MORPHISMS AND FACES OF PSEUDO-EFFECTIVE CONES

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Abstract. Let $\pi : X \to Y$ be a morphism of projective varieties and suppose that α is a pseudo-effective numerical cycle class satisfying $\pi_*\alpha = 0$. A conjecture of Debarre, Jiang, and Voisin predicts that α is a limit of classes of effective cycles contracted by π. We establish new cases of the conjecture for higher codimension cycles. In particular we prove a strong version when X is a fourfold and π has relative dimension one.

1. Introduction

Let $\pi : X \to Y$ be a morphism of projective varieties over an algebraically closed field. The pushforward of cycles induces a map $\pi_* : N_k(X) \to N_k(Y)$ on numerical groups with $\mathbb{R}$-coeffficients, and one would like to understand how $\ker(\pi_*)$ reflects the geometry of the map π. In the special case when $\alpha \in N_k(X)$ is the class of a closed subvariety Z, then α lies in the kernel of π_* precisely when $\dim \pi(Z) < \dim(Z)$. A similar statement holds when α is the class of an effective cycle. However, the geometry of arbitrary elements of $\ker(\pi_*)$ is more subtle.

An important idea of [DJV13] is that this geometric interpretation of elements of $\ker(\pi_*)$ should be extended beyond the effective classes. Recall that the pseudo-effective cone $\overline{\mathrm{Eff}}_k(X)$ is the closure of the cone in $N_k(X)$ generated by the classes of effective k-cycles. The following is the numerical analogue of the homological statement in [DJV13].

Conjecture 1.1. *Let $\pi : X \to Y$ be a morphism of projective varieties over an algebraically closed field. Suppose that $\alpha \in \overline{\mathrm{Eff}}_k(X)$ satisfies $\pi_*\alpha = 0$. Then*

> ***Weak Conjecture:*** *α is in the vector space generated by k-dimensional subvarieties that are contracted by π.*
>
> ***Strong Conjecture:*** *α is in the closure of the cone generated by k-dimensional subvarieties that are contracted by π.*

The improvement to *pseudo*-effective classes is crucial for understanding the geometry of π. For example, the interplay between morphisms from X and faces of the Mori cone $\overline{\mathrm{NE}}(X) = \overline{\mathrm{Eff}}_1(X)$ is an essential tool in birational geometry. The Strong Conjecture predicts that for higher dimensional cycles there is still a distinguished way of constructing a face of $\overline{\mathrm{Eff}}_k(X)$ from a morphism π, allowing us to deduce geometric facts from intersection theory. The first cases of the conjecture were settled by [DJV13, Theorem 1.4] which proves the Strong Conjecture for divisor classes and curve classes over $\mathbb{C}$.

Example 1.2. Let E be an elliptic curve without complex multiplication and define $X = E \times E$. Then the Neron-Severi space $N_1(X)$ is 3-dimensional with a basis given by the classes F_1, F_2 of fibers of the two projections and by the diagonal class Δ. The pseudo-effective cone $\overline{\mathrm{Eff}}_1(X)$ is a round cone.

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Consider the first projection $\pi : X \to E$. The curves contracted by π are all numerically equivalent with class F_1. However, the kernel of π is two-dimensional: it is generated by F_1 and by $\Delta - F_2$. In particular, the kernel is *not* spanned by the subvarieties contracted by π. Nevertheless, the intersection $\ker(\pi_*) \cap \overline{\mathrm{Eff}}_1(X)$ consists of a unique ray $\mathbb{R}_{\geq 0} F_1$, so that the Strong Conjecture holds for this map π.

Consider the face $\ker \pi_* \cap \overline{\mathrm{Eff}}_k(X)$ of $\overline{\mathrm{Eff}}_k(X)$. The Strong Conjecture for $\pi : X \to Y$ predicts that effective classes are dense in this face. It is then not surprising that the Strong Conjecture has close ties with other well-known problems predicting the existence of special cycles, as in the following example.

Example 1.3. Let S be a smooth complex surface with $q = p_g = 0$. Let Δ denote the diagonal on $S \times S$ and let F_1 and F_2 be the fibers of the projections π_1 and π_2. Bloch's conjecture predicts that the diagonal $\Delta - F_1$ is $\mathbb{Q}$-*linearly* equivalent to a sum of cycles that are contracted by π_2. In contrast, the Weak Conjecture (applied to the morphism π_2) predicts that $\Delta - F_1$ is *numerically* equivalent to a sum of cycles that are contracted by π_2. In this case the Weak Conjecture for π_2 can be verified by Hodge Theory so that S admits a "numerical" diagonal decomposition. (See Example 3.6 for details.)

More generally, suppose that X is a smooth complex variety satisfying $H^{i,0}(X) = 0$ for $i > 0$. Then the Strong Conjecture for currents has implications for the Generalized Hodge Conjecture on $X \times X$ when applied to the projection maps. This is discussed in more detail in [DJV13, §6]. □

As our main result, we prove the Strong Conjecture for arbitrary classes when X is a fourfold and π has relative dimension one. (This is a special case of the more general results described below.) The proof for surface classes involves new concepts and techniques concerning the positivity of higher (co)dimension cycles. The basic principle underlying our work is that it is best to consider the Strong and Weak conjectures separately for "movable" classes and "rigid" classes. This is motivated by proof of the divisor case in [DJV13], which relies on the σ-decomposition of [Nak04] in a fundamental way.

We first discuss the Strong Conjecture for movable classes. In [FL13] we introduced the movable cone of k-cycles $\overline{\mathrm{Mov}}_k(X)$ which is the closure in $N_k(X)$ of the cone generated by classes of effective cycles that deform in irreducible families which cover X. Since movable cycles deform to cover all of X, morally they should not reflect the pathologies of special fibers of π. Thus it should be easier to settle the Strong and Weak Conjectures for movable classes, and in fact, the conjectures in §4.1 predict stronger statements. A result in this direction is the following

Theorem 1.4 (cf. 6.11). *Let $\pi : X \to Y$ be a morphism of projective varieties over $\mathbb{C}$ of relative dimension e. Fix an ample divisor H on Y. Suppose that $\alpha \in \overline{\mathrm{Mov}}_k(X)$ for some $k \geq e$. If α satisfies*

$$\alpha \cdot \pi^* H^{k-e+1} = 0,$$

which in particular implies that $\pi_* \alpha = 0$, then the Strong Conjecture holds for α. In particular the Strong Conjecture holds for all movable classes when $e = 1$.

For perspective, note that when α is a movable class satisfying $\alpha \cdot \pi^* H^{k-e} = 0$, then $\alpha = 0$ by Theorem 1.6 below. Theorem 1.4 handles one additional step.

A crucial technical step is to improve our understanding of the "dual positive classes" defined in [FL14]. With this improvement, the proof technique is similar to [Leh11], where the second author proves the analogous theorem for divisors. In fact, we prove a somewhat stronger statement, allowing us to prove many cases of the Strong Conjecture for fourfolds (see Corollary 6.15).

We next discuss the Strong and Weak Conjectures for "rigid" classes. To capture the notion of rigidity, we use the Zariski decomposition for numerical cycle classes introduced by the authors in [FL15]. A Zariski decomposition of a pseudo-effective class α is an expression

$$\alpha = P(\alpha) + N(\alpha)$$

where $P(\alpha)$ is a movable class that retains all the "positivity" of α and $N(\alpha)$ is *pseudo*-effective. (This decomposition is an analogue of the σ-decomposition of [Nak04].) [FL13, Conjecture 5.19] predicts that any negative part $N(\alpha)$ is the pushforward of a pseudo-effective class from a proper subscheme of X. We establish this conjecture in a special case:

Definition 1.5. Let $\pi : X \to Y$ be a dominant morphism of projective varieties of relative dimension e. Suppose that $\alpha \in \overline{\mathrm{Eff}}_k(X)$. We say that α is π-*exceptional* if there is an ample divisor H on Y such that $\alpha \cdot \pi^* H^r = 0$ for some $r \leq k - e$.

When α is the class of a subvariety Z, this definition simply means that the codimension of $\pi(Z)$ is greater than the codimension of Z, thus extending the familiar notion for divisors. In general, this definition identifies the classes that are forced to be "rigid" by the geometry of the morphism π. A typical example is *any pseudo*-effective class in the kernel of π_* for a birational map π.

Theorem 1.6 (cf. 5.14)**.** *Let $\pi : X \to Y$ be a dominant morphism of projective varieties. If α is a π-exceptional class, then:*

(1) $\alpha = 0 + N(\alpha)$ is the unique Zariski decomposition for α.
(2) α is the pushforward of a pseudo-effective class from a proper subscheme of X.

Condition (2) implies that the Strong or Weak Conjecture for a π-exceptional class can be concluded from a statement in lower dimensions. For example, since the Strong Conjecture is known for complex threefolds, we immediately obtain the Strong Conjecture for exceptional classes on fourfolds over $\mathbb{C}$. This inductive relationship goes both ways:

Proposition 1.7 (cf. 5.19 and 5.23)**.** *The Strong (resp. Weak) Conjecture holds for birational maps $\pi : X \to Y$ of varieties of dimension n if and only if the Strong (resp. Weak) Conjecture holds in dimension $\leq n - 1$.*

Example 1.8. To illustrate our techniques, in Example 5.18 we revisit the results of [CC14] and [Sch15] which describe the geometry of higher codimension cycles on moduli spaces of pointed curves. These papers identify classes that lie on extremal rays of the effective cone. Using the results above, we show that their arguments actually establish extremality in the pseudo-effective cone. (See also [CC14, Remark 2.7].)

1.1. **Organization.** Section 2 recalls the basic properties of numerical groups and positive cones. We explain the basic features of the Strong and Weak Conjectures in Section 3. In particular, applying ideas from [FL14], we recover many of the results of [DJV13], extending some of them over an arbitrary algebraically closed field. We also give many examples. Section 4 describes how the Zariski decomposition is related to the Strong and Weak Conjectures; the negative and positive parts are analyzed in Sections 5 and 6 respectively.

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2. Background on numerical equivalence

By *variety* we mean a reduced, irreducible, separated scheme of finite type over an algebraically closed field of arbitrary characteristic. Unless otherwise stated, $\pi : X \to Y$ is a morphism of projective varieties over the fixed ground field.

A *k-cycle* on a projective variety X is a formal sum $Z = \sum_{i=1}^{r} a_i V_i$ where the V_i are k-dimensional closed subvarieties of X and the a_i are coefficients in $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ (in which case we say that Z is a $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$-cycle respectively). The *support* of Z is $|Z| = \cup_i V_i$. When for all i we have $a_i \geq 0$, we say that the cycle is *effective*.

We let $N_k(X)_{\mathbb{Z}}$ denote the quotient of the group of $\mathbb{Z}$-k-cycles by the relation of *numerical equivalence* as in [Ful84, Chapter 19]. $N_k(X)_{\mathbb{Z}}$ is a lattice inside the *numerical group*

$$N_k(X) := N_k(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

If Z is a k-cycle with $\mathbb{R}$-coefficients, its class in $N_k(X)$ is denoted $[Z]$.

The *numerical dual group* is the vector space $N^k(X)$ dual to $N_k(X)$. Any weighted degree k homogeneous polynomial in Chern classes induces an element of $N^k(X)$ by intersecting against k-cycle Chow classes, and $N^k(X)$ is spanned by such elements. The Chern class action on Chow groups descends to intersection maps $\cap : N^r(X) \times N_k(X) \to N_{k-r}(X)$. We also use "$\cdot$" to denote these intersections.

Given a morphism $\pi : X \to Y$, the pushforward of cycles descends to numerical groups. Dually, we obtain pullbacks for numerical dual groups. When π is a flat equidimensional map of relative dimension e and Y is smooth, the flat pullback of cycles descends to yield a flat pullback map $\pi^* : N_k(Y) \to N_{k+e}(X)$. Unlike for Chow groups, we do not know if the smoothness of Y can be removed.

Convention 2.1. For the rest of the paper, the term cycle will always refer to a cycle with $\mathbb{Z}$-coefficients, and a numerical class will always refer to a class with $\mathbb{R}$-coefficients, unless otherwise qualified.

2.1. Families of cycles.

Definition 2.2. Let X be a projective variety. A family of effective k-cycles (always with $\mathbb{Z}$-coefficients) on X consists of a variety W, a reduced closed subscheme $U \subset W \times X$, and an integer $a_i \geq 0$ for each component U_i of U such that for each component U_i of U the first projection map $p : U_i \to W$ is flat (projective) equidimensional dominant of relative dimension k. We will only consider families of cycles where each $a_i > 0$.

The fiber over a closed point of W defines a cycle $\sum_i a_i U_{i,w}$ on X. As we vary $w \in W$, the resulting cycles are algebraically equivalent. We denote the corresponding numerical class by $[p]$. A family of cycles can always be extended to a projective base by using a flattening argument (see [FL13, Remark 2.13]).

Construction 2.3 (Strict transform families)**.** Let X be a projective variety and let $p : U \to W$ be a family of effective k-cycles on X. Suppose that $\phi : X \dashrightarrow Y$ is a birational map. We define the strict transform family of effective k-cycles on Y as follows.

First, modify U by removing all irreducible components whose image in X is contained in the locus where ϕ is not an isomorphism. Then define the reduced closed subset U' of

$W \times Y$ by taking the strict transform of the remaining components of U. Over an open subset $W^0 \subset W$, the projection map $p' : (U')^0 \to W^0$ is flat equidimensional on each component of $(U')^0$. Each component of U' is the transform of a unique component of U, and we assign it the same coefficient.

2.2. **Cones of cycles.**

Definition 2.4. Let X be a projective variety, and let $k \geq 0$. The *pseudo-effective* cone $\overline{\mathrm{Eff}}_k(X)$ in $N_k(X)$ is the closure of the cone generated by classes of irreducible subvarieties of X. A class is *big* when it lies in the interior of $\overline{\mathrm{Eff}}_k(X)$. We use the notation $\alpha \preceq \beta$ if $\beta - \alpha \in \overline{\mathrm{Eff}}_k(X)$. The *nef* cone $\mathrm{Nef}^k(X)$ is the dual cone in $N^k(X)$.

Note that pseudo-effectiveness is preserved by pushforward, and dually nefness is preserved by pullback. In [FL14] the authors prove the following:

Theorem 2.5. *Let X be a projective variety of dimension n. For any k, the cone $\overline{\mathrm{Eff}}_k(X)$*

a) generates $N_k(X)$ as a vector space,
b) contains no lines (i.e. is salient, or pointed, or strict), and
c) contains the complete intersection $(h_1 \cdot \ldots \cdot h_{n-k}) \cap [X]$ in its strict interior for any ample classes $h_i \in N^1(X)$.

