A SNAPSHOT OF THE MINIMAL MODEL PROGRAM

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ABSTRACT. We briefly review the main goals of the minimal model program. We then discuss which results are known unconditionally, which are known conditionally, and which are still open.

1. Introduction

This survey paper reviews the main goals and results of the Minimal Model Program (henceforth MMP). The paper has three parts. In Section 2, we give a very gentle introduction to the main ideas and conjectures of the MMP. We emphasize why the results are useful for many different areas of algebraic geometry.

In Sections 3-6, we take a "snapshot" of the MMP: we describe which results are currently known unconditionally, which are known conditionally, and which are wide open. We aim to state these results precisely, but in a manner which is as useful as possible to as wide a range of mathematicians as possible. Accordingly, this paper becomes more and more technical as we go. The hope is this paper will be helpful for mathematicians looking to apply the results of the MMP in their research.

In Section 7, we briefly overview current directions of research which use the MMP. Since the foundations of the MMP are discussed previously, this section focuses on applications. The choice of topics is not comprehensive and is idiosyncratically based on my own knowledge.

These notes are *not* intended to be a technical introduction to the MMP. There are many good introductions to the techniques of the MMP already: [KM98], [Mat02], [HK10], [Kol13b], and many others. These notes are also *not* intended to be a historical introduction. We will focus solely on the most recent results which are related to Principle 2.2: the existence of minimal models, termination of flips, and the abundance conjecture. Thus we will not cover in any length the many spectacular technical developments required as background. In particular, we will unfortunately omit most of the foundational results from the 1980's due to Kawamata, Kollár, Mori, Reid, Shokurov, and many others. We will also not cover the analytic side of the picture in any depth. To partially amend for this decision, we give a fairly complete list of references at the end.

Throughout we will work over $\mathbb{C}$ unless otherwise specified. Varieties are irreducible and reduced.

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2. Main idea of the MMP

Warning 2.1. This entire section is filled with inaccuracies, imprecisions, over-simplifications, and outright falsehoods. The few terms which may be new to a general audience will be defined rigorously in the next section.

I will focus on the guiding principle:

Principle 2.2. Let X be a smooth projective variety over a field. The geometry and arithmetic of X are governed by the "positivity" of the canonical bundle $\omega_X := \bigwedge^{\dim X} \Omega^1_X$.

We will participate in the traditional abuse of notation by letting K_X denote any Cartier divisor satisfying $\mathcal{O}_X(K_X) \simeq \omega_X$. Such a divisor is only unique up to linear equivalence, but since our statements are all linear-equivalence invariant this abuse is harmless in practice.

We first work out the case of dimension 1. Let C be a smooth integral projective curve with canonical bundle $\omega_C = \Omega_C$. The central feature of the curve is its genus:

$$g_C = \dim H^0(C, \Omega_C).$$

(Hodge theory shows that this coheres with the classical definition via Betti numbers.) By Riemann-Roch and Serre duality this is equivalent to saying that:

$$\deg(\Omega_C) = 2g_C - 2.$$

As is well-known, curves split into three categories based upon their genus or curvature. In line with Principle 2.2, it is most natural to describe this trichotomy in terms of the degree of the canonical bundle – only then do we see why the trichotomy is the right one.

$\deg(K_C)$	< 0	$= 0$	> 0
g_C	0	1	≥ 2
examples	$\mathbb{P}^1$	elliptic curves	smooth plane curves of degree > 3
universal cover/$\mathbb{C}$	$\mathbb{P}^1$	$\mathbb{C}$	$\mathbb{H}$
automorphisms	PGL_2	$\approx$ itself	finite
rational points over # field	dense after deg 2 ext	dense and thin after ext	finite

It would be very nice to have a similar trichotomy in higher dimensions. Of course this is too optimistic – a complex manifold of higher dimensions can have different curvatures in different directions – but we will soon see that there is some hope.

For arbitrary varieties X, we first need to decide what properties of K_X best correspond to the conditions $\deg(K_C) < 0, = 0, > 0$ for curves. Ample divisors are the most natural generalization of "positive degree" divisors for curves (and dually for antiampleness and "negative degree"). In the setting of the MMP, the best analogue of "degree 0" turns out to be the condition that K_X is torsion – some multiple of K_X is linearly equivalent to the 0 divisor. When K_X is ample, torsion, or antiample we say that our variety has "pure type".

With these changes, the trichotomy we found for curves seems to hold up well in higher dimensions. However, many of statements are still only conjectural – these will be designated with a question mark in the table below.

K_X	antiample	$\sim_{\mathbb{Q}} 0$	ample
examples	$\mathbb{P}^n$ Fanos	abelian varieties hyperkählers Calabi-Yaus	high degree hypersurfaces in $\mathbb{P}^n$
fundamental group/$\mathbb{C}$	1	almost abelian	??
rational curves on $X/\mathbb{C}$	dense	contained in countable union of proper subsets	not dense?
rational points over # field	potentially dense?	??	not dense?

While this conceptual picture is very appealing, at first glance it seems to only address a very limited collection of varieties. The main conjecture of the MMP is that *any* variety admits a "decomposition" into these varieties of pure types: at least after passing to a birational model, we can find a fibration with pure type fibers.

