$ is an isomorphism of $k$-vector spaces. (This is proved in RPoV Lemma 1.3.9.)

### 0.3. **Group cohomology.**

**Exercise 0.5.** Let $C_m$ denote the cyclic group of order $m$ with generator $\sigma$. Show that for any $C_m$-module $A$ we can identify $H^1(C_m, A)$ with the following quotient:

$$H^1(C_m, A) = \{a \in A | (1 + \sigma + \ldots + \sigma^{m-1})a = 0\}/(\sigma - 1)A$$

**Exercise 0.6.** Suppose $G$ is a finite group and $A$ is a $G$-module. Prove that $H^1(G, A)$ is torsion, and in fact, every element of $H^1(G, A)$ is killed by $|G|$. (The same statement is true for the higher cohomology groups as well, and if you know some group cohomology you can try to prove it using the restriction and corestriction maps.)

**Exercise 0.7.** Let $G$ be a group and suppose we have a short exact sequence of $G$-modules

$$0 \to A \to B \to C \to 0$$

Prove that we have an exact sequence

$$0 \to A^G \to B^G \to C^G \to H^1(G,A) \to H^1(G,B) \to H^1(G,C)$$

**Exercise 0.8.** RPoV Exercise 1.9

**Exercise 0.9.** RPoV Exercise 1.10

0.4. **Brauer groups.**

**Exercise 0.10.** Using the definition of the group operation in the Brauer group directly, show that $\mathbb{H} \cdot \mathbb{H} = 1$ in $\mathrm{Br}(\mathbb{R})$. (That is, don't appeal to the Frobenius theorem in your proof.)

**Exercise 0.11.** Let $k$ be a field of characteristic different from 2. Given $a, b \in k^\times$, the quaternion algebra $A_{a,b}$ is constructed analogously to the usual quaternions: we define

$$A_{a,b} = k \oplus k \cdot i \oplus k \cdot j \oplus k \cdot ij$$

where $i^2 = a$, $j^2 = b$, $ji = -ij$.

(1) Prove that $A_{a,b}$ is an Azumaya algebra of index 2 over $k$.
(2) Show that $A_{a,b} \cong M_2(k)$ if and only if there are $x, y, z \in k$ which are not all zero and satisfy
$$ax^2 + by^2 = z^2.$$
Thus the theory of quaternion algebras is closely related to the theory of bilinear forms.
(3) Explicitly identify a splitting field $k \subset L \subset A_{a,b}$.

In some cases the quaternion algebras define all the 2-torsion elements in the Brauer group, but not always!

**Exercise 0.12.** RPoV Exercise 1.15

**Exercise 0.13.** RPoV Exercise 1.20