Moreover:

d) If $\alpha \in \overline{\mathrm{Eff}}_k(X)$ satisfies $\deg_h(\alpha) := h^k \cap \alpha = 0$ for some ample $h \in N^1(X)$, then $\alpha = 0$.
e) For any ample $h \in N^1(X)$ there exists a norm $|\cdot|$ on $N_k(X)$ such that $|\alpha| = \deg_h(\alpha)$ for any $\alpha \in \overline{\mathrm{Eff}}_k(X)$.
f) If $\pi : X \to Y$ is a surjective morphism of projective varieties, then $\pi_* \overline{\mathrm{Eff}}_k(X) = \overline{\mathrm{Eff}}_k(Y)$.

It is also useful to identify the cones of "moving cycles". These cones are well-studied for divisors and curves; the set-up for arbitrary cycles was considered in [FL13].

Definition 2.6. Let X be a projective variety. A family of effective k-cycles $p : U \to W$ is *strictly movable* if every component of U dominates X. The cycles defined by this family are called movable cycles. When U is an irreducible variety we say that p is *strongly movable*.

The closure of the cone generated by classes of cycle theoretic general fibers of strongly movable families is the *movable* cone $\overline{\mathrm{Mov}}_k(X)$. Its elements are called *movable* classes.

[FL13] proves the basic properties of movable classes:

Theorem 2.7. *Let X be a projective variety. For every statement of Theorem 2.5, the analogue for the movable cones $\overline{\mathrm{Mov}}_k(X)$ is also valid. Furthermore, if $\alpha \in \overline{\mathrm{Mov}}_k(X)$ then:*

g) For any pseudo-effective Cartier divisor E on X we have $\alpha \cdot E \in \overline{\mathrm{Eff}}_{k-1}(X)$.
h) For any nef Cartier divisor N on X we have $\alpha \cdot N \in \overline{\mathrm{Mov}}_{k-1}(X)$. If N is big and nef, then $\alpha \cdot N = 0$ if and only if $\alpha = 0$.
i) If $\pi : X \to Y$ is a generically finite map and $\pi_* \alpha = 0$, then $\alpha = 0$.

2.3. **Basepoint free cone.** While nef divisors satisfy many desirable geometric properties, nef classes of higher codimension may fail to behave as well. The basepoint free cone is introduced in [FL14] as a better analogue of the nef cone of divisors.

Definition 2.8. A basepoint free family of effective k-cycles on a projective variety X consists of

- an equidimensional quasi-projective scheme U,

- a (necessarily equidimensional) flat morphism $s : U \to X$,
- and a proper morphism $p : U \to W$ of relative dimension k to a quasi-projective variety W such that each component of U surjects onto W.

Note that the term "family" here differs from that in Definition 2.2, since U is not necessarily a subset of $W \times X$, so that the fibers U_w are not necessarily cycles on X.

The basepoint free cone $\mathrm{BPF}_k(X) \subset N_k(X)$ is the closure of the cone generated by the classes $F_p := (s|_{U_w})_*[U_w]$, where U_w is the fiber of a basepoint free family p as above over a general $w \in W$. If X is smooth, we define the basepoint free cone $\mathrm{BPF}^k(X) \subset N^k(X)$ using the isomorphism $\cap[X]$.

Remark 2.9. The terminology indicates that the classes are "basepoint free" in the following sense: suppose that α is the class of a basepoint free family. Then for every subvariety $V \subset X$ there is an effective cycle of class $\alpha \cap [X]$ that intersects V in the expected dimension. (To see this, let d denote the codimension of V. Then $s^{-1}V$ has codimension at least d in U by flatness, and $s^{-1}(V) \cap U_w$ has codimension at least d in U_w. Then $V \cap |F_p|$ has codimension at least d in $|F_p|$ by upper-semicontinuity of fiber dimensions. See [Kle74] for more arguments of this kind, particularly 1 Lemma.)

Despite the "homological" feel of the definition, $\mathrm{BPF}_k(X)$ is not preserved by pushforward, but it is preserved by pullbacks between smooth varieties.

Theorem 2.10 ([FL14], Lemma 5.4, Lemma 5.6, Corollary 5.7)**.** *Let $\pi : Y \to X$ be a morphism of projective varieties and let $p : U \to W$ be a basepoint free family of cycles on X.*

1. *The base change $p_Y : U \times_X Y \to W$ is also a basepoint free family. If X is smooth, we have the relation $[F_{p_Y}] = \pi^*[F_p]$.*
2. *Suppose X is smooth. For any top-dimensional (effective) cycle T supported on a general fiber U_w of p, there is a canonical (effective) cycle with support equal to $Y \times_X |T|$ whose pushforward to Y represents $\pi^*(s|_T)_*[T] \cap Y$.*

In particular, if both X and Y are smooth, $\pi^* \mathrm{BPF}^k(X) \subset \mathrm{BPF}^k(Y)$. Furthermore, if Y is smooth then the intersection of basepoint free classes on Y is basepoint free.

We will often use a special case of this construction. Suppose $\pi : Y \to X$ is birational and $p : U \to W$ is a family of k-cycles admitting a flat map to X. In this case we can consider p both as a family of cycles and as a basepoint free family. In this situation, the base change family on Y coincides with the strict transform family as defined earlier. (Since every component of U maps dominantly onto X, so also every component of $U \times_X Y$ dominates Y. Thus both families are defined via base change.) In particular, for a general member of p the numerical class of the strict transform cycle is the pullback of the class of the cycle.

Theorem 2.11 ([FL14], Theorem 1.3, Theorem 1.6)**.** *Let X be a smooth projective variety. Then*

1. $\mathrm{BPF}^k(X)$ *generates $N^k(X)$ as a vector space and* $\mathrm{BPF}^k(X) \subset \mathrm{Nef}^k(X)$.
2. *If $h_1, \ldots, h_k \in N^1(X)$ are ample divisor classes, then $h_1 \cdot \ldots \cdot h_k$ belongs to the strict interior of* $\mathrm{BPF}^k(X)$.
3. *If E is a globally generated vector bundle, then the Chern and dual Segree classes $c_k(E)$ and $s_k(E^\vee)$ (see* [Ful84, §3.1]*) are basepoint free.*

3. Reductions and particular cases

We now study the basic features of the Strong and Weak Conjectures in the numerical setting.

Definition 3.1. Let $\pi : X \to Y$ be a morphism of projective varieties. Let

- $\overline{\mathrm{Eff}}_k(\pi)$ denote the closed convex cone in $N_k(X)$ generated by effective k-classes of X contracted by π,
- $N_k(\pi)$ denote the subspace of $N_k(X)$ generated by effective k-classes of X contracted by π,

Remark 3.2. We can rephrase our conjectures and properties of interest as follows:

i) Weak Conjecture: $\ker \pi_* \cap \overline{\mathrm{Eff}}_k(X) \subseteq N_k(\pi)$.
ii) Strong Conjecture: $\ker \pi_* \cap \overline{\mathrm{Eff}}_k(X) = \overline{\mathrm{Eff}}_k(\pi)$.

Collectively we call the Strong and Weak Conjectures the *Pushforward Conjectures* and denote them by PC. (More precisely, to say a property holds for the PC means that it holds for both the Strong Conjecture and the Weak Conjecture.) The following proposition is the basic tool used in this section.

Proposition 3.3. *Let $\pi : X \to Y$ be a morphism of projective varieties. If $\alpha \in \overline{\mathrm{Eff}}_k(X)$ and h is an ample class on Y, then $\pi_*\alpha = 0$ if and only if $\alpha \cdot \pi^* h^k = 0$.*

Proof. This is a consequence of Theorem 2.5.d and of the projection formula. □

3.1. Examples.

Example 3.4. One way to establish the Weak Conjecture for a morphism π is to show that *all* of the kernel of π_* is spanned by the classes of contracted cycles. This property is studied in [FL15] which calls it the GK property for π. For example, this property is well-known for projective bundle maps.

[FL15] shows that if the fibers of a morphism have trivial Chow groups then π satisfies the GK property (and thus satisfies the Weak Conjecture):

Theorem 3.5. *Let $\pi : X \to Y$ be a dominant morphism of projective varieties over an uncountable algebraically closed field, with Y smooth. Suppose that every fiber F (over a closed point) of π satisfies $\dim_{\mathbb{Q}}(A_0(F)_{\mathbb{Q}}) = 1$. Then the GK property and the Weak Conjecture hold for π.*

An important special case is when $\pi : X \to Y$ is a birational morphism over $\mathbb{C}$ with Y smooth.

Example 3.6. Let S be a smooth surface such that $A_0(S) = \mathbb{Z}$. By the work of [Mum68] and [Roĭ72], this implies that $p_g = 0$ and $\mathrm{Alb}(S)$ is trivial. Examples include any rational surface S and conjecturally any surface with $q = p_g = 0$.

Suppose that Y is another smooth surface. There is an isomorphism

$$N_2(S \times Y) \cong \mathbb{R} \oplus (N_1(S) \otimes N_1(Y)) \oplus \mathbb{R}.$$

For surfaces over $\mathbb{C}$, this follows easily from Hodge theory and the Kunneth formula. (In fact, this argument also works for any surface S satisfying $q = p_g = 0$.) Over an arbitrary algebraically closed field, this follows from [FL15, Theorem 1.3].

Note that the Weak Conjecture holds for the second projection map and for the first projection map as well. If $S = Y$, we can also see right away that S admits a numerical decomposition of the diagonal, in the sense that Δ is numerically equivalent to a sum of cycles whose supports are contracted by a projection map. These two properties are closely related via an argument of [DJV13]: note that if $\alpha \in N_2(S \times Y)$ satisfies $\pi_{1*}\alpha = 0$ and $\alpha \cdot F_2 > 0$, then there are ample divisors A on S and A' on Y such that $\alpha + [\pi_1^*A \cdot \pi_2^*A'] \in \overline{\mathrm{Eff}}_2(S \times Y)$. Thus in the special case where $S = Y$, the Weak Conjecture applied to $\Delta - F_1 + \pi_1^*A \cdot \pi_2^*A'$ implies that Δ has a numerical decomposition of the diagonal.

We now discuss the Strong Conjecture for the map $\pi = \pi_2 : S \times Y \to Y$. We first need to understand the geometry of π-vertical surfaces. Suppose that Z is an irreducible π-vertical surface so $\pi(Z)$ is a curve C on Y. Let C' be a normalization of C and Z' denote the strict transform of Z on $S \times C'$. If Z' does not dominate S, then it is the pullback of a divisor on S. If it does dominate S, then it induces a morphism $S \to \mathrm{Jac}(C')$. But by assumption on the Albanese map this morphism is trivial. So after twisting by the pullback of a line bundle from S, the divisor Z' is the pullback of a divisor on C'.

To prove the Strong Conjecture, it suffices to consider the case when $\alpha \in \overline{\mathrm{Eff}}_2(X) \cap \ker \pi_*$ is extremal.

Claim 3.7. The Strong Conjecture holds for an extremal class α if and only if the projection $\alpha^{(1)}$ of α onto the $N_1(S) \otimes N_1(Y)$ component of $N_2(S \times Y)$ has shape $a \otimes b$ with $a \in N_1(S)$ and $b \in N_1(Y)$. (If the SC is true, then by Lemma 5.1 we can write α as a limit of cycles Z_i, each with irreducible support that does not dominate Y. The above argument shows that $Z_i = a_i \otimes b_i$ or $Z_i \in \mathbb{R}_+F_2$, and by passing to limits $\alpha = \alpha^{(1)} = a \otimes b$, or α is a multiple of F_2 and $\alpha^{(1)} = 0$. Conversely, if $\alpha = a \otimes b + cF_2$, then $c \geq 0$ because it identifies with $\pi_*\alpha$. Let η be an arbitrary nef class in $N_1(S)$. Then $(a \cdot \eta)b = \pi_{2*}(\alpha \cdot \pi_1^*\eta) \in \overline{\mathrm{Eff}}_1(Y)$ and up to signs we can assume that a and b are both psef. Consequently $\alpha = \alpha^{(1)} + cF_2$ is a sum of psef cycles, both contracted by π and SC is straightforward.)

By the claim, the SC holds if and only if $\overline{\mathrm{Eff}}_2(S \times Y) \cap (N_1(S) \otimes N_1(Y) \oplus \mathbb{R}[F_2])$ is generated by F_2 and by classes $a \otimes b$, where $a \in \overline{\mathrm{Eff}}_1(S)$ and $b \in \overline{\mathrm{Eff}}_1(Y)$. This holds for example when either $\overline{\mathrm{Eff}}_1(S)$ or $\overline{\mathrm{Eff}}_1(Y)$ is simplicial, e.g. when the Picard number of S or Y is at most two. (Say $\overline{\mathrm{Eff}}_1(S)$ is simplicial, generated by a basis $a_1, \ldots, a_\rho$ of $N^1(S)$. Then the dual basis a_i^* generates $\mathrm{Nef}^1(S)$. Writing α in the unique way as $\alpha = \sum_i a_i \otimes b_i + cF_2 \in \overline{\mathrm{Eff}}_2(S \times Y)$, we see $c \geq 0$ and $\pi_{2*}(\alpha \cdot \pi_1^*(a_i^*)) = b_i \in \overline{\mathrm{Eff}}_1(Y)$.) □

Example 3.8 (Surface classes on the self products of a very general abelian surface)**.** Let (S, θ) be a very general principally polarized complex abelian surface. Put $X = S \times S$. A result of Tankeev and Ribet ([DELV11, Proposition 3.1]) and [DELV11, Proposition 3.2] imply that $N^1(X)$ has a basis given by $\{\theta_1, \theta_2, \lambda\}$, where θ_1 and θ_2 are the pullbacks of θ via the two projections, and $\lambda = c_1(\mathcal{P})$, where $\mathcal{P}$ is the Poincaré line bundle on X. Furthermore, the product map $\mathrm{Sym}^2 N^1(X) \to N^2(X)$ is an isomorphism. The positive cones of divisors on X are computed by [DELV11, Proposition 3.9]:

$$\overline{\mathrm{Eff}}^1(X) = \mathrm{Nef}^1(X) = \left\{a_1\theta_1 + a_2\theta_2 + a_3\lambda \mid a_1 \geq 0,\ a_2 \geq 0,\ a_1a_2 \geq a_3^2\right\}.$$

The pseudoeffective cone $\overline{\mathrm{Eff}}_2(X)$ of surface classes on X is computed by [DELV11, Theorem 4.1]: A class

$$\alpha = a_1\theta_1^2 + a_2\theta_1\theta_2 + a_3\theta_2^2 + a_4\theta_1\lambda + a_5\theta_2\lambda + a_6\lambda^2$$

is pseudoeffective if and only if

$$
\begin{aligned}
a_1,\ a_2,\ a_3 &\geq 0, & a_2 &\geq 2|a_6|,\\
a_1(a_2+2a_6) &\geq a_4^2, & a_3(a_2+2a_6) &\geq a_5^2,\\
a_1a_3 &\geq a_6^2, & (a_1a_3-a_6^2)(a_2+2a_6)+2a_4a_5a_6 &\geq a_3a_4^2+a_1a_5^2.
\end{aligned}
$$