Conjecture 2.3 (Guiding conjecture of the MMP)**.** *Any smooth projective variety X admits either:*

(i) a birational model $\psi : X \dashrightarrow X'$ and a morphism $\pi : X' \to Z$ with connected fibers to a variety of smaller dimension such that the general fiber F of π has K_F antiample, or
(ii) a birational model $\psi : X \dashrightarrow X'$ and a morphism $\pi : X' \to Z$ with connected fibers to a variety of smaller dimension such that the general fiber F of π has K_F torsion, or
(iii) a birational model $\psi : X \dashrightarrow X'$ and a birational morphism $\pi : X' \to Z$ such that K_Z is ample.

We will refer to the outcomes respectively as cases (i), (ii), (iii). It is clear why Conjecture 2.3 is so powerful – it suggests that we can leverage results for pure type varieties to study any variety via a suitable fibration.

Historically, this conjecture has its roots in the Kodaira-Enriques classification of surfaces, which categorizes birational equivalence classes of surfaces exactly according to the ability to find a morphism with fibers of a given pure type. (The fact that the birational map ψ may not be defined everywhere is a new feature in higher dimensions.)

Remark 2.4. Even when X is a smooth surface, the varieties predicted by Conjecture 2.3 may have certain "mild" singularities. In this section we will ignore singularities completely, but the reader should remember they are there.

Implicit in the statement of Conjecture 2.3 is that the three cases have a hierarchy ordered by negativity: we look for an antiample fibration, then (failing to find any) a torsion fibration, then (failing that too) we expect to be in case (iii). The justification is that it is quite easy to construct subvarieties with ample canonical divisor – for example, take complete intersections of sufficiently positive very ample divisors. The "most special" subvarieties are those with antiample canonical divisor, and we should look for these subvarieties first. (As we will see soon, this hierarchy is more properly motivated by the birational properties of K_X.)

The apparent asymmetry of case (iii) is also justified by this logic. Every variety admits many rational maps with fibers of general type, and the existence of such maps tells us essentially nothing about the variety. In contrast, Conjecture 2.3 has real geometric consequences.

Remark 2.5. Another common perspective on the MMP is that it identifies a "distinguished set" of representatives of each fixed birational equivalence class of varieties. In dimension 2, the Kodaira-Enriques classification identifies a unique smooth birational model of any surface and we obtain a birational classification of surfaces. In higher dimensions, the analogous constructions are not unique, and so this perspective is slightly less useful.

Conjecture 2.3 suggests the following questions:

(a) How can we identify the target Z, or equivalently, the rational map $\phi = \pi \circ \psi : X \dashrightarrow Z$?
(b) How can we identify the rational map ψ and the birational model X'? What properties of X' distinguish it as the "right" birational model?
(c) How can we predict the case (i), (ii), (iii) of X based on the geometry of K_X?

We will answer these questions in the following subsections.

2.1. **Canonical models.** We first turn to Question (a): how to identify the variety Z? In other words, how can we naturally choose a rational map $\phi : X \dashrightarrow Z$ which captures the essential geometric features of X? For now we will focus on cases (ii) and (iii); case (i) is somewhat different.

Rational maps are constructed from sections of line bundles on X. For arbitrary varieties we only really have access to one line bundle: the canonical bundle. Furthermore, the canonical bundle encodes fundamental information about the curvature of our variety X. Thus it is no surprise that our "canonical map" ϕ should be constructed from the canonical divisor K_X.

A fundamental principle of birational geometry is that the geometry of a divisor L is best reflected not by sections of L but by working with all multiples of L simultaneously. We obtain access to this richer structure by the following seminal theorem of [BCHM10].

Theorem 2.6 ([BCHM10] Corollary 1.1.2)**.** *Let X be a smooth projective variety. Then the pluricanonical ring of sections*

$$R(X, K_X) := \bigoplus_{m \geq 0} H^0(X, mK_X)$$

is finitely generated.

Suppose that some multiple of K_X has sections (which should happen – and can only happen – in cases (ii) and (iii)). Since the pluricanonical ring is finitely generated, we can take its Proj.

Definition 2.7. Suppose that some multiple of K_X has sections. The canonical model of X is defined to be $\operatorname{Proj} R(X, K_X)$.

As suggested by the notation, we obtain a rational map $\phi : X \dashrightarrow \operatorname{Proj} R(X, K_X)$ which is canonically determined by X. It is expected, but not yet proved, that this rational map exactly coincides with the map $\phi : X \dashrightarrow Z$ predicted by Conjecture 2.3 in cases (ii) and (iii).

Remark 2.8. It is worth mentioning briefly how case (i) fits into the picture. In this case, no multiple of K_X has sections and $\operatorname{Proj} R(X, K_X)$ is empty. Our solution is simply to add on a sufficiently large ample divisor A to ensure that multiples of $K_X + A$ have sections, and then to work with $\operatorname{Proj} R(X, K_X + A)$. Of course, there are now many choices of rational map corresponding to the possible choices of ample divisor A. This is reflected in the richer birational MMP structure for varieties falling into case (i).