Let $\pi : X \to S$ be the first projection and suppose that $\alpha \in \overline{\mathrm{Eff}}_2(X)$ satisfies $\pi_*\alpha = 0$. By Proposition 3.3, this is equivalent to requiring that $\alpha \cdot \theta_1^2 = 0$. It is easy to see that $\theta_1^3 = \theta_1^2\lambda = 0$ (see [DELV11, §4] for a complete list of relations in $N^*(X)$). Writing α with coefficients as above, we see $\alpha \cdot \theta_1^2 = 0$ is equivalent to $a_3 = 0$. The inqualities describing $\overline{\mathrm{Eff}}_2(X)$ then imply that $a_5 = a_6 = 0$ as well. Consequently $\alpha = a_1\theta_1^2 + a_2\theta_1\theta_2 + a_4\theta_1\lambda = \theta_1(a_1\theta_1 + a_2\theta_2 + a_4\lambda)$. The relations between the coefficients are

$$a_1,\ a_2 \geq 0 \quad \text{and} \quad a_1a_2 \geq a_4^2.$$

This precisely means that $a_1\theta_1 + a_2\theta_2 + a_4\lambda$ is a pseudoeffective divisor. Thus any pseudo-effective surface class with $\pi_*\alpha = 0$ is a product of nef divisors, and the Strong Conjecture for surface classes for $\pi : X \to S$ is an easy consequence. □

Example 3.9 (Trivial Grassmann bundles)**.** Let G be a product of finitely many Grassmann varieties, and let Y be a projective variety. Then the strong conjecture is true for the first projection $\pi : Y \times G \to Y$. (Put $n = \dim Y$ and $g = \dim G$. If G is a Grassmann variety, then $\overline{\mathrm{Eff}}^k(G) = \mathrm{Nef}^k(G)$ is a simplicial cone generated by the nonzero classes of Schubert cycles s_λ corresponding to partitions λ of k. A similar result holds true when G is replaced by a product of Grassmann varieties, but Schubert cycles are replaced by products $s_{\underline{\lambda}}$ of Schubert cycles from each factor, and $\lambda \vdash k$ is replaced by tuples $\underline{\lambda}$ of partitions $\lambda^{(i)} \vdash k_i$, where $k = \sum_i k_i$. We use the notation $\underline{\lambda} \vdash k$ and $k = |\underline{\lambda}|$. Since G is a homogeneous space, the intersection of pseudoeffective classes is pseudoeffective. Furthermore $\overline{\mathrm{Eff}}_k(G)$ and $\overline{\mathrm{Eff}}^k(G)$ are generated by dual bases of $N_k(G)$ and $N^k(G)$ respectively.

Let $\alpha \in N_k(Y \times G)$. Then by [Ful84, §15.6]

$$\alpha = \sum_{\underline{\lambda}} \pi^*\beta_{\underline{\lambda}} \cdot \rho^* s_{\underline{\lambda}},$$

where ρ is the second projection, and $\beta_{\underline{\lambda}} \in N_{k+|\underline{\lambda}|-g}(Y)$. If α is pseudoeffective, then by again using the homogeneity of G, we have that $\alpha \cdot \rho^* s_{\underline{\mu}}$ is pseudoeffective for all tuples of partitions $\underline{\mu}$. Then the same is true of $\pi_*(\alpha \cdot \rho^* s_{\underline{\mu}})$. For duality and dimension reasons, using the projection formula, one sees that as $\underline{\mu}$ varies, the previous formula recovers the pseudoeffectivity of $\beta_{\underline{\lambda}}$ for all $\underline{\lambda}$. Consequently $\overline{\mathrm{Eff}}_k(Y \times G)$ is generated by $\pi^* \overline{\mathrm{Eff}}_{k+r-g}(Y) \cdot \rho^* \overline{\mathrm{Eff}}^r(G)$ for all r.

Note that $\pi_*\alpha = 0$ if and only if $\beta_{\underline{\lambda}} = 0$ when $|\underline{\lambda}| = g$ (there exists only one such $\underline{\lambda}$, since $\dim N^g(G) = 1$).

Assume now that $\alpha \in \overline{\mathrm{Eff}}_k(Y \times G) \cap \ker \pi_*$. For all $\underline{\lambda}$ with $|\underline{\lambda}| < g$, we have that $\beta_{\underline{\lambda}}$ is a limit of effective classes, and then the same is true of $\pi^*\beta_{\underline{\lambda}} \cdot \rho^* s_{\underline{\lambda}}$, but furthermore all these effective classes are π-contracted (the nonempty fibers have dimension $g - |\underline{\lambda}| > 0$).) □

3.2. **Reduction steps.** [DJV13] shows that the (homological) SC/WC follow from certain special cases. Using Proposition 3.3, we can recover all the reduction steps of [DJV13] in the setting of numerical equivalence using essentially the same proofs.

Remark 3.10. If $\dim Y < k$, then $N_k(\pi) = N_k(X)$ and PC holds trivially.

The following are corollaries of Proposition 3.3.

Corollary 3.11 (The finite case)**.** *Let $\pi : X \to Y$ be a finite morphism of projective varieties. Let $\alpha \in \overline{\mathrm{Eff}}_k(X)$. Then $\pi_*\alpha = 0$ if and only if $\alpha = 0$.*

Proof. As in [DJV13, Proposition 2.1], let h be an ample class on Y. Then π^*h is ample on X by the finiteness of π. The result follows from Theorem 2.5.d and from Proposition 3.3. $\square$

Corollary 3.12 (Reduction to X nonsingular)**.** *The PC for $\pi : X \to Y$ follow from the PC for π precomposed with any surjective morphism $f : X' \to X$ with X' a projective variety. In particular, the PC follow from the case when X is nonsingular.*

Proof. By Theorem 2.5.f, there exists $\alpha' \in \overline{\mathrm{Eff}}_k(X')$ such that $f_*\alpha' = \alpha$. Moreover, $f_* \overline{\mathrm{Eff}}_k(\pi \circ f) = \overline{\mathrm{Eff}}_k(\pi)$ and $f_*N_k(\pi \circ f) = N_k(\pi)$. We obtain the last statement by letting $f : X' \to X$ be a nonsingular alteration. $\square$

Corollary 3.13. *Assume $\dim Y = k$, and let $\alpha \in \overline{\mathrm{Eff}}_k(X)$ with $\pi_*\alpha = 0$. Let $\pi' : X' \to Y'$ be the main component of the base change of π over a generically finite cover of Y, and let α' be a pseudo-effective lift of α. If the PC hold for π' and α', then they also hold for π and α.*

Proof. Denote by f the generically finite morphism $Y' \to Y$. Then $f_* : N_k(Y') \to N_k(Y)$ is the multiplication by $\deg f$ via the isomorphisms $N_k(Y) \simeq \mathbb{R}\cdot[Y]$ and $N_k(Y') \simeq \mathbb{R}\cdot[Y']$. $\square$

Corollary 3.14 (Reduction to π surjective)**.** *Let $\pi : X \to Y$ be a morphism of projective varieties. Let $Z = \pi(X)$ with closed embedding $\imath : Z \hookrightarrow Y$. Denote by $\pi' : X \to Z$ the induced surjective morphism. If the PC hold for π', then they hold for π as well.*

Proof. Corollary 3.11 implies that $\pi_*\alpha = 0$ if and only if $\pi'_*\alpha = 0$. Clearly $\overline{\mathrm{Eff}}_k(\pi) = \overline{\mathrm{Eff}}_k(\pi')$ and $N_k(\pi) = N_k(\pi')$. $\square$

Corollary 3.15 (Reduction to Y a projective space)**.** *If the PC are true when π is surjective and $Y = \mathbb{P}^m$, then they are always true.*

Proof. As in [DJV13, Proposition 3.1], let $Z = \mathrm{Im}(\pi)$, and let $Z \to \mathbb{P}^m$ be a finite morphism. Apply the Corollary 3.11 for $Z \to Y$ and $Z \to \mathbb{P}^m$, and Corollary 3.14 for $X \to Z$. $\square$

Corollary 3.16 (Reduction to connected fibers)**.** *If the PC hold for dominant morphisms with connected fibers, then they are always true.*

Proof. As in [DJV13, Corollary 2.4], apply the Corollaries 3.11 and 3.14 to a Stein factorization of $X \to \mathrm{Im}(\pi)$. $\square$

3.3. **Counterexamples to potential generalizations of the pushforward conjectures.** We show that the weak conjecture may fail if we try to weaken or alter the data of the problem.

Example 3.17 (Positivity is necessary)**.** Let $X = E \times E$, with E an elliptic curve, and denote by $\pi : X \to E$ the first projection. Let δ be the class of the diagonal in X, and let f_1 and f_2 be the classes of the fibers of the projections. These three classes are linearly independent in $N_1(X)$. Observe that $\pi_*(\delta - f_2) = 0$, but $\delta - f_2$ is not in $N_1(\pi) = \mathbb{R}\cdot f_1$. $\square$

The next example shows that one can not replace pseudoeffectivity by a possibly weaker positivity condition.

Example 3.18 (Pseudoeffectivity is necessary)**.** We use the example of [DELV11] discussed in Example 3.8, keeping the notation there, to find a counterexample to the "Weak Nef Conjecture". That is, we find a nef class α with $\pi_*\alpha = 0$ which fails to be contained in the subspace generated by nef classes with π-vertical support, or even in $N_k(\pi)$. By [DELV11, Example 4.3], the class $\alpha = 8\theta_1\theta_2 + 3\lambda^2$ is nef. Note that $\alpha \cdot \theta_1^2 = 0$, and by checking the equations defining $\overline{\mathrm{Eff}}^2(X)$ in Example 3.8 we see that α is not pseudoeffective. We have that $\pi_*\alpha = 0$ (this follows from the fact that $N^2(S)$ is one-dimensional). However, the computation in Example 3.8 shows that α is not in the linear span of even the pseudoeffective classes in $\ker \pi_*$. □

Example 3.19 (Semiample intersections are necessary)**.** For a morphism $\pi : X \to Y$, we have by Proposition 3.3 that $\alpha \in \overline{\mathrm{Eff}}_k(X) \cap \ker \pi_*$ if and only if $\alpha \cdot \pi^* h^k$, where h is an arbitrary ample class in $N^1(Y)$. Denoting $\eta = \pi^* h$, we can rephrase the pushforward conjectures as follows:

Question: Let X be a projective variety, and let $\eta \in \mathrm{Nef}^1(X)$ be a semiample class. Let $\alpha \in \overline{\mathrm{Eff}}_k(X)$ with $\alpha \cdot \eta^k = 0$. Is α a linear combination (or limit) of effective classes α_i with $\alpha_i \cdot \eta^k = 0$?

None of the pushforward conjectures are true if one only asks that η is nef, instead of semiample. Mumford verifies the full strength of the hypotheses of Kleiman's ampleness criterion by constructing an example (cf. [Laz04, Example 1.5.2]) of a surface X with a nef Cartier divisor class η such that $\eta^2 = 0$ (so η is not ample), and η dots positively against any effective curve class. In particular there are no nonzero effective curve classes that have zero intersection with η, although $\alpha = \eta$ is psef and $\alpha \cdot \eta^1 = 0$. □

3.4. **Curves and divisors.** [DJV13] proves the Strong Conjecture over $\mathbb{C}$ for curve classes and divisor classes. (Although [DJV13] works with the cohomological versions of the conjectures, they are equivalent to our versions by the Hodge Conjecture for divisors and curves.) Our techniques allow for the removal of the smoothness assumptions.

3.4.1. *The Strong Conjecture for curves.* The proof of the Strong Conjecture for curves in [DJV13] works in arbitrary characteristic. Since the argument is short, we reproduce it here.

Theorem 3.20 ([DJV13] Theorem 4.1)**.** *Let $\pi : X \to Y$ be a morphism of projective varieties. Consider a class $\alpha \in \overline{\mathrm{Eff}}_1(X)$ with $\pi_*\alpha = 0$. Then α is in the closed convex cone generated by the irreducible curves contracted by π.*

Proof. Assume that $\alpha \notin \overline{\mathrm{Eff}}_1(\pi)$. Then there exists a Cartier divisor class D such that $D \cdot \alpha < 0$ and $D \cdot (\overline{\mathrm{Eff}}_1(\pi) \setminus \{0\}) > 0$. In particular, D is π-ample. Let then h be a large ample class on Y such that $D' := D + \pi^* h$ is ample. Then $D' \cdot \alpha = D \cdot \alpha$ by the projection formula. The latter is negative, which contradicts the ampleness of D'. □

3.4.2. *The Strong Conjecture for divisors.* The key tool for the proof of the Strong Conjecture for divisors in [DJV13] is the σ-decomposition of [Nak04]. We give a shorter argument that relies on the results of [Leh11]; this argument will come in useful again in Section 4.3.

Theorem 3.21. *Let $\pi : X \to Y$ be a morphism of projective varieties over $\mathbb{C}$. Let n denote the dimension of X. Assume that $\alpha \in \overline{\mathrm{Eff}}_{n-1}(X)$ satisfies $\pi_*\alpha = 0$. Then α is in the closed convex cone generated by irreducible subvarieties contracted by π.*

As in [DJV13], using Remark 3.10, we reduce to the case when $c := \mathrm{reldim}(\pi) \in \{0, 1\}$. The case $c = 0$ follows from much more general results concerning generically finite maps in Section 5.

Proposition 3.22. *Let $\pi : X \to Y$ be a (dominant) generically finite morphism of projective varieties of dimension n over an algebraically closed field. Assume that $\alpha \in \overline{\mathrm{Eff}}_{n-1}(X)$ satisfies $\pi_*\alpha = 0$. Then α is the class of an effective Weil divisor contracted by π.*

Proof. We can assume that α is nonzero and spans an extremal ray of $\overline{\mathrm{Eff}}_k(X)$. The proof of Proposition 5.19 shows that $\alpha = \imath_*\alpha'$ for some $\alpha' \in \overline{\mathrm{Eff}}_{n-1}(E_1)$, where E_1 is an irreducible divisor on X. But $\overline{\mathrm{Eff}}_{n-1}(E_1) = \mathbb{R}_{\geq 0}[E_1]$, therefore α is effective. $\square$

The case $c = 1$ requires more work. For a pseudo-effective divisor L on a smooth projective variety X over $\mathbb{C}$, we let $L = P_\sigma(L) + N_\sigma(L)$ denote the σ-decomposition of [Nak04]. This decomposition is a numerical invariant, so for a pseudo-effective divisor class α we can write $\alpha = P_\sigma(\alpha) + N_\sigma(\alpha)$.