There are two reasons why Theorem 2.6 does not solve Conjecture 2.3. First, it is unclear whether any multiples of K_X have sections; if not, we can not hope to obtain any geometry from the canonical map. Second, it is a priori unclear whether ϕ has the special structure predicted by the conjecture: does it factor as a birational map $\psi : X \dashrightarrow X'$ followed by a fibration with pure type fibers?

2.1.1. *Obstructions from negativity.* It is worth exploring this second point in more detail. Let us identify the obstruction to the existence of a factorization of rational maps $\phi = \pi \circ \psi$, where ψ is birational and π has pure type fibers.

Suppose first for simplicity that the rational map $\phi : X \dashrightarrow \operatorname{Proj} R(X, K_X)$ is defined everywhere. Due to the properties of the Proj construction, there is an ample divisor A on $\operatorname{Proj} R(X, K_X)$ and a positive integer m such that

$mK_X = \phi^* A + E$, where E is an effective divisor contained in the base locus of mK_X. We consider two cases:

- $\dim \operatorname{Proj} R(X, K_X) < \dim X$. By adjunction, for a general fiber F we have $mK_F \sim E|_F$. Thus E represents the obstruction to the torsionness of K_F.
- $\dim \operatorname{Proj} R(X, K_X) = \dim X$. If the singularities of the canonical model are sufficiently mild, its canonical divisor is well defined. Then the canonical divisor of the canonical model necessarily coincides with the ample divisor A. The mildness of the singularities of the canonical model are controlled by E.

In either case we see that the divisor E represents an "obstruction" to the conclusion of Conjecture 2.3.

In the general case (when ϕ is only rational), a similar argument shows that the base locus of multiples of K_X prevents the fibers from having pure type. We are naturally led to try to "remove" the base locus of multiples of K_X via a birational transformation. (In fact, essentially the only way of proving that a divisor has finitely generated section ring is to find a birational model of X on which the strict transform of the divisor has no base locus.)

The problem of "removing" the base locus of multiples of K_X is traditionally divided into two steps. First, we find a birational model X' such that $K_{X'}$ has non-negative intersection against every curve. This condition on $K_{X'}$ is necessary, but not sufficient, for ensuring an empty base locus. We then hope to use this condition to prove that multiples of $K_{X'}$ have no base locus.

2.2. **Running the MMP.** We next turn to Question (b): how to find the birational model $\psi : X \dashrightarrow X'$ predicted by Conjecture 2.3? As discussed above, we would like to remove the base locus of K_X. This is accomplished by an inductive procedure known as "running the minimal model program." In brief, we would like to repeatedly contract curves which have negative intersection against the canonical divisor.

First, we need to know that if X carries K_X-negative curves – that is, curves with negative intersection against K_X – then there is a morphism $f : X \to Y$ with connected fibers to a normal projective Y which contracts some of these curves. This foundational result is known as the Cone Theorem, and can be applied whenever the singularities of X are sufficiently mild (the version we will use in this section is due to [Kaw84a]).

The MMP procedure starts when f is a birational map. There are two possibilities to consider:

- (Divisorial contraction) The morphism f could contract an irreducible divisor. In this case the variety Y only has mild singularities, and we can continue the MMP by applying the Cone Theorem to Y.
- (Flipping contraction) The morphism f could contract a locus of codimension ≥ 2. In this case the variety Y has harsh singularities: we cannot apply the Cone Theorem to Y. The key insight is that we

should not work with Y, but with a related variety X^+: by [HM10a], [BCHM10] there is a diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & {}_f\searrow \quad \swarrow_{f^+} & \\ & Y & \end{array}$$

where X^+ has mild singularities, X and X^+ are isomorphic in codimension 1, and the contracted curves for f are K_X-negative but the contracted curves for f^+ are K_{X^+}-positive. Thus in passing from X to X^+ we have eliminated some "negativity" of the canonical divisor. Since X^+ again only has mild singularities, we can continue by applying the MMP by applying the Cone Theorem to X^+. This diagram is known as a flip diagram, and the rational map $X \dashrightarrow X^+$ is known as a flip.

The main conjecture of the MMP is that we can do only a finite sequence of such birational transformations. It is not hard to see that there can be only finitely many steps in the MMP which contract a divisor: each such step drops the Picard rank, while flips preserve the Picard rank. However, it is much more subtle to determine whether there can be an infinite sequence of flips:

Conjecture 2.9 (Termination of flips)**.** *There is no infinite sequence of flips.*

2.2.1. *End result of the MMP.* Assuming that we can only do a finite sequence of birational steps as above, the process will end with a birational model $\psi : X \dashrightarrow X'$. This X' must satisfy one of the following two conditions.

The first possibility is that X' contains $K_{X'}$-negative curves, but the corresponding contraction $f : X' \to Z$ maps to a variety of smaller dimension. This f achieves the desired conclusion for case (i): the fibers of f have anti-ample canonical divisor, and the birational map ψ is the map we are looking for.