Proposition 3.23. *Let $\pi : X \to Y$ be a surjective projective morphism of relative dimension $c = 1$ over $\mathbb{C}$. Let $\alpha \in \overline{\mathrm{Eff}}_{n-1}(X)$ satisfy $\pi_*\alpha = 0$. Then α is in $\overline{\mathrm{Eff}}_{n-1}(\pi)$.*

Proof. Arguing just as in Corollaries 3.12, 3.13, and 3.16, we may assume that X and Y are nonsingular and that π has connected fibers. If b is the class of a fiber of π, then by Corollary 3.3 we have $\alpha \cdot b = 0$, so the restriction of α to the general fiber of π is numerically trivial. By [Leh11, Theorem 1.3], there exists a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

with X' and Y' nonsingular, with f and g birational, and a pseudo-effective divisor class β on Y' such that $P_\sigma(f^*\alpha) = P_\sigma(\pi'^*\beta)$. It is enough to prove the proposition for π' and $f^*\alpha$. Thus we may assume $\pi = \pi'$ and

$$P_\sigma(\alpha) = P_\sigma(\pi^*\beta)$$

for some pseudo-effective divisor class β on Y.

Write $\alpha = P_\sigma(\alpha) + N_\sigma(\alpha)$. Since $N_\sigma(\alpha)$ is automatically an effective class, it suffices to show that $P_\sigma(\alpha) = P_\sigma(\pi^*\beta)$ lies in $\overline{\mathrm{Eff}}_{n-1}(\pi)$. Write β as a limit of big classes $\beta_i := \beta + \frac{1}{i}\delta$, where δ is a big divisor class on Y. Then $\pi^*\beta_i$ is an effective class, therefore we have a σ-decomposition

$$\pi^*\beta_i = P_\sigma(\pi^*\beta_i) + N_\sigma(\pi^*\beta_i),$$

where the positive and negative parts are both effective classes. The class $\pi^*\beta_i$ is in $\ker \pi_*$ by the projection formula. It follows that $P_\sigma(\pi^*\beta_i)$ and $N_\sigma(\pi^*\beta_i)$ both belong to $\overline{\mathrm{Eff}}_{n-1}(\pi)$. By [Nak04, III.1.7.(2) Lemma], $P_\sigma(\pi^*\beta)$ is the limit of the sequence $P_\sigma(\pi^*\beta_i)$, so that $P_\sigma(\pi^*\beta)$ also belongs to the closed cone $\overline{\mathrm{Eff}}_{n-1}(\pi)$. $\square$

Proof of Theorem 3.21. It follows immediately from the two propositions. $\square$

4. The Movable Strong/Weak Conjectures and Zariski decompositions

As explained in the introduction, we can study the Strong/Weak Conjectures for pseudo-effective cycles by using the Zariski decomposition of [FL13] to decompose a pseudo-effective class into a "movable" part and a "rigid" part and studying the conjecture for each part separately. Just as for divisors, this approach makes the problem more tractable and allows us to prove a number of special cases. While the Zariski decomposition is not used in an essential way for the rest of the paper, it places our results in a more general conceptual framework.

4.1. **Movable Strong/Weak Conjectures.** It seems likely that whenever the Strong/Weak Conjectures hold, a stronger version will hold for movable classes.

Definition 4.1. Let $\pi : X \to Y$ be a morphism of projective varieties. Let

- $\overline{\mathrm{Mov}}_k(\pi)$ denote the closed convex cone in $N_k(X)$ generated by movable effective k-classes of X contracted by π,
- $M_k(\pi)$ denote the subspace of $N_k(X)$ generated by movable effective k-classes of X contracted by π.

The analogues of the Strong and Weak Conjectures for movable classes are:

Movable Strong Conjecture (MSC). *Let $\pi : X \to Y$ be a morphism of projective varieties. Let $\alpha \in \overline{\mathrm{Mov}}_k(X)$ satisfy $\pi_*\alpha = 0$. Then α belongs to $\overline{\mathrm{Mov}}_k(\pi)$.*

Movable Weak Conjecture (MWC). *Let $\pi : X \to Y$ be a morphism of projective varieties. Let $\alpha \in \overline{\mathrm{Mov}}_k(X)$ satisfy $\pi_*\alpha = 0$. Then α belongs to $M_k(\pi)$.*

There are no immediate implications between these conjectures and the Strong/Weak Conjectures.

These conjectures seems substantially easier than their non-movable counterparts. The key insight is that movable cycles should not reflect the properties of special fibers of a map, inviting arguments that only rely on general fibers. For example, Theorem 2.7i implies that the Movable Strong Conjecture is true for generically finite maps π (compare with Corollary 3.11). This should be contrasted with Proposition 5.23, which shows the the Strong Conjecture in dimension n for birational maps is equivalent to the Strong Conjecture in all lower dimensions (and for maps of arbitrary relative dimension).

We denote the Movable Strong Conjecture and Movable Weak Conjecture collectively as the Movable Pushforward Conjectures (MPC). We next carry out the reduction steps of Section 3 for the MPC.

Proposition 4.2. *If the MPC hold for maps $\pi : X \to Y$ where*

- *X is smooth,*
- *π is surjective,*
- *π has connected fibers*

then the MPC are always true.

Proof. The proof is identical to the PC case. □

Remark 4.3. Over $\mathbb{C}$, we can also assume that Y is smooth. (Let $\pi : X \to Y$ be a surjective projective morphism with connected fibers with X smooth, and let $\pi' : X' \to Y'$ be a birational model of π with X' and Y' both smooth, and with birational morphisms

$f_X : X' \to X$ and $f_Y : Y' \to Y$. We can assume that π and π' coincide over an open subset of Y. Consequently π' also has connected fibers. If α' is a movable f_X-lift of α (cf. Theorem 2.7.f), then $(f_Y)_*(\pi'_*\alpha') = 0$ implies $\pi'_*\alpha' = 0$ by Theorem 2.7.i. If the MPC are true for π' and α', then they are also true for π and α.)

Example 4.4. Example 3.19 also shows that generalization of the MPC from semiample to nef divisors can not hold.

4.2. **The Movable Strong Conjecture for curves.** The following result is due to [Pet12].

Theorem 4.5 ([Pet12], 6.8 Theorem)**.** *Let $\pi : X \to Y$ be a surjective morphism with connected fibers from a smooth variety X to a normal variety Y over $\mathbb{C}$. Let F be a general fiber of π. Then:*

i) The inclusion map $i_* : N_1(F) \to N_1(X)$ maps $\overline{\mathrm{Mov}}_1(F)$ into $\overline{\mathrm{Mov}}_1(X)$.
ii) $\overline{\mathrm{Mov}}_1(X) \cap \ker(\pi_*)$ is the closure of the cone generated by movable curves C satisfying $\pi_*C = 0$.
iii) $i_* : \overline{\mathrm{Mov}}_1(F) \to \overline{\mathrm{Mov}}_1(X) \cap \ker(\pi_*)$ is surjective.

Note that the MSC for curves is an immediate consequence whenever the hypotheses of the theorem hold. We use this theorem to derive the curve case of the MSC for arbitrary maps over $\mathbb{C}$.

Theorem 4.6. *Let $\pi : X \to Y$ be a morphism of projective varieties over $\mathbb{C}$. Then the MPC hold for curve classes on X.*

Proof. Note that the MWC and MSC can be detected after precomposing by any surjective map. So we may suppose X is smooth. Then apply Theorem 2.7 to the normalization of the Stein factorization of π to reduce to the case when Y is normal and π has connected fibers. This case is then settled by Theorem 4.5. □

Corollary 4.7. *Let $\pi : X \to Y$ be a surjective morphism of projective varieties over $\mathbb{C}$ of relative dimension 1. Let α be a movable curve class with $\pi_*\alpha = 0$. Then α is proportional to the class of a general fiber.*

Proof. By the projection formula, we may precompose by a generically finite map to assume that X is smooth. We may also replace Y by the normalization of the Stein factorization, since the general fiber of the Stein factorization is proportional to the general fiber of the original map. Thus we may assume Y is normal and π has connected fibers. Then the general fiber of π is a smooth curve F and $\overline{\mathrm{Mov}}_1(F)$ is clearly spanned by $[F]$. Conclude by Theorem 4.5. □

4.3. **The Movable Strong Conjecture for divisors.** We verify the MSC for divisors using the same strategy as the verification of the SC for divisors.

Theorem 4.8. *Let $\pi : X \to Y$ be a morphism of projective varieties over $\mathbb{C}$. Then the MPC hold for divisor classes on X.*

Proof. Let α be a movable divisor class on X satisfying $\pi_*\alpha = 0$. By Remark 3.10 we reduce to the case when $c := \mathrm{reldim}(\pi) \in \{0, 1\}$.

When π is birational, Theorem 2.7 shows that $\alpha = 0$. So the MSC holds in this case.

When π has relative dimension 1, we reduce just as in Proposition 3.23 to the case when X and Y are smooth and $\alpha = P_\sigma(\pi^*\beta)$ for some psef class β on Y. The argument in the proof of Proposition 3.23 shows that α is the limit of the movable effective classes $P_\sigma(\pi^*\beta_i)$ where the β_i are big effective classes approximating β. So the MSC holds in this case as well. □

4.4. **Zariski decompositions for cycle classes.** [FL13] constructs a Zariski decomposition for cycle classes of arbitrary codimension. The starting point is the mobility function, a measure of the "positivity" of a cycle class.

Definition 4.9 ([Leh13]). Let X be a projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Z}}$. The mobility of α is

$$\mathrm{mob}(\alpha) := \limsup_{m\to\infty} \frac{\max \left\{ b \in \mathbb{Z}_{\geq 0} \,\middle|\, \begin{array}{c} \text{For any } b \text{ general points on } X \text{ there is an} \\ \text{effective } \mathbb{Z}\text{-cycle of class } m\alpha \text{ containing them} \end{array} \right\}}{m^{\frac{n}{n-k}}/n!}$$

Building on [DELV11, Conjecture 6.5], in [Leh13], the second author shows that the mobility extends to a homogeneous continuous function on all of $N_k(X)$. In particular, the mobility is positive precisely for big classes α.

Definition 4.10 ([FL13]). Let X be a projective variety. A Zariski decomposition for a big class α is a sum $\alpha = P(\alpha) + N(\alpha)$ where $P(\alpha)$ is movable, $N(\alpha)$ is pseudo-effective, and $\mathrm{mob}(P(\alpha)) = \mathrm{mob}(\alpha)$.

Let $\mathcal{P} \subset N_k(X) \times N_k(X)$ denote the set of all pairs $(\beta, P(\beta))$ for big classes β and positive parts $P(\beta)$ of β. For α pseudo-effective, a decomposition $\alpha = P(\alpha) + N(\alpha)$ is a Zariski decomposition whenever $(\alpha, P(\alpha))$ lies in the closure of $\mathcal{P}$. [FL13] verifies that for big cycles this definition coheres with the previous one.

[FL13] establishes the existence of Zariski decompositions.

Theorem 4.11 ([FL13], Theorem 1.6). *Let X be a projective variety and let $\alpha \in N_k(X)$ be a pseudo-effective class. Then α admits a Zariski decomposition $\alpha = P(\alpha) + N(\alpha)$.*

The following conjecture of [FL13] describes the fundamental geometry underlying negative parts of Zariski decompositions.

Conjecture 4.12. *Let X be an integral variety and let $\alpha \in \overline{\mathrm{Eff}}_k(X)$. For some/any Zariski decomposition $\alpha = P(\alpha) + N(\alpha)$, the negative part $N(\alpha)$ is the pushforward of a pseudo-effective cycle on a proper subscheme.*

Then we have:

Theorem 4.13. *Assume Conjecture 4.12. Then:*

(1) The MWC in dimension $\leq n$ implies the WC in dimension $\leq n$.
(2) The MSC in dimension $\leq n$ implies the SC in dimension $\leq n$.

Proof. (1) We prove this by induction on dimension. Assume the MWC in dimension $\leq n$. Suppose $\pi : X \to Y$ is a morphism where X has dimension n and $\alpha \in \overline{\mathrm{Eff}}_k(X)$ satisfies $\pi_*\alpha = 0$. Write $\alpha = P + N$ for the Zariski decomposition of α and note that both P and N push forward to 0. The MWC implies that $P \in \overline{\mathrm{Eff}}_k(\pi)$. Conjecture 4.12 implies that there is a subscheme $i : W \hookrightarrow X$ of dimension $\leq n-1$ such that $\alpha = i_*\beta$ for some pseudo-effective β. We may replace W by the disjoint union of its reduced components without changing this property. Then by the induction assumption, the fact that $\pi_*\beta = 0$ implies that $\beta \in \overline{\mathrm{Eff}}_k(\pi|_W)$. Hence the pushforward is in $\overline{\mathrm{Eff}}_k(\pi)$.

(2) is the same argument. $\square$

5. Exceptional classes

The prototypical example of a negative class for the σ-decomposition is an exceptional divisor. In this section we define and study an analogous notion for arbitrary cycle classes.

5.1. **Cone lemmas.**

We start by recalling a useful lemma concerning cones.

Lemma 5.1. *Let C be a closed full-dimensional salient convex cone generated by a set $\{c_i\}$ inside a finite dimensional vector space. Let $\alpha \in C$ span an extremal ray. Then there exists a subsequence $\{c_j\}$ of $\{c_i\}$ and positive real numbers r_j such that $\alpha = \lim_{j\to\infty} r_j c_j$.*

Proof. Let V be the finite dimensional vector space spanned by C. Denote $m := \dim V$, and let $w_1, \ldots, w_m$ be a basis for $V^\vee$ consisting of elements of $C^\vee$. Such a basis exists, because $C^\vee$ is full-dimensional inside $V^\vee$ by the assumptions on $C \subset V$.

The function $|\cdot| : V \to \mathbb{R}_{\geq 0}$ defined by $|v| = \sum_{k=1}^m |\langle v, w_k\rangle|$ is a norm on V, and its restriction to C is given by $|v| = \langle v, w\rangle$, where $w = \sum_{k=1}^m w_k$. Note that $w \in (C^\vee)^{\mathrm{int}}$.

By rescaling, we can assume that α and the c_i are all of length 1 with respect to the norm defined above. Working inside

$$P := C \cap \{v \mid \langle v, w\rangle = 1\},$$

we reduce to the following:

Lemma 5.2. *Let P be the closed convex hull of a bounded set of points $\{c_i\}$ inside a finite dimensional affine space. If $\alpha \in P$ is not a strict convex combination of (at least two) points in P, then α is in the closure of the set $\{c_i\}$.*

Proof. Let V be the affine span of P, and denote $m := \dim V + 1$. There exists a sequence v_k converging to α, with each v_k a convex combination of a finite number of the c_i. We can arrange that each such combination involves at most m of the c_i. Write

$$v_k = r_{k1}c_{k1} + \ldots + r_{km}c_{km}$$

with $1 \geq r_{k1} \geq \ldots \geq r_{km} \geq 0$ and $r_{k1} + \ldots + r_{km} = 1$ for all k. In particular, $r_{k1} \in [\frac{1}{m}, 1]$ for all k. By considering subsequences, we can assume that r_{kj} and c_{kj} are convergent for every $j \in \{1, \ldots, m\}$. Denote the limits by $r_j \geq 0$ and $\bar{c}_j \in P$. Then α is a convex combination of $\bar{c}_j$, which implies that $\alpha = \bar{c}_1$. □

To conclude the proof of Lemma 5.1, note that P is contained in the unit ball for the norm $|\cdot|$ constructed above. □

Proposition 5.3. *Let X be a projective variety and let $\alpha \in \overline{\mathrm{Eff}}_k(X)$ be an extremal ray. Suppose that there is an effective Cartier divisor E such that $\alpha \cdot E$ is not pseudo-effective. Then there is some component E_1 of E and a pseudo-effective class $\beta \in N_k(E_1)$ such that α is the pushforward of β under the inclusion map.*

Proof. We use Lemma 5.1 to write $\alpha = \lim_{j\to\infty} \alpha_j$ where $\alpha_j = r_j[S_j]$ for some positive real number r_j and some irreducible k-dimensional subvariety S_j.

Let $\{E_i\}_{i=1}^r$ denote the components of E. Note that there is some index i such that an infinite subsequence of the S_j is contained in E_i. If this were not the case, then the classes $E \cdot S_j$ would be pseudo-effective for sufficiently large j, hence $E \cdot \alpha$ would also be pseudo-effective. Therefore up to passing to a subsequence and renumbering the i's, we can assume that $S_j \subset E_1$ for all j. Define the pseudo-effective classes

$$\alpha'_j = r_j[S_j] \text{ in } N_k(E_1).$$

Then letting $\imath : E_1 \to X$ denote the closed embedding, we have $\alpha_j = \imath_* \alpha'_j$. Fix an ample class h on X. By the projection formula,

$$(h|_{E_1})^k \cdot \alpha'_j = h^k \cdot \alpha_j.$$

Theorem 2.5.e implies that α'_j is a bounded sequence in $N_k(E_1)$ so that we can extract a convergent subsequence. If α' denotes the limit, then by continuity, $\imath_* \alpha' = \alpha$. □

5.2. **Exceptional classes.**

Definition 5.4. Let $\pi : X \to Y$ be a morphism of projective varieties of relative dimension e and fix $k \geq e$. We say that $\alpha \in \overline{\mathrm{Eff}}_k(X)$ is an exceptional (pseudo-effective) class of π if it satisfies the following equivalent conditions:

- there is some (equivalently any) ample divisor H on Y such that $\alpha \cdot \pi^* H^{k-e} = 0$, or equivalently
- there is some (equivalently any) ample divisor A on X such that $\pi_*(\alpha \cdot A^e) = 0$.

Example 5.5. For effective classes, being π-exceptional is a geometric condition. Suppose that Z is a subvariety of X. Then $[Z]$ is π-exceptional if and only if $\mathrm{reldim}(X/Y) < \mathrm{reldim}(Z/\pi(Z))$. (Indeed, keeping the notation from Definition 5.4, note that the condition $[Z] \cdot \pi^* H^{k-e} = 0$ implies $\pi|_Z^* H^{k-e} = 0$. The injectivity of $\pi|_Z^*$ for dominant maps then says $k - e > \dim \pi(Z)$.)

Example 5.6. Suppose E is an effective divisor. Then $[E]$ is exceptional for a morphism π precisely when $\mathrm{codim}(\pi(E)) \geq 2$, that is, when E is exceptional by the usual definition.

In fact, it follows from Theorem 5.14 below that any pseudo-effective π-exceptional divisor class α is represented by an effective divisor that is exceptional in the traditional sense.

Example 5.7. Let $\pi : X \to Y$ be birational, or only generically finite and dominant. Then a pseudo-effective class is π-contracted if and only if it is π-exceptional.

Example 5.8. A pseudo-effective curve class can only be exceptional for generically finite dominant maps.

Lemma 5.9. *Suppose $\pi : X \to Y$ is an equidimensional morphism of projective varieties. Then there are no π-exceptional classes on X besides the 0 class.*

Proof. Let Z be a complete intersection on X of codimension $e = \mathrm{reldim}(\pi)$ with embedding morphism $i : Z \to X$. Let $f = \pi|_Z$. By Lemma 5.10, if we choose Z general we may ensure that f is finite. If α is π-exceptional, then $f_*(i^*\alpha) = 0$. By Lemma 5.11 below, $i^*\alpha$ is pseudoeffective. By Corollary 3.11 and the projection formula, $\alpha \cdot [Z] = 0$ on X, which implies $\alpha = 0$ since $[Z]$ is in the interior of $\mathrm{Nef}^e(X)$. □

Lemma 5.10. *Let $\pi : X \to Y$ be a map of projective varieties, with equidimensional fibers of relative dimension ≥ 1. Fix a very ample divisor H on X. For some sufficiently large integer m, the general member A of $|mH|$ has that the fibers of $\pi : A \to Y$ are equidimensional.*

Proof. For degree reasons there is an upper bound on the number of components of a fiber of π. We first show that the supports of the irreducible components of a general fiber of π are parameterized by a quasi-projective variety Z dominating Y generically finitely. In particular $\dim Z = \dim Y$. For the claim, let k denote the base field, and note that there exists a finite extension $K(Y) \subset K$ such that every irreducible component of X_K as a scheme over $\mathrm{Spec}\, K$ is geometrically irreducible. Let $\mathcal{T}$ be the support of a component of X_K that dominates the generic fiber of π. Then there exists a projective morphism $\rho : T \to Z$ over k with Z affine, $K(Z) = K$, with $\mathcal{T}$ the generical fiber of ρ, and Z finite dominant over an affine open subset of Y. Furthermore ρ has irreducible and reduced fibers, each a component of a general fiber of π.

Returning to the lemma, it suffices to show that there are elements of $|mH|$ which do not contain a component of a fiber of π over any general closed point of Y. Fix a general fiber F of π and let $F_1, \ldots, F_s$ be its components. Consider the exact sequence:

$$H^0(X, \mathcal{I}_{F_i} \otimes \mathcal{O}_X(mH)) \hookrightarrow H^0(X, \mathcal{O}_X(mH)) \to H^0(F_i, \mathcal{O}_{F_i}(mH)) \to H^1(X, \mathcal{I}_{F_i} \otimes \mathcal{O}_X(mH))$$

For m sufficiently large, the last term vanishes for every i and the dimension of the next to last term is larger than $\dim Z = \dim Y$ for every i. Furthermore, since F is general the same statements will hold for any general fiber of π. Thus, $\dim |mH|$ is strictly greater than the dimension of the space of divisors containing a component of F plus $\dim Y$. Constructing the incidence correspondence, one sees that the general element of $|mH|$ does not contain a component of any general fiber. After removing the proper closed subset of $|mH|$ parametrizing divisors which contain a component of the special fibers, we obtain the conclusion of the theorem. $\square$

Lemma 5.11. *Let X be a projective variety, and D a complete intersection (irreducible) subvariety of codimension d with embedding morphism $\imath : D \hookrightarrow X$. If $\alpha \in \overline{\mathrm{Eff}}_k(X)$, then $\alpha|_D := \imath^*\alpha \in \overline{\mathrm{Eff}}_{k-d}(D)$.*

Proof. By induction we can assume that D is a divisor. By continuity and additivity of intersections, we can assume that $\alpha = [Z]$ for some irreducible subvariety Z of X. If D and Z meet properly, then $[Z]|_D$ is the class of an effective cycle supported on $D \cap |Z|$. Otherwise $Z \subset D$ and then $[Z]|_D$ is the pushforward of $c_1(\mathcal{O}_Z(D)) \cap [Z]$ from Z to D. But $\mathcal{O}_Z(D)$ is ample and the conclusion follows. $\square$

The key lemma controlling the behavior of exceptional classes is the following.

Lemma 5.12. *Let $\pi : X \to Y$ be a dominant morphism of projective varieties. There is a birational model $f_X : X' \to X$ and an effective Cartier divisor E on X' satisfying the following condition: for any k and for any $0 \neq \alpha' \in \overline{\mathrm{Eff}}_k(X')$ such that $f_{X*}\alpha'$ is a π-exceptional class, $E \cdot \alpha'$ is not pseudo-effective. Furthermore the support of E does not dominate Y.*

In the special case when π is generically finite, we may take $X' = X$.

Proof. Let $\pi' : X' \to Y'$ be a flattening of π and let f_X, f_Y denote the corresponding birational maps. Set $e = \mathrm{reldim}(\pi)$ and $\alpha = f_{X*}\alpha'$. As in Definition 5.4 the condition that α is π-exceptional is equivalent to

$$f_{Y*}\pi'_*(\alpha' \cdot f_X^* A^e) = 0$$

for any ample class A on X. Let F be an effective f_Y-anti-ample Cartier divisor on Y'. Then $E = \pi'^*F$ is an effective f_X-anti-ample Cartier divisor on X' (since f_X is the composition of a finite inclusion map with a base change of f_Y).

Suppose that $E \cdot \alpha'$ is pseudo-effective. By Lemma 5.13, there is an ample divisor A' on X' satisfying $\alpha' \cdot f_X^* A^e \succeq \alpha' \cdot A'^e$. Thus $f_{Y*}\pi'_*(\alpha' \cdot (A')^e) = 0$ so that α' is $f_Y \circ \pi'$-exceptional. Furthermore, since

$$F \cdot \pi'_*(\alpha' \cdot (A')^e) = \pi_*(E \cdot \alpha' \cdot (A')^e)$$

is pseudo-effective and f_Y is generically finite, Lemma 5.13 implies $\pi'_*(\alpha' \cdot (A')^e) = 0$, i.e. α' is π'-exceptional. But this is impossible by Lemma 5.9.

To see the final statement, note that by considering the Stein factorization of π one immediately reduces to the birational case; but then the flattening of π is the identity map of X. $\square$

Lemma 5.13. *Let $\pi : X \to Y$ be a generically finite dominant morphism of projective varieties, and let E be an effective π-antiample Cartier divisor on X. Let $\alpha \in \overline{\mathrm{Eff}}_k(X)$ and let H be an ample divisor on Y such that $\pi^*H - E$ is ample. If $\alpha \cdot [E] \in \overline{\mathrm{Eff}}_{k-1}(X)$, then $\alpha \cdot \pi^*H^e \succeq \alpha \cdot (\pi^*H - E)^e$ for any $1 \leq e \leq k$.*

Proof. Observe that

$$\pi^*H^e - (\pi^*H - E)^e = E \cdot \left(\sum_{i=1}^{e} \pi^*H^{i-1}(\pi^*H - E)^{e-i} \right),$$

and $\sum_{i=1}^{e} \pi^*H^{i-1}(\pi^*H - E)^{e-i}$ is a positive combination of complete intersections of nef divisors. $\square$

Theorem 5.14. *Let $\pi : X \to Y$ be a morphism of projective varieties of relative dimension l and let α be a π-exceptional class.*

(1) α admits a unique Zariski decomposition $\alpha = N(\alpha)$ and $P(\alpha) = 0$.
(2) α is the pushforward of a pseudo-effective class on a proper subvariety of X. (In other words, Conjecture 4.12 holds for α.)

In fact, there is a proper closed subset $W \subset X$ such that every *π-exceptional class is the pushforward of a pseudo-effective class on W.*

Proof. (1) Write $\alpha = P(\alpha) + N(\alpha)$ for a Zariski decomposition of α. Since α is π-exceptional, so are $P(\alpha)$ and $N(\alpha)$. Choose a flattening $\pi' : X' \to Y'$ of π and let $\alpha' \in \overline{\mathrm{Mov}}_k(X')$ be a preimage of $P(\alpha)$. Lemma 5.12 implies that there is an effective Cartier divisor E on X' such that $\alpha' \cdot E$ is not pseudo-effective if $\alpha' \neq 0$. This is impossible by Theorem 2.7.i, showing that $P(\alpha) = 0$ as well.

(2) It suffices to consider the case when α lies on an extremal ray. Choose a flattening $\pi' : X' \to Y'$ of π. Then there is some α' lying on an extremal ray of $\overline{\mathrm{Eff}}_k(X')$ that is a preimage of α and an effective Cartier divisor E satisfying $E \cdot \alpha'$ is not pseudo-effective. Proposition 5.3 shows that α' is the pushforward of a pseudo-effective class from a subvariety E_1 of X'. Then α is the pushforward of a pseudo-effective class from the image of E_1 in X.

To see the final statement, note that by Lemma 5.12 we can choose E_1 independently of α in (2); set $W = f_X(|E_1|)$. $\square$

5.3. **Contractibility index.** It turns out to be useful to quantify "how close" a π-contracted class is to being π-exceptional.

Definition 5.15. Let $\pi : X \to Y$ be a dominant morphism of projective varieties. Suppose H is an ample divisor on Y. For a class $\alpha \in \overline{\mathrm{Eff}}_k(X)$, the contractibility index of α is the largest non-negative integer $c \leq k+1$ such that $\alpha \cdot \pi^*H^{k+1-c} = 0$. The definition is independent of the choice of H. We denote the π-contractibility index of α by $\mathrm{contr}_\pi(\alpha)$.

If $V \subset X$ is a subvariety, we define the contractibility index of V to be the contractibility index of $[V]$.

Note that when $c = 0$ we have $\alpha \cdot \pi^*H^{k+1} = 0$ for dimension reasons, so that the π-contractibility index is well-defined. The following properties are immediate:

- A pseudo-effective class $\mathrm{contr}_\pi(\alpha) > 0$ precisely when $\pi_*\alpha = 0$.
- The contractibility index is (by definition) at most $k+1$, and $0 \in \overline{\mathrm{Eff}}_k(X)$ is the only pseudo-effective class achieving this maximal value.
- The contractibility index of $\alpha \in \overline{\mathrm{Eff}}_k(X)$ is at least $k - \dim Y$.
- The contractibility index of $[X] \in N_{\dim X}(X)$ is the relative dimension of π. More generally, if α is the class of an irreducible cycle Z, then $\mathrm{contr}_\pi(\alpha) = \mathrm{reldim}(\pi|_Z)$.
- A pseudo-effective class α is π-exceptional precisely when its contractibility index is greater than $\mathrm{reldim}(\pi)$.

The following theorem was inspired by a question of Dawei Chen.