The second possibility is that $K_{X'}$ is not negative against any curve in X'. Conjecturally $\psi : X \dashrightarrow X'$ is exactly the birational map predicted by cases (ii) and (iii) (see the next section for more details). At the very least, we have successfully eliminated the "intersection-theoretic" contributions to the base locus of K_X. This condition on X' is useful enough to merit its own terminology:

Definition 2.10. A variety X' is called a minimal model if $K_{X'}$ has non-negative intersection against every curve. We say X' is a minimal model for X if it is a minimal model achieved by running the MMP for X (but a more precise definition will be given in Definition 4.2).

Note that this usage is slightly different than the classical usage for surfaces involving (-1)-curves. For smooth surfaces, the non-existence of (-1)-curves is necessary but not sufficient to be a minimal model in our sense. An important feature which only appears in dimension > 2 is that X might admit many different minimal models if we make different choices of morphism while applying the Cone Theorem.

The termination of flips conjecture would imply:

Conjecture 2.11 (Existence of minimal models)**.** *Suppose that X is not uniruled. Then X admits a minimal model.*

Let us briefly contrast the notion of minimal and canonical models:

(1) Minimal models are expected to exist whenever X is not uniruled, and canonical models exist when some multiple of K_X has sections. These two conditions on X are expected to be equivalent by the Abundance Conjecture below, and should correspond to cases (ii) and (iii).
(2) A minimal model for X is always birational to X, but a canonical model for X may have smaller dimension (in case (ii)).
(3) There can be many minimal models for X, but there is only one canonical model for X.
(4) There should be a morphism (defined everywhere) from any minimal model for X to the canonical model for X, as predicted by the Abundance Conjecture below.
(5) If X' is a minimal model for X then $K_{X'}$ may have vanishing intersection against some curves, but (conjecturally) all such curves should be contracted by the morphism to the canonical model.

2.3. **Abundance Conjecture.** Finally, suppose that by running the MMP we have found a minimal model X' whose canonical divisor $K_{X'}$ has non-negative degree against every curve. We hope that $K_{X'}$ has no base locus. This is the content of the following conjecture:

Conjecture 2.12 (Abundance Conjecture)**.** *Suppose that $K_{X'}$ has non-negative degree against every curve. Then there is a morphism $\pi : X' \to Z$ and an ample divisor A on Z such that some multiple of $K_{X'}$ is linearly equivalent to π^*A.*

Note that if the Abundance Conjecture is true, then Z will be the canonical model:

$$\begin{aligned} Z &= \operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X', mK_{X'}) \\ &= \operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, mK_X) \end{aligned}$$

due to the birational invariance of plurigenera.

As suggested by the notation, the morphism $\pi : X' \to Z$ in the Abundance Conjecture should be the same as the morphism π in Conjecture 2.3. Let us analyze the connection more carefully. Assuming the Abundance Conjecture:

- In case (ii), we will necessarily have $\dim(Z) < \dim(X)$. Using adjunction, we see that some multiple satisfies $mK_X|_F \sim 0$ for a general fiber F. So F has torsion pure type as desired.
- In case (iii), Z will be necessarily be birational to X. Since canonical divisors relate well over birational maps, the identification $K_X \sim_{\mathbb{Q}} \pi^*A$ actually allows us (after a more careful setup and argument) to identify $A = K_Z$. So X will be birational to a variety Z with ample canonical divisor.

According to the Abundance Conjecture, a numerical property for K_X – having non-negative intersection against every curve – implies a section property – some multiple has no base locus. Such implications are quite rare: usually one can not deduce holomorphic information from intersection theory. In fact, this is an important theme in the study of the canonical bundle which applies in much more generality:

Principle 2.13. The behavior of sections of multiples of K_X is governed by intersection theoretic properties of ω_X.

We finally answer Question (c): how can we determine which case (i), (ii), (iii) X falls into? The case depends on the birational positivity of K_X. According to Principle 2.13, we can use either numerical or sectional forms of positivity. Although we have not yet seen the relevant definitions, the *conjectural* picture is summarized below:

	Case (i)	Case (ii)	Case (iii)
Sectional property	$\kappa(X) = -\infty$	$0 \leq \kappa(X) < \dim X$, $\kappa(X) = \dim Z$	$\kappa(X) = \dim X$
Numerical properties	$K_X \notin \overline{\mathrm{Eff}}^1(X)$, equivalently $\nu(K_X) = -\infty$	K_X on the boundary of $\overline{\mathrm{Eff}}^1(X)$, $\nu(K_X) = \dim Z$	$K_X \in \overline{\mathrm{Eff}}^1(X)^\circ$, equivalently $\nu(K_X) = \dim X$
Curve properties	uniruled, equivalently dominated by K_X-negative curves	dominated by K_X-trivial curves (but not negative ones)	neither of the previous conditions

3. Background

A pair (X, Δ) is a normal variety X and an effective $\mathbb{R}$-Weil divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.

We refer to any standard reference for a summary of the various singularity types: terminal, canonical, Kawamata log terminal (henceforth klt), log canonical (henceforth lc), etc.

3.1. **Birational geometry of divisors.** The Neron-Severi space $N^1(X)$ is the vector space of $\mathbb{R}$-Cartier divisors up to numerical equivalence. The dual space is $N_1(X)$. We define the pseudo-effective and nef cones:

- the pseudo-effective cone of divisors $\overline{\mathrm{Eff}}^1(X)$ is the closure of the cone in $N^1(X)$ generated by the classes of effective Cartier divisors;
- the pseudo-effective cone of curves $\overline{\mathrm{Eff}}_1(X)$ is the closure of the cone in $N_1(X)$ generated by the classes of effective curves
- the nef cone of divisors $\mathrm{Nef}^1(X)$ is the dual of $\overline{\mathrm{Eff}}_1(X)$;
- the nef cone of curves $\mathrm{Nef}_1(X)$ is the dual of $\overline{\mathrm{Eff}}^1(X)$.

The nef cone of divisors can also be interpreted as the closure of the cone generated by all ample Cartier divisors (see [Kle66]). The nef cone of curves can also be interpreted as the closure of the cone generated by all irreducible curves which deform to cover X (see [BDPP13]).

We say that an $\mathbb{R}$-Cartier divisor D is

- pseudo-effective, if its numerical class is contained in $\overline{\mathrm{Eff}}^1(X)$.
- big, if its numerical class is contained in $\overline{\mathrm{Eff}}^1(X)^\circ$.
- nef, if its numerical class is contained in $\mathrm{Nef}^1(X)$.
- ample, if its numerical class is contained in $\mathrm{Nef}^1(X)^\circ$.

The movable cone of divisors $\mathrm{Mov}^1(X)$ is the closure of the cone generated by all divisors whose base locus has codimension ≥ 2. Equivalently, it is the closure of the cone generated by all divisors such that every irreducible component deforms to cover X.

Suppose X has dimension n. Given an $\mathbb{R}$-Cartier divisor D, its Iitaka dimension is

$$\kappa(D) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \,\middle|\, \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^k} > 0 \right\}.$$

unless every $H^0(X, \lfloor mD \rfloor) = 0$, in which case we formally set $\kappa(D) = -\infty$. It is well-known that

- D is big if and only if $\kappa(D) = n$.
- if D is not pseudo-effective then $\kappa(D) = -\infty$;

however, the converse of the latter statement is false. By combining the seminal results of [MM86] and [BDPP13], a smooth variety X is uniruled if and only if K_X is not pseudo-effective.

We define the numerical dimension of an $\mathbb{R}$-Cartier divisor in a similar way. Choose any sufficiently ample Cartier divisor A. Then

$$\nu(D) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \,\middle|\, \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor) + A)}{m^k} > 0 \right\}.$$

unless D is not pseudo-effective, in which case we formally set $\nu(D) = -\infty$. It is well-known that D is big if and only if $\nu(D) = n$. On the other extreme, divisors satisfying $\nu(D) = 0$ are very "rigid": they are numerically equivalent to a unique effective divisor E, and even after perturbing by a small ample divisor, we can not deform E away from its support. The prototypical example of a divisor D satisfying $\nu(D) = 0$ is an exceptional divisor of a blow-up.

There is always an inequality $\kappa(D) \leq \nu(D)$. A divisor D is said to be abundant if $\kappa(D) = \nu(D)$. When D is pseudo-effective, this condition turns out to be equivalent to a very natural geometric statement: after a birational modification, the positive part in the Nakayama-Zariski decomposition of D is the pullback of a big divisor on a variety of dimension $\kappa(D) = \nu(D)$.

We refer to [Nak04], [Leh12], and [Eck15] for more details on these invariants.

3.2. Birational contractions.

A birational contraction is a birational map $\phi : X \dashrightarrow Y$ which does not extract any divisor (or equivalently, the inverse map ϕ^{-1} does not contract any divisor to a smaller dimensional locus). This implies that the divisor theory on Y is "controlled" by the divisor theory on X.

One reason why birational contractions are useful is that any birational map $\psi : X \dashrightarrow X'$ constructed via a sequence of flips and divisorial contractions is a birational contraction. In fact, there is an additional property which (more-or-less) identifies the outcomes of the MMP amongst all birational contractions.

Definition 3.1. Let (X, Δ) be a pair. A birational contraction $\phi : X \dashrightarrow Y$ is called a $(K_X + \Delta)$-negative birational contraction if $(Y, \phi_*\Delta)$ is a pair and for some (equivalently any) resolution

$$\begin{array}{ccc} & W & \\ {}^{g_X}\swarrow & & \searrow^{g_Y} \\ X & \overset{\phi}{\dashrightarrow} & Y \end{array}$$

we have that $g_X^*(K_X + \Delta) = g_Y^*(K_Y + \phi_*\Delta) + \sum a_i E_i$ where each a_i is positive and E_i varies over all the g_Y-exceptional divisors.

If we allow some $a_i = 0$, the contraction is called $(K_X + \Delta)$-non-positive.

4. Main conjectures

We now state precisely the main conjectures of the MMP. We focus almost exclusively on establishing when the loosely phrased Principle 2.2 and Conjecture 2.3 hold for a variety X and on related issues.