Theorem 5.16. *Let $\pi : X \to Y$ be a dominant morphism of projective varieties. Fix a positive integer m. Let $k = k(m)$ be the largest integer such that there is a subvariety of dimension k of contractibility index $\geq m$. Then*

(1) There are only finitely many subvarieties $V_1, \ldots, V_s$ of dimension k and contractibility index $\geq m$.

(2) Any $\alpha \in \overline{\mathrm{Eff}}_k(X)$ of contractibility index $\geq m$ is a non-negative linear combination of the $[V_i]$.

Proof. Set $n = \dim X$. The proof is by induction on the codimension $n-k$. The base case $n-k=0$ is obvious.

Now suppose $n-k>0$. In particular m is greater than the contractibility index of X, so that any class with contractibility index $\geq m$ is π-exceptional. Theorem 5.14 guarantees that there is a proper subscheme $i : W \hookrightarrow X$ such that every π-exceptional class is pushed forward from W. Let $\{W_i\}$ be the irreducible components of W. Note that for any pseudo-effective class β on a component W_i, the contractibility index for $\pi|_{W_i}$ is the same as the contractibility index of the pushforward of β to X.

In particular, for each W_i any subvariety with contractibility index $\geq m$ has dimension no more than k. Applying the induction assumption to each W_i in turn, we immediately obtain (1) for X. Suppose now that $\alpha \in \overline{\mathrm{Eff}}_k(X)$ has contractibility index $\geq m$. Since α is π-exceptional, it is pushed forward from W, and hence is a sum of pushforwards of pseudo-effective classes of contractibility index $\geq m$ from the W_i. Applying (2) to each W_i, we obtain (2) for X by pushing forward. □

If $\pi : X \to Y$ is birational and $Z \subset X$ is the exceptional locus of π, then there are no nonzero effective π-exceptional (equivalently π-contracted by Example 5.7) classes of dimension bigger than $\dim Z$. As a consequence of the theorem above, the same is true for pseudo-effective classes. In particular the SC holds in this case.

Corollary 5.17. *Let $\pi : X \to Y$ be a birational morphism and let $Z = \mathrm{Exc}(\pi) \subset X$. If $d = \dim Z$, then $\overline{\mathrm{Eff}}_k(\pi) = 0$ for all $k > d$ and $\overline{\mathrm{Eff}}_d(\pi)$ is polyhedral, generated by the components of Z that are contracted by π.*

Example 5.18. [CC14] and [Sch15] identify cycle classes which lie on extremal rays of the *effective* cone of various moduli spaces of curves. We will focus on the first paper. The key tools for establishing extremality are [CC14, Proposition 2.1, Proposition 2.2, Proposition 2.5]. We explain how these are extended to the pseudo-effective setting.

Let $\pi : X \to Y$ be a morphism of projective varieties.

[CC14, Proposition 2.1] shows that if $\alpha \preceq \beta$ are effective classes on X, then $\mathrm{contr}_\pi(\alpha) \geq \mathrm{contr}_\pi(\beta)$. The analogous fact for pseudo-effective classes is immediate.

[CC14, Proposition 2.2] shows that if only finitely many k-dimensional subvarieties of X have contractibility index $\geq m$ then the classes of these subvarieties generate an extremal face of the effective cone. The analogous statement for pseudo-effectivity can be proven using Theorem 5.16, as was observed in [CC14, Remark 2.7]. (To see this, it suffices to show that if there is a subvariety $V \subset X$ of dimension $d > k$ and contractibility index $\geq m$ then there are infinitely many k-dimensional subvarieties of contractibility index $\geq m$. If $\dim(\pi(V)) \geq d-k$, then the preimage of any sufficiently general codimension $d-k$ subvariety of $\pi(v)$ will have contractibility index $\geq m$. If $\dim(\pi(V)) < d-k$, then every k-dimensional subvariety will have contractibility index $\geq m$.)

[CC14, Proposition 2.5] considers the case when π is generically finite, $Z \subset X$ is a subvariety which contains the π-exceptional locus, and the pushforward $N_k(Z) \to N_k(X)$ is injective. Then (under some additional hypotheses) any effective class $\alpha \in N_k(X)$ which is π-exceptional, and the pushforward of a class lying on an extremal ray of $\mathrm{Eff}_k(Z)$ is also extremal in $\overline{\mathrm{Eff}}_k(X)$. The analogous statement for pseudo-effectivity is also true, even without the additional hypotheses, by essentially the same argument. (Suppose that β_i are pseudo-effective classes on X satisfying $\sum \beta_i = \alpha$. Then each β_i is also π-exceptional; arguing as in Lemma 5.12, we see that any such β_i is the pushforward of a pseudo-effective class on $N_k(Z)$. Using the injectivity of the pushforward, we deduce that $\sum \beta_i = \alpha$ as classes on $N_k(Z)$ – by extremality, each β_i must be proportional to α in $N_k(Z)$, hence also in $N_k(X)$.)

Using these strengthened versions, one can extend many of the results of [CC14] to the pseudo-effective cone. We explain the argument in a couple cases to illustrate the method; we refer to [CC14] for the relevant notation and preliminaries.

- [CC14, Theorem 6.1] shows that the boundary codimension 2 strata D_{S_1,S_2,S_3} on $\overline{M}_{0,n}$ have classes which are extremal in the effective cone. They are also extremal in the pseudo-effective cone. In the case when $s_1, s_3 > 2$, [CC14] constructs birational morphisms f_A, f_B with exceptional loci the irreducible divisors Δ_{S_1,S_1^c}, Δ_{S_3,S_3^c} respectively, and the intersection of these two divisors is exactly D_{S_1,S_2,S_3}. By applying the pseudo-effective analogue of [CC14, Proposition 2.5] to $\overline{M}_{0,n}$ and Δ_{S_1,S_1^c}, we see that it suffices to show that D_{S_1,S_2,S_3} is extremal in Δ_{S_1,S_1^c}. We show this by applying Theorem 5.16 to the restriction of f_B to Δ_{S_1,S_1^c}. The other cases are proved similarly.
- [CC14, Theorem 7.2] shows that the effective cone of codimension 2 cycles on $\overline{M}_{1,n}$ is not finite polyhedral. The analogous statement for $\overline{\mathrm{Eff}}_{n-1}(\overline{M}_{1,n})$ is also true. Indeed, one just needs to replace [CC14, Proposition 2.5] by the corresponding pseudo-effective extension above. The same argument in the context of [CC14, Theorem 8.2] shows that the pseudo-effective cone of codimension 2 cycles on $\overline{M}_{2,n}$ is not finite polyhedral.
- The Keel-Vermeire lifts in [Sch15, Section 4] are extremal in $\overline{\mathrm{Eff}}_2(\overline{M}_{0,7})$. Their extremality in the effective cone is proved by [Sch15, §3 Lifting Lemma]. This is essentially equivalent to [CC14, Proposition 2.5], and the strengthened version above applies.

5.4. **Further reduction steps.** Theorem 5.14 allows us to make some further reductions to the pushforward conjectures.

Proposition 5.19 (Reducing dimension in the relative dimension zero case)**.** *Let $\pi : X \to Y$ be a generically finite dominant morphism of projective varieties of dimension n. Let E_i be* *the components of an effective π-antiample Cartier divisor E of class e on X. If the PC are true for $\pi|_{E_i}$, then they are also true for π.*

Proof. Let $\alpha \in \overline{\mathrm{Eff}}_k(X)$ with $\pi_*\alpha = 0$. By Example 5.7, α is π-exceptional. We may assume that α is also extremal. The last statement in Lemma 5.12 shows that α is the pushforward of a pseudo-effective class from a component of E, whence the result. ☐

In particular, to deduce the SC/WC by an inductive argument, we may always precompose by generically finite maps.

Proposition 5.20 (Reduction to the flat case)**.** *The general case of the PC is implied by the flat case.*

Proof. Let $\pi : X \to Y$ be a morphism of projective varieties, and let $\alpha \in \overline{\mathrm{Eff}}_k(X) \cap \ker \pi_*$ which we can assume to be extremal. Let $\pi' : X' \to Y'$ be a flattening of π with birational morphisms $f_X : X' \to X$ and $f_Y : Y' \to Y$. Let α' be an extremal pseudo-effective preimage of α on X'. It is enough to show that the PC hold for the pair $(\pi \circ f_X, \alpha')$. Let F be an effective f_Y-anti-ample Cartier divisor on Y', and let H be an ample class on Y such that $f_Y^*H - F$ is ample on Y'. Put $E = \pi'^*F$.

If $\alpha' \cdot E \in \overline{\mathrm{Eff}}_{k-1}(X')$, then by the projection formula, $\pi'_*\alpha' \cdot F \in \overline{\mathrm{Eff}}_{k-1}(Y')$. Proposition 3.3 and Lemma 5.13 then show that $\pi'_*\alpha' = 0$. If the PC hold for the pair (π', α'), then they also hold for (π, α).

If $\alpha' \cdot E$ is not pseudo-effective, then Proposition 5.3 shows that α' is pushed forward from one of the irreducible components of the support of E. Conclude by induction on $\dim X$. ☐

Finally we show that the Strong or Weak Conjecture follows from the birational case. The argument which works by increasing dimension is based on a relative version of a classical cone construction.

Lemma 5.21. *Let $\pi : X \to Y$ be a projective morphism. Let H be a sufficiently π-ample divisor on X, and let*

$$\mathcal{R}(H) := \mathcal{O}_Y \oplus \bigoplus_{m\geq 1} \pi_*\mathcal{O}_X(mH). \tag{1}$$

Put $T = \mathrm{Spec}_{\mathcal{O}_Y}\mathcal{R}(H)$ with induced affine morphism $\rho : T \to Y$. The natural map

$$\mathcal{R}(H) \twoheadrightarrow \mathcal{O}_Y$$

induces a section $i : Y \hookrightarrow T$ of ρ. Then $Z := \mathrm{Bl}_{i(Y)}T$ admits a morphism $f : Z \to X$ that is isomorphic to the bundle map for the geometric line bundle $Z' := \mathrm{Spec}_{\mathcal{O}_X} \bigoplus_{m\geq 0} \mathcal{O}_X(mH)$. Moreover if E denotes the exceptional divisor of the blow-up, then $f|_E : E \to X$ is an isomorphism whose inverse is the zero section of f, and the induced map $E \to Y$ is naturally isomorphic to π.

Proof. If H is sufficiently π-ample, then we have a surjective morphism

$$\pi^*\mathcal{R}(H) \twoheadrightarrow \bigoplus_{m\geq 0} \mathcal{O}_X(mH).$$

This shows that Z' is a subvariety of $T \times_Y X$. Similarly there is a surjective morphism

$$\rho^*\mathcal{R}(H) \twoheadrightarrow \mathcal{O}_T \oplus \bigoplus_{m\geq 1} \mathcal{I}^m,$$

where $\mathcal{I}$ is the ideal sheaf of $i(Y)$ in T (because $\mathcal{I}$ is generated by $\rho^*\pi_*\mathcal{O}_X(H)$ as an ideal in $\mathcal{O}_T$). Then Z is also a subvariety of $T \times_Y X$. Both Z and Z' are irreducible. To check that $Z = Z'$, it is then enough to verify this over general points of Y. But over a general point on Y, we simply have the classical cone construction. The assertions on E are verified similarly. □

Remark 5.22. If one replaces $\mathcal{O}_Y$ with $\pi_*\mathcal{O}_X$ in degree 0 in the formula for $\mathcal{R}(H)$ in (1), then $T = \mathrm{Spec}_{\mathcal{O}_Y}\mathcal{R}(H)$ is the cone over the Stein factorization of π.

Proposition 5.23. *If SC (or WC) holds for birational maps, then it holds in general. More precisely, if the conjecture is valid for birational maps in dimension $n+1$, then it is valid for all morphisms $\pi : X \to Y$ of projective varieties with $\dim X = n$.*

Proof. Let $\pi : X \to Y$ be a morphism of projective varieties, with $\dim X = n$. Consider $\eta : W \to X$ a projective bundle of rank 1 over X that compactifies the geometric line bundle from Lemma 5.21. Let $E \subset W$ be the zero section of η. By Lemma 5.21, the divisor $E \subset W$ can be blown-down as $\sigma : W \to S$, with $\sigma(E) = Y$, and $\sigma|_E^Y$ is naturally isomorphic to π. The proposition is a consequence of the following identifications:

$$N_k(X) = E \cdot \eta^* N_k(X) \subset N_k(W),$$
$$N_k(\pi) = N_k(\sigma),$$

which are compatible with the respective pseudo-effective cones. □

6. Movable classes

We next study the Strong and Weak Conjectures for movable classes. The main result of the section is Theorem 6.10 which proves the Strong Conjecture for movable classes that are "almost exceptional", in the sense that their contractibility index (5.3) is one away from the condition for being exceptional (the exceptional case is covered by the proof of Theorem 5.14 which shows that exceptional nonzero classes are never movable). Throughout this section we will often work over the complex numbers; this allows us to understand the behavior of birational maps via the following proposition.

Proposition 6.1 ([FL15] Proposition 3.8)**.** *Suppose that $\pi : X \to Y$ is a birational morphism of varieties over $\mathbb{C}$ with Y smooth. Then the kernel of $\pi_* : N_k(X) \to N_k(Y)$ is spanned by classes of effective k-cycles contracted by π.*

6.1. **Movability and restrictions.**
We first identify criteria guaranteeing that the restriction of a movable class to a subvariety is still movable. Throught this section, when π is a birational map we will use interchangably the terms "π-exceptional" and "π-contracted". (Note that these two terms mean the same thing for birational maps by Example 5.7.)

Lemma 6.2. *Let $\pi : Y \to X$ be a morphism of smooth varieties that is a composition of blow-ups along smooth centers. Then $N_k(Y)$ is spanned by $\pi^* N_k(X)$ and by a finite set of π-exceptional effective k-cycles, each of which is the pushforward of a basepoint free class on a π-exceptional divisor.*

Proof. Using Theorem 2.10 inductively, it suffices to consider the case when π is the blow-up along a single smooth center T. Since T is smooth, each $N_j(T)$ is spanned by basepoint free classes. Using [Ful84, Theorem 3.3.(b)] and Proposition 6.1, we see the kernel of the pushforward map is spanned by the intersection of divisors of a fixed ample divisor class with the pullbacks of these classes. These intersections represent effective, basepoint free, π-exceptional cycles by Theorem 2.10. □

Proposition 6.3. *Let X be a smooth projective variety over $\mathbb{C}$ and let α be the class of a strongly movable family of k-cycles $t : R \to S$. Suppose U is a scheme admitting a flat dominant map $s : U \to X$ and a proper map $q : U \to W$ to an integral variety W. Then for every component F' of a general fiber F of q we have that $\alpha|_{s(F')}$ is movable.*

In particular, for any (reduced) component Z of a general member of a basepoint free family on X, $\alpha|_Z$ is movable on Z.