Remark 4.1. I will precisely state the theorems proved in the literature, with the somewhat annoying consequence of frequently switching back and forth between singularity types. I will also indicate when log canonical pairs are known to be irredeemably worse behaved.

While the extensions to worse singularities (semi log canonical pairs) are very important, they introduce an additional layer of technicality not in keeping with the spirit of this paper and will not be discussed. We will also not work in the relative setting, for the same reason, despite the important additional theoretical flexibility it provides. Essentially all of the results stated below go through unchanged.

We split the main conjectures into three parts: existence of a (good) minimal model, termination of flips, and various flavors of the abundance conjecture. Note the existence of a minimal model usually comes down to the termination of a "special" sequence of flips, which distinguishes the problem from the termination of arbitrary sequences of flips.

4.0.1. *Existence of a good minimal model.* The definition of a minimal model is supposed to encode the end result of the MMP, without keeping track of the steps taken. (Recall that running the MMP is more-or-less the same as identifying a $(K_X + \Delta)$-negative contraction). The definition of a *good* minimal model furthermore encodes the expected existence of the "pure type map" given by the canonical model. These are exactly the structures predicted by Conjecture 2.3 in cases (ii) and (iii).

Definition 4.2. Let (X, Δ) be a lc pair. A minimal model of (X, Δ) is a $(K_X + \Delta)$-negative birational contraction $\psi : X \dashrightarrow X'$ such that $(X', \psi_*\Delta)$ is a lc pair and $K_{X'} + \psi_*\Delta$ is nef.

A good minimal model of (X, Δ) is a minimal model $\psi : X \dashrightarrow X'$ such that $K_{X'} + \psi_*\Delta$ is semiample – that is, $\mathbb{R}$-linearly equivalent to the pullback of an ample divisor under some morphism $\pi : X' \to Z$.

In dimension ≥ 3, the following conjectures have their roots in [Mor82].

Conjecture 4.3. *(Existence of minimal models) Let (X, Δ) be a lc pair. If $K_X + \Delta$ is pseudo-effective, then (X, Δ) admits a minimal model.*

Conjecture 4.4. *(Existence of good minimal models) Let (X, Δ) be a lc pair. If $K_X + \Delta$ is pseudo-effective, then (X, Δ) admits a good minimal model.*

A minimal model can only exist if $K_X + \Delta$ is pseudo-effective. To understand the non-pseudo-effective case, by analogy with the notion of a minimal model, we must identify the expected result in case (i) of Conjecture 2.3:

Definition 4.5. Let (X, Δ) be a lc pair. A birational Mori fiber space structure for (X, Δ) is a $(K_X + \Delta)$-negative birational contraction $\psi : X \dashrightarrow X'$ and a morphism $\pi : X' \to Z$ with connected fibers such that $\dim(Z) < \dim(X)$ and $(K_{X'} + \psi_*\Delta)|_F$ is antiample for a general fiber F of π. (It is also common to insist that π have relative Picard rank 1, in which case we may insist that $K_{X'} + \psi_*\Delta$ is antiample along every fiber. We will however use the more general version.)

There is no need for a "existence of Mori fiber spaces conjecture" in the klt case since (as discussed below) the existence has already been proved by [BCHM10].

4.0.2. *Termination of flips.*

Definition 4.6. A flip consists of a lc pair (X, Δ) and a diagram of birational maps

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle f^+} & \\ & Y & \end{array}$$

such that f and f^+ have exceptional locus of codimension at least 2, $(X^+, \phi_*\Delta)$ is a lc pair, $K_X + \Delta$ is f-antiample, and $K_{X^+} + \phi_*\Delta$ is f^+-ample.

Flips are known to exist in the lc case by [Bir12a, Corollary 1.2], [HX12, Corollary 1.8]. The termination of flips conjecture predicts that there is no infinite sequence of flips.

Warning 4.7. A flop is the similar diagram where both maps are crepant. While klt flops exist, lc flops need not exist (see [Kol08, Exercise96]). Furthermore, even for klt flops there can be a non-terminating sequence by [Rei83] or [Ogu14].

4.0.3. *Abundance conjecture.* The abundance conjecture predicts that the asymptotic sectional properties of the canonical divisor are controlled by its numerical properties. Results of this type were first proved for varieties of general type by [Ben83] and [Kaw84c] in dimension 3 and by [Sho85] in general. They were also first proved for varieties of intermediate Kodaira dimensions by [Kaw85b]. Two common versions of the abundance conjecture are:

Conjecture 4.8 (Semi-ample abundance conjecture)**.** *If (X, Δ) is a lc pair such that $K_X + \Delta$ is nef, then $K_X + \Delta$ is $\mathbb{R}$-semi-ample.*

Conjecture 4.9 (Non-vanishing conjecture)**.** *If (X, Δ) is a lc pair such that $K_X + \Delta$ is pseudo-effective, then $\kappa(K_X + \Delta) \geq 0$.*

These phrasings are useful but slightly unsatisfactory, since they implicitly rely on the existence of a minimal model for use in applications. My preference is for the following version, which again predicts that the sectional properties are controlled by numerical properties but in a cleaner "birational" sense.