Proof. Let $\pi : Y \to X$ be a birational map from a smooth model Y that flattens the map $t : R \to X$. After possibly passing to a higher model, we may assume that π is a compositiion of smooth blow-ups. Let α' be the class of the strict transform family of t on Y; note that α' is a basepoint free class. By Lemma 6.2, we can write $\alpha' = \alpha + [V]$ where V is a (not necessarily effective) linear combination of π-exceptional cycles that are general members of basepoint free families on exceptional divisors.

Set $U' := U \times_X Y$. Since the natural morphism $s_Y : U' \to Y$ is flat, every component of U' dominates Y. Thus, the s_Y-image of the general fiber of $p_Y : U' \to W$ is the strict transform of the s-image of a general fiber of p.

Suppose that Z is a d-dimensional component of a general fiber of p and Z' is its strict transform on Y. We next verify that:

(1) There is a cycle of class $[V \cdot Z']$ supported on $V \cap Z'$.
(2) $V \cap Z'$ is a π-exceptional cycle.

The two properties together show that $\pi_*[V \cdot Z'] = 0$.

Arguing by induction on the number of blow-ups, it suffices to consider the case when π is a blow-up of a smooth center W, E is the exceptional divisor, and V is a π-exceptional cycle that is basepoint free in E. To verify (1), note that since E is a Cartier divisor whose support does not contain Z', $[Z']|_E$ is represented by a cycle T supported on $Z' \cap E$. We then apply Theorem 2.10 to T and to components of V as basepoint free cycles on E. To verify (2), note that since s and s_Y are flat, codimension is preserved upon taking preimages. In particular, the codimension of $W \cap Z$ in Z is the same as the codimension of W in X, and $E \cap Z'$ has codimension 1 in Z'. Since the fibers of the blow-up of W in X are irreducible, the only way this can happen is if Z' contains every fiber of π that it intersects. Then since V is contracted by π, $V \cap Z'$ is also contracted by π.

Thus

$$\begin{aligned}\alpha|_Z &= (\pi|_{Z'})_*(\pi^*\alpha|_{Z'}) \\ &= (\pi|_{Z'})_*(\alpha'|_{Z'} + [V]|_{Z'}) \\ &= (\pi|_{Z'})_*(\alpha'|_{Z'}).\end{aligned}$$

Since α' is basepoint free, the restriction to Z' is also basepoint free, and hence its pushforward is movable. □

Immediately from Proposition 6.3 we obtain:

Corollary 6.4. *Let X be a smooth projective variety over $\mathbb{C}$ and let $\alpha \in \overline{\mathrm{Mov}}_k(X)$. Suppose U is a scheme admitting a flat dominant map $s : U \to X$ and a proper map $q : U \to W$ to an integral variety W. Then for every component F' of a very general fiber F of q we have that $\alpha|_{s(F')}$ is movable.*

In particular, for any (reduced) component Z of a very general member of a basepoint free family on X, $\alpha|_Z$ is movable on Z.

6.2. **Almost exceptional classes: special case.** We show in Corollary 6.7 that the class of a general fiber of a dominant morphism of projective varieties $\pi : X \to Y$ of relative dimension k over $\mathbb{C}$ is the only class in $\overline{\mathrm{Mov}}_k(X)$ satisfying $\alpha \cdot \pi^* h = 0$ for some (equivalently any) ample divisor class h on Y. This will play an important role in the proof of Theorem 6.10 in the next subsection.

Lemma 6.5. *Let X be a smooth projective variety of dimension n over $\mathbb{C}$, and let $\alpha \in \overline{\mathrm{Mov}}_k(X)$. Then there exists Z smooth projective of dimension $n-k+1$ with a morphism $f : Z \to X$ such that $f_* : N_{n-k}(Z) \to N_{n-k}(X)$ is surjective, and $f^*\alpha \in \overline{\mathrm{Mov}}_1(X)$.*

Proof. Choose a finite set of r very ample vector bundles E_i on X such that $N^*(X)$ is generated *as a ring* by the Segre classes $s_j(E_i^\vee)$, and in particular $N^k(X) = N_{n-k}(X)$ is generated as a vector space by monomials of weight k in these Segre classes. In this list repeat each bundle k times (this will make it possible to write any monomial in dual Segre classes of bundles in this set as a monomial in dual Segree classes of different (occurrences) of these bundles in the set). Put $P = \prod_X \mathbb{P}(E_i)$, and let $\pi : P \to X$ denote the projection map of relative dimension denoted by p. Let ξ_i be the pullbacks to P of $c_i(\mathcal{O}_{\mathbb{P}(E_i)}(1))$. Using the proof of [Ful84, Proposition 3.1.(b)], we see that $N_{n-k}(X)$ is generated by $\pi_*(\prod_{j=1}^{p+k} \xi_{i_j})$, where i_j are arbitrary indices in the set $\{1, \ldots, r\}$ (they may repeat, and the number of occurrences the index i determines which Segre class of $E_i^\vee$ appears in the resulting monomial in dual Segre classes of the E_i's).

By Bertini, we can choose smooth representatives $Q_{\underline{i}}$, one for each $\prod_{j=1}^{p+k} \xi_{i_j}$. We can also ensure that they are disjoint as long as $2(p+k) > n+p$, which can be achieved for example by adding $\mathcal{O}(A)^{\oplus n+1}$ to the list, for some very ample A on X. Put $T_{\underline{i}} := \pi_* Q_{\underline{i}}$. These are cycles on X whose classes generate $N_{n-k}(X)$ by the previous paragraph.

Let $\widetilde{P}$ be the blow-up of P along all Q_i. Let Z be a general complete intersection of dimension $n-k+1$ on $\widetilde{P}$. This is smooth, and for each i it contains an $(n-k)$-dimension cycle $\widetilde{Q}_{\underline{i}}$ that dominates $Q_{\underline{i}}$ (because the exceptional divisors are projective bundles over Q_i of relative dimension $p+k-1$, which is the codimension of Z). If $f : Z \to X$ denotes the induced map, then $f_* : N_{n-k}(Z) \to N_{n-k}(X)$ is surjective. It remains to show that $f^*\alpha$ is movable.

Let $\sigma : \widetilde{P} \to P$ be the blow-up map. Since the intersection of an ample class with a movable class is movable (see [FL13, Lemma 3.12]), and a smooth pullback of a movable class from a smooth variety is movable (see [FL13, Lemma 3.6.(2)]), it is enough to check that $\sigma^*\pi^*\alpha$ is movable. Let q_j be a sequence of strictly movable families such that $\alpha = \lim_{j\to\infty}[q_j]$. For very general choices of $Q_{\underline{i}}$, the general member of $\pi^* q_j$ meets $Q_{\underline{i}}$ properly for all $\underline{i}$ and for all j. Then $\sigma^*\pi^*[q_j]$ represents the strict transform in $\widetilde{P}$ of the strictly movable family $\pi^* q_j$ by [Ful84, Corollary 6.7.2]. Consequently $\sigma^*\pi^*\alpha$ is still movable. $\square$

Remark 6.6. By the first two paragraphs of the argument, any smooth projective variety X over $\mathbb{C}$ admits a basis of $N^k(X)$ consisting of classes of basepoint free families whose total space U and general fiber are irreducible. (Indeed, we can take the $Q_{\underline{i}}$ to be irreducible by the Bertini theorems, so that the $T_{\underline{i}}$ are also irreducible. Complete intersections $Q_{\underline{i}}$ of globally generated line bundles are cycles in basepoint free families on P, and then so are their flat pushforwards $T_{\underline{i}}$ on X.)

Moreover, if $f : X \to Y$ is a dominant morphism to another projective variety Y, with $\dim Y \geq n-k$, we can also arrange that $\dim f(T_{\underline{i}}) = n-k$ for every i. (Indeed we claim that we can choose $T_{\underline{i}}$ such that $[T_{\underline{i}}]$ belongs to the interior of $\mathrm{BPF}^k(X)$, hence (cf. Theorem 2.11) in particular to the interior of $\mathrm{Nef}^k(X)$. Assuming this, if $\dim f(T_{\underline{i}}) < n-k$, then $[T_{\underline{i}}] \cdot f^*h^{n-k} = 0$ for any ample $h \in N^1(Y)$, which implies $f^*h^{n-k} = 0$, leading to the contradiction $\dim Y < n-k$. For the claim, replace first each E_i by $E_i \otimes \det E_i$, and add the ample line bundles $\det E_i$ to the initial list. These have the same linear span of dual Segre monomials. Since complete intersections belong to the interior of $\mathrm{BPF}^r(X)$ for all r, and dual Segre monomials of globally generated bundles are basepoint free (cf. Theorem 2.11), by Theorem 2.10 it is enough to check that $s_j((E_i \otimes \det E_i)^\vee)$ belongs to the interior of $\mathrm{BPF}^j(X)$. By [Ful84, Example 3.1.1], the class $s_j((E_i \otimes \det E_i)^\vee)$ is a positive combination of classes in $\mathrm{BPF}^j(X)$, one of which is a positive multiple of the interior class $c_1^j(E_i)$.) $\square$

Corollary 6.7. *Let $\pi : X \to Y$ be a surjective morphism of projective varieties over $\mathbb{C}$ with relative dimension k. Let $\alpha \in \overline{\mathrm{Mov}}_k(X)$ be such that $\alpha \cdot \pi^*H = 0$ for an ample divisor H on Y. Then α is proportional to the class of a fiber.*

Proof. First suppose X is smooth. Choose Z as in Lemma 6.5 with morphism $f : Z \to X$. Note that $\pi \circ f : Z \to Y$ has relative dimension 1. Then $f^*\alpha$ is a movable curve class that pushes forward to 0 on Y. Thus it is proportional to the class of a fiber by Corollary 4.7. So for any divisor D on Z we have

$$D \cdot \alpha = c \deg(\pi \circ f|_D).$$

But since $f_* : N_{n-k}(Z) \to N_{n-k}(X)$ is surjective, the same proportionality relationship holds for $(n-k)$-cycles on X. So α is proportional to the class of a general fiber of the map π.

When X is singular, let $\phi : X' \to X$ be a smooth birational model and let α' be a movable preimage of α. By Theorem 2.7.i, $\alpha' \cdot \phi^*\pi^*H = 0$. Applying the smooth case to α', we see that α' is proportional to the class of a general fiber of $\pi \circ \phi$. Pushing forward, we see that α is also proportional to the class of a general fiber of π. $\square$

6.3. **Almost exceptional classes: general case.** In Theorem 6.10 and its corollary we prove the "almost exceptional" case of the Movable Strong Conjecture over $\mathbb{C}$, and discuss how this settles most of the cases of the Strong Conjecture for morphisms from complex fourfolds.

Lemma 6.8. *Let $g : X \to Y$ be a finite dominant map of projective varieties with Y smooth. Let $\alpha \in N^{n-k}(X)$. Suppose that there is a finite collection of $(n-k)$-dimensional subvarieties $\{W_i\}$ of Y containing general points of Y, such that if Z_1 and Z_2 are $(n-k)$-dimensional integral subvarieties of X both mapping to the same W_i, then*

$$\frac{\alpha \cdot Z_1}{\deg(Z_1/W_i)} = \frac{\alpha \cdot Z_2}{\deg(Z_2/W_i)}.$$

Then there is some $\beta \in N^{n-k}(Y)$ so that for any Z above one of the W_i we have $\alpha \cdot Z = g^*\beta \cdot Z$. If $\alpha \cap [X]$ is movable, we may ensure that $\beta \cap [Y]$ is also movable.

Proof. Let $d = \deg(X/Y)$. Set $\beta = \frac{1}{d} \cdot g_*(\alpha \cap [X])$; since Y is smooth we can think of $\beta \in N^{n-k}(Y)$. If E is an irreducible subvariety with $g(E) = W_i$, we have

$$\begin{aligned} g^*\beta \cdot E = \beta \cdot g_* E &= \frac{1}{d} \cdot g_*(\alpha \cap [X]) \cdot \deg(E/W_i) W_i \\ &= \frac{\deg(E/W_i)}{d} \alpha \cdot (g^*[W_i] \cap [X]). \end{aligned}$$

We check that $g^*[W_i] \cap [X] = [g^{-1}W_i]$. It is enough to check the equality in Chow groups. Note that $g^{-1}W_i$ has the expected dimension because Y is smooth and g is finite. Using the restriction sequence for Chow groups, the finiteness of g, and the generality assumption on W_i, it is enough to check the equality of Chow classes over the flat locus of g. Over the flat locus the equality is the definition of the flat pullback of W_i via the compatibility between flat and smooth pullback (see [Ful84, Proposition 8.1.2.(a)]).

Using this equality, we find

$$\begin{aligned} g^*\beta \cdot E &= \frac{\deg(E/W_i)}{d} \alpha \cdot \sum_{g(E_j) = W_i} \operatorname{ramdeg}(E_j/W_i) E_j \\ &= \frac{\deg(E/W_i)}{d} \cdot \sum_{g(E_j) = W_i} \operatorname{ramdeg}(E_j/W_i) \cdot \frac{\deg(E_j/W_i)}{\deg(E/W_i)} \cdot \alpha \cdot E. \end{aligned}$$

Since the W_i contain general points, the smooth locus of any component of the preimage intersects the smooth locus of X, so that by [Ful84, Example 4.3.7]

$$d = \sum_{g(E_j) = W_i} \operatorname{ramdeg}(E_j/W_i) \cdot \deg(E_j/W_i).$$

This implies that $g^*\beta \cdot E = \alpha \cdot E$. The final statement follows since β is proportional to the pushforward of α and movability is preserved by pushforward. □

Lemma 6.9. *Let $\pi : X \to Y$ be a generically finite dominant map of smooth projective varieties over $\mathbb{C}$. Let $\alpha \in N_k(X)$. Let $T_1, \ldots, T_r$ be $(n-k)$-cycles on Y which are components of general members of bpf families. Suppose that for each T_i there is a constant s_i so that*

$$\alpha \cdot Z = s_i \deg(Z/T_i)$$

for any subvariety Z lying above T_i. Then there is a class $\beta \in N^{n-k}(Y)$ such that $\alpha \cdot Z = \pi^*\beta \cdot Z$ for any Z lying above some T_i.

If α is movable, then so is $\beta \cap [Y]$.