Conjecture 4.10 (Abundance conjecture)**.** *Let (X, Δ) be a lc pair. Then $K_X + \Delta$ is abundant: $\kappa(K_X + \Delta) = \nu(K_X + \Delta)$.*

It is common to write C_n as a shorthand to denote "the statement of Conjecture C for varieties of dimension at most n." The easy relationships between the conjectures are:

- Termination of $\text{flips}_n \implies$ Existence of minimal models_n.
- Existence of minimal models_n + Semi-ample $\text{abundance}_n \iff$ Existence of good minimal models_n.
- Existence of good minimal $\text{models}_n \implies$ Non-vanishing_n + Semi-ample abundance_n.

As we will see below, there are other (quite difficult) implications between the various statements in the literature.

5. Unconditional results

We now discuss what is known about the main conjectures (and also mention a few useful technical corollaries). We divide these results into "unconditional" and "conditional" results. In this section we discuss "unconditional" results, meaning theorems that can directly be verified on a single variety. In the following section we discuss "conditional" results, which usually means results which "follow from induction on dimension, given some assumptions."

5.1. **Existence of good minimal models.** The most important unconditional advances in the MMP are due to [BCHM10]. One of the key technical advances in [BCHM10] is a special case of the termination of flips. Recall that while running the MMP, we allowed ourselves to choose any $(K_X+\Delta)$-negative face of $\overline{\mathrm{Eff}}_1(X)$ at each step. [BCHM10] shows that if we instead limit our choice by "scaling an ample divisor", then under some conditions the sequence of flips must terminate.

The first result establishes Conjecture 2.3 in the case when $K_X + \Delta$ is not pseudo-effective (thus establishing that "case (i)" holds whenever it is possible).

Theorem 5.1 ([BCHM10], Corollary 1.3.2)**.** *Let (X, Δ) be a dlt pair. Suppose that $K_X + \Delta$ is not pseudo-effective. Then (X, Δ) admits a birational Mori fiber space structure.*

The next result establishes Conjecture 2.3 in the case when $K_X + \Delta$ is big (thus establishing that "case (iii)" holds whenever it possible).

Theorem 5.2 ([BCHM10], Theorem 1.2)**.** *Let (X, Δ) be a klt pair. Suppose that $K_X + \Delta$ is big. Then (X, Δ) admits a good minimal model.*

In fact, something more is true: the techniques of [BCHM10] can be applied so long as Δ itself is big. Thus there is one additional case: when K_X itself is not big, but upon adding a big Δ it lands on the boundary of the pseudo-effective cone. This additional case is applicable to uniruled varieties.

Theorem 5.3 ([BCHM10], Theorem 1.2)**.** *Let (X, Δ) be a dlt pair. Suppose that $K_X+\Delta$ is pseudo-effective and Δ is big $\mathbb{R}$-Cartier. Then (X, Δ) admits a good minimal model.*

Since we will refer to these results later, we state:

Condition 5.4. (X, Δ) is a pair such that either Δ is big $\mathbb{R}$-Cartier or $K_X + \Delta$ is big.

Note that the only unsettled cases of the existence of good minimal models for klt pairs are all "case (ii)": situations where $K_X + \Delta$ lies on the boundary of the pseudo-effective cone. These are the situations where $0 \leq \kappa(K_X + \Delta) \leq \nu(K_X + \Delta) < n$.

Perhaps the next most natural case to consider is when $K_X + \Delta \equiv 0$; is it then true that $K_X + \Delta \sim_{\mathbb{Q}} 0$ as predicted by the Abundance Conjecture? The answer is yes for any log canonical (or even semi-log canonical) pair as proved by [Gon13]. Other results in this direction have been proved in [Kaw85a], [Nak04], [Gon11], [CKP12], [Kaw13b]. In fact, a stronger result for dlt pairs was proved by [Nak04], [Dru11], [Gon11].

Theorem 5.5 ([Nak04] V.4.8 Theorem, [Dru11] Corollaire 3.4, [Gon11] Theorem 1.2)**.** *Let (X, Δ) be a dlt pair. Suppose that $\nu(K_X + \Delta) = 0$. Then (X, Δ) admits a good minimal model.*

In other words, if $K_X + \Delta$ is numerically rigid, then it is actually $\mathbb{R}$-linearly equivalent to an exceptional divisor for a birational map. It is interesting that there are complete results on both extremes – both for the most positive divisors ($\nu = \dim X$) and the most rigid divisors ($\nu = 0$).

Note that the condition $\nu = 0$ implies the condition $\kappa = 0$, but not conversely; indeed, if one could establish the $\kappa = 0$ case then the existence of good minimal models would follow for varieties of positive Kodaira dimension.

It turns out that the numerical dimension is a very useful tool for working with minimal models. The most general statement, which subsumes most of the statements we have made already, is due to [Lai10]. (The statement cited is for terminal varieties but the argument works as well for klt varieties.)