Proof. Let $\pi' : X' \to Y'$ be a flattening via a birational map $f_Y : Y' \to Y$ with Y' smooth. Let $f_X : X' \to X$ denote the corresponding birational map. For each T_i denote its strict transform on Y' by T_i'. Since T_i is a component of a general member of a bpf family, $[T_i'] = f_Y^*[T_i]$ by Theorem 2.10. Note that any $(n-k)$-dimensional subvariety Z on X dominating some T_i is again a component of a basepoint free family, since it is a component of the base-change of p (see [FL14, Lemma 5.6]). Thus, the pullback $f_X^*[Z]$ coincides with the class of its strict transform, so that

$$\pi'_* f_X^*[Z] = \deg(Z/T_i)[T_i'] = f_Y^* \pi_*[Z].$$

Consider the pullback $f_X^*\alpha$. It still satisfies the intersection compatibility with degree for cycles lying over the T_i'. So Lemma 6.8 shows that $\beta' := \frac{1}{\deg \pi} \pi'_*(f_X^*\alpha)$ has the property that

$$\pi'^*\beta' \cdot [Z'] = f_X^*\alpha \cdot [Z']$$

for any cycle Z' lying above one of the T'_i. Define $\beta = f_{Y*}\beta'$. Then for any Z lying above one of the T_i

$$\begin{aligned}\pi^*\beta \cdot [Z] &= f_{Y*}\beta' \cdot \pi_*[Z] \\ &= \beta' \cdot \deg(Z/T_i)[T'_i] \\ &= \beta' \cdot \pi'_* f_X^*[Z] \\ &= \alpha \cdot [Z].\end{aligned}$$

As for the final statement, we see that $\beta \cap [Y]$ is movable since it is constructed to be proportional to the pushforward of α. $\square$

Finally we prove the main theorem of this section.

Theorem 6.10. *Let $\pi : X \to Y$ be a surjective morphism with connected fibers of smooth projective varieties over $\mathbb{C}$ of relative dimension e. Suppose $\alpha \in \overline{\mathrm{Mov}}_k(X)$ for some $k \geq e$ and that $\alpha \cdot \pi^* H^{k-e+1} = 0$ for some ample divisor H on Y. Then there is a diagram*

$$\begin{tikzcd} X' \arrow[r, "f_X"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "f_Y"] & Y \end{tikzcd}$$

with f_X and f_Y birational, Y' smooth, and π' flat and a class $\beta \in \overline{\mathrm{Mov}}_{k-e}(Y')$ such that $f_{X*}\pi'^*\beta = \alpha$.

Note that the classes satisfying the condition $\alpha \cdot \pi^* H^{k-e+1} = 0$ are "almost exceptional": the contractibility index (cf. §5.3) is (at most) one away from the condition for being exceptional.

Proof. Let n be the dimension of X and d the dimension of Y so that $e = n - d$. Consider a flattening $\pi' : X' \to Y'$ of π with Y' smooth:

$$\begin{tikzcd} X' \arrow[r, "f_X"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "f_Y"] & Y \end{tikzcd}$$

Let $\psi : \widetilde{X} \to X'$ be a resolution and let $\rho : \widetilde{X} \to Y'$ denote the composition of ψ and π'. Let $\tilde{\alpha}$ be a movable preimage of α on $\widetilde{X}$. From Theorem 2.7.i we see that

$$\tilde{\alpha} \cdot (\rho^* f_Y^* H)^{k-e+1} = 0.$$

Writing $f_Y^* H = A + E$ for an effective Cartier divisor E and an ample divisor A, and using the movability of $\tilde{\alpha}$ and Lemma 5.13, we also obtain

$$\tilde{\alpha} \cdot \rho^* A^{k-e+1} = 0.$$

Thus $\tilde{\alpha}$ is "almost exceptional" for the map ρ. By pushing forward, we observe $(\psi_*\tilde{\alpha}) \cdot \pi'^* A^{k-e+1} = 0$, so that $\psi_*\tilde{\alpha}$ is "almost exceptional" for the map π'.

Let $\{p_i : U_i \to W_i\}$ be a finite collection of basepoint free families whose classes span $N^k(X)$. We can choose them such that the U_i's and the general fiber of each p_i are irreducible and their images on X are not contracted by π (see Remark 6.6). We will do a series of constructions to p_i; at each step, we will replace W_i by an open subset which for simplicity we also denote by W_i. The strict transform families $p'_i : U'_i \to W_i$ on X' are still basepoint free by Theorem 2.10 and the images on X' of general fibers are not contracted by π'. Consider the diagram

$$\begin{tikzcd} U_i' \times_{Y'} X' \arrow[r, "t_i"] \arrow[d, "\pi_i'"] \arrow[dd, bend right, "q_i"'] & X' \arrow[d, "\pi'"] \\ U_i' \arrow[r, "\pi' \circ s_i'"] \arrow[d, "p_i'"] & Y' \\ W_i & \end{tikzcd}$$

Note that the map t_i is flat and the map q_i is proper, making $U_i' \times_{Y'} X'$ a basepoint free family on X'. We then take the pushout diagram to $\widetilde{X}$ to obtain a basepoint free family $r_i : \widetilde{U}_i \to W_i$ with a flat map $\widetilde{s}_i : \widetilde{U}_i \to \widetilde{X}$. For each i, let F_i be a very general fiber of r_i. Since π' is flat equidimensional with irreducible general fiber, Theorem 2.10 implies that F_i is irreducible. Since the general cycles in p_i do not contract in Y, it follows that $\dim \widetilde{s}_i(F_i) = n - k + e$ and $\dim \rho(\widetilde{s}_i(F_i)) = n - k$.

We claim that there is some $\beta \in \overline{\mathrm{Mov}}_{k-e}(Y')$ such that $\tilde{\alpha} \cdot Z = \rho^*\beta \cdot Z$ for every $(n-k)$-cycle Z contained in some $\widetilde{s}_i(F_i)$. This will conclude the proof of the theorem: since each $\widetilde{s}_i(F_i)$ contains the strict transform of the corresponding cycle of p_i and since the classes of these strict transforms are the pullbacks of a basis of $N^k(X)$, we see that $f_{X*}\pi'^*\beta = f_{X*}\psi_*\tilde{\alpha} = \alpha$.

With $V = \widetilde{s}_i(F_i)$, we have $\dim V = (n - k + e)$, and $\dim \rho(V) = n - k$. By Corollary 6.4, we see that $\tilde{\alpha}|_V$ is a movable class in $N_e(V)$. Setting $r = k - e + 1$, with $\imath : V \hookrightarrow \widetilde{X}$ denoting the natural map, for $C \gg 0$, we have

$$\begin{aligned} \imath_*\imath^*(\tilde{\alpha} \cdot \rho^*A) &= \tilde{\alpha} \cdot [V] \cdot \rho^*A^{r-(k-e)} \\ &\preceq \tilde{\alpha} \cdot C\rho^*A^r = 0, \end{aligned}$$

so that $\tilde{\alpha}|_V$ satisfies the conditions for Corollary 6.7. The conclusion is that $\tilde{\alpha}|_V$ is proportional to the class of a fiber of $\rho|_V$. Thus $\tilde{\alpha} \cdot [V] \in N_e(X)$ is proportional to the class of a fiber of ρ via some constant b.

As we vary i, the argument above yields constants b_i which are a priori unrelated. Let $\{T_i\}$ be the $(n-k)$-dimensional subvarieties of Y' that are the images of the V_i. Note that each T_i is the image of a general member of a basepoint free family on Y' (constructed as the flat image of the p_i').

Let M be a very general complete intersection of ample divisors on $\widetilde{X}$ of dimension d. By very generality, $\tilde{\alpha}|_M$ is movable. The previous paragraph shows that for each T_i, intersections of $\tilde{\alpha}|_M$ against cycles with support contained in each $\rho|_M^{-1}(T_i)$ are proportional via some constant b_i. Lemma 6.9 shows that there is a movable class β on Y' such that $\rho^*\beta|_M \cdot Z = \tilde{\alpha}|_M \cdot Z$ for any cycle Z lying above one of the T_i. But since intersections are compatible against degree for *any* subvariety of the V_i, we see that $\tilde{\alpha} \cdot Z = \rho^*\beta \cdot Z$ for any cycle Z with support contained in $\widetilde{s}_i(F_i)$ which has dimension $(n + e - k)$. □

Corollary 6.11. *Let $\pi : X \to Y$ be a surjective morphism of projective varieties over $\mathbb{C}$ of relative dimension e. Suppose $\alpha \in \overline{\mathrm{Mov}}_k(X)$ for some $k \geq e$ and that $\alpha \cdot \pi^*H^{k-e+1} = 0$ for some ample divisor H on Y. Then the MSC holds for α. In particular the MSC is true when $e = 1$.*

The proof is the same as in Proposition 3.23 which handles the divisor case.

Proof. Since we know the MSC for generically finite maps, arguing as in Remark 4.3 we may assume that X and Y are smooth and π has connected fibers. Applying Theorem 6.10 we obtain a smooth birational model Y' and a class $\beta \in \overline{\mathrm{Mov}}_{k-e}(Y')$. Let $\{Z_i\}_{i=1}^{\infty}$ be a sequence of strictly movable cycles whose classes limit to β. Since π' is flat, each $\pi'^*[Z_i] = [\pi'^{-1}Z_i]$ is the class of an effective π'-contracted cycle which is strictly movable by [FL13, Lemma 3.6]. The image of each π'^*Z_i under $(f_X)_*$ is a movable π-contracted cycle, and the corresponding classes limit to α. □

In fact, we can weaken the hypotheses of Theorem 6.10 without changing the proof. We now explain this stronger version.

Remark 6.12. We have not used the full strength of the movability condition on α in this section. In Lemmas 6.8 and 6.9, one can replace movability with any notion invariant under pushforward by surjective morphisms. Theorem 6.10 uses three properties of movability:

i) Movable classes admit movable preimages by surjective maps.
ii) Corollary 6.4.
iii) Movable curves are movable in the sense of [BDPP13].

Consider the following partial substitute:

Definition 6.13. Say that a class $\alpha \in \overline{\mathrm{Eff}}_k(X)$ is *weakly movable* if there exists a sequence $\{V_i\}_{i=1}^{\infty}$ of effective k-cycles on X such that $\alpha = \lim_i [V_i]$, and for each (reducible) divisor E on X, infinitely many of the V_i's meet $|E|$ properly.

The following remark explains how the analogue of Theorem 6.10 for weakly movable classes is proved.

Remark 6.14. Weakly movable classes are invariant under pushforward by surjective morphisms and have the following additional characteristics:

i) Extremal classes in $\overline{\mathrm{Eff}}_k(X)$ that are not the pushforward of any pseudoeffective class on a (reducible) divisor on X are weakly movable and admit weakly movable preimages by surjective *generically finite* maps.
ii) The analogue of Corollary 6.4 holds for weakly movable classes when the basepoint free family has irreducible general fiber that is not contracted by the map to X, i.e. it produces a nonzero class.
iii) Weakly movable curves are movable in the sense of [BDPP13].

(If $\pi : X \to Y$ is surjective, $\alpha \in \overline{\mathrm{Eff}}_k(X)$ is weakly movable, and $\{V_i\}$ is a sequence of cycles that verifies its movability, then $\{\pi_* V_i\}$ is a sequence of cycles that verifies the weak movability of $\pi_*\alpha$.

If $\alpha \in \overline{\mathrm{Eff}}_k(Y)$ is extremal and not pushed from a divisor on Y, then there exists a sequence of cycles V_i on Y having irreducible support, and such that every subsequence is dense in Y. The sequence $\{V_i\}$ verifies the weak movability of α. Furthermore, any extremal pseudoeffective preimage $\beta \in \overline{\mathrm{Eff}}_k(X)$ with $\pi_*\beta = \alpha$ is likewise not pushed from a divisor on X. Therefore β is also weakly movable and this proves i).

The justification of ii) is a standard relative Hilbert scheme argument. Let α be weakly movable and let V_i be k-cycles that verify its weak movability. Let $p : U \to W$ be a projective morphism with irreducible general fiber to W integral and let $s : U \to X$ be an equidimensional flat morphism. For very general $w \in W$, the fiber U_w sits in general position relative to all V_i's. Consider the relative Hilbert scheme $\mathcal{H}_j$ parameterizing pairs (w, D_w), where D_w is a divisor on U_w that contains a component of $s^{-1}V_i \cap U_w$ for all $i \geq j$. If the weak movability of restrictions (computed as proper intersections in the sense of [Ful84, §7]) fails, then by the uncountability of the base field, some component of some $\mathcal{H}_j$ dominates W. One then uses the universal family over an appropriate subvariety of this component to construct a divisor whose image in X is a divisor that meets all but finitely many of the V_i's improperly.

It is an immediate consequence of the definition that a weakly movable curve class has nonnegative intersection with any effective Cartier divisor. Then iii) follows by [BDPP13].)

Corollary 6.15. *The Strong Conjecture holds for surjective morphisms from fourfolds to threefolds over $\mathbb{C}$. More generally, it holds for almost exceptional classes on fourfolds regardless of the target.*

Proof. Let $\pi : X \to Y$ be a morphism from a fourfold, and let $\alpha \in \overline{\mathrm{Eff}}_k(X)$ satisfy $\pi_*\alpha = 0$. It is enough to treat the case of surface classes ($k = 2$), since curves and divisors are covered by [DJV13, Theorem 1.4]. By Remark 3.10 and Corollary 3.14 we may assume that π is surjective and furthermore that its relative dimension is $e \in \{0, 1, 2\}$. If $e = 0$, i.e. π is generically finite, then α is exceptional, and hence pushed forward from a subscheme of X by Theorem 5.14. We obtain the Strong Conjecture for α from the Strong Conjecture for threefolds using the arguments in Theorem 4.13.

When $e = 1$, the condition $\pi_*\alpha = 0$ is equivalent to $\alpha \cdot \pi^*h^{k-e+1} = \alpha \cdot \pi^*h^2 = 0$ for some h ample on Y. In particular α is almost exceptional. We may assume that α is extremal and is not pushed from any divisor on X, otherwise we reduce to the case when α is a divisor class on a threefold. Then α is weakly movable and Remarks 6.12 and 6.14 show that the proof of Theorem 6.10 carries through.

The same argument works when $e = 2$ if α is almost exceptional. □

Remark 6.16. The only unsettled case of the Strong Conjecture in dimension 4 over $\mathbb{C}$ is that of a surjective morphism $\pi : X \to Y$ to a surface and of classes $\alpha \in \overline{\mathrm{Eff}}_2(X)$ with $\alpha \cdot \pi^*h^2 = 0$, but $\alpha \cdot \pi^*h \neq 0$, where h is an ample divisor class on Y.

Question 6.17. Are weakly movable classes movable?

As mentioned above, this is true for curves by [BDPP13] (which holds in arbitrary characteristic; see [FL13, Section 2.2]). It is also true for divisors: using Remark 6.14.(ii) one reduces to the case of smooth varieties. But on smooth varieties the weakly movable condition for L implies that $N_\sigma(L) = 0$ so that L is movable (see [Nak04] and [Mus13]). In general this question is closely related to Conjecture 4.12.

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