Theorem 5.6 ([Lai10] Theorem 4.4)**.** *Let (X, Δ) be a klt pair. Suppose that $K_X + \Delta$ is pseudo-effective and abundant: $\kappa(K_X + \Delta) = \nu(K_X + \Delta)$. Then (X, Δ) admits a good minimal model.*

Note that no induction assumption is necessary in the statement. This theorem is most useful in situations where abundance is known automatically: if $K_X + \Delta$ is big, if $\nu(K_X + \Delta) = 0$, or if $\kappa(K_X + \Delta) = n - 1$. Recently an additional step has been taken:

Theorem 5.7 ([LP16] Theorem B)**.** *Let X be a terminal normal variety with K_X nef. If $\nu(K_X) = 1$ and $\chi(X, \mathcal{O}_X) \neq 0$ then $\kappa(K_X) \geq 0$.*

Also, after much hard work the main conjectures of the MMP are known for all varieties of small dimension:

Theorem 5.8.
- *Termination of flips is known in dimension ≤ 3. See* [Mor88], [Kol89], [Sho93], [Kaw92c].

- *All forms of the Abundance Conjecture (and hence existence of good minimal models) are known in dimension* ≤ 3*. See* [Kaw92a]*,* [Miy88b]*,* [Miy88a]*,* [KMM94]*.*
- *The existence of minimal models (via a special case of termination of flips) is known in dimension* ≤ 4*. See* [AHK07]*,* [Sho09]*,* [Bir09b]*.*
- *The existence of minimal models is known for klt pairs* (X, Δ) *with* $\kappa(K_X + \Delta) \geq 0$ *in dimension* ≤ 5*. See* [Bir10a]*.*

5.2. **Structure of minimal models.** Once we have established the existence of a good minimal model for (X, Δ), it is natural to ask if the set of all good minimal models has any kind of structure. It is important to distinguish between two options:

- We can consider all minimal models of (X, Δ) as abstract varieties, up to isomorphism.
- We can consider possible ways to construct a minimal model for (X, Δ) by running the MMP $\psi : X \dashrightarrow X'$. We identify $\psi : X \dashrightarrow X'$ and $\phi : X \dashrightarrow X''$ only if the rational map $\psi^{-1} \circ \phi$ extends to an isomorphism. This is the same as counting the number of distinct subcones $\psi^{-1}\mathrm{Amp}^1(X') \subset \overline{\mathrm{Eff}}^1(X)$ defined by outcomes of the MMP.

For surfaces, we only have a finite number of minimal models using either of the counting methods. (Note that this is not true for the "classical" definition of minimal model, where we only insist that there are no (-1)-curves. The blow-up of $\mathbb{P}^2$ in 9 points gives a counter-example.) But in higher dimensions the situation is more subtle.

5.2.1. *Structure of minimal models.* We first simply consider minimal models as abstract varieties. The strongest known results are for varieties of general type by [BCHM10] (which proves something stronger as we will soon see).

Theorem 5.9 ([BCHM10], Corollary 1.1.5)**.** *Let* (X, Δ) *be a klt pair with* $K_X + \Delta$ *big. Then there are only finitely many minimal models for* (X, Δ) *as abstract varieties, up to isomorphism.*

For intermediate Kodaira dimensions, the situation is much more subtle. In particular, the following question of Kawamta is open:

Question 5.10. Does every variety X only admit finitely many minimal models as abstract varieties, up to isomorphism?

Certain cases of this question are known.

Theorem 5.11 ([Kaw97b] Theorem 4.5)**.** *Let* X *be a smooth projective variety of dimension* 3 *with* $\kappa(X) > 0$*. Then* X *only has finitely many minimal models as abstract varieties, up to isomorphism.*

Remark 5.12. One can also phrase a stronger question: are there only finitely many minimal models in a fixed birational equivalence class? (That is, are there only finitely many terminal $\mathbb{Q}$-factorial normal varieties X' with $K_{X'}$ nef?) Both Kawamata's original question and the results of [Kaw97b] are phrased in this generality.

5.2.2. *Set of MMP outcomes.* In this section we discuss possible outcomes of the MMP $\phi : X \dashrightarrow X'$, and not just the abstract varieties X'. As discussed already, the "first step" of the MMP is to choose a K_X-negative extremal ray or face of $\overline{\mathrm{Eff}}_1(X)$. A key insight of Mori is to interpret a step of the MMP via the curve classes it contracts in $\overline{\mathrm{Eff}}_1(X)$. The precise statement is known as the Cone Theorem. Building on [Kaw84c, Theorem 4.5] and [Kol84, Theorem 1], we have:

Theorem 5.13 ([Fuj11] Theorem 1.4)**.** *Let (X, Δ) be a lc pair. There are countably many $(K_X+\Delta)$-negative rational curves C_i such that $0 < -(K_X+\Delta)\cdot C_i < 2\dim X$ and*

$$\overline{\mathrm{Eff}}_1(X) = \overline{\mathrm{Eff}}_1(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

The rays $\mathbb{R}_{\geq 0}[C_i]$ only accumulate along the hyperplane $(K_X+\Delta)^{\perp}$.

The $K_X+\Delta$-negative faces of $\overline{\mathrm{Eff}}_1(X)$ are in bijection with $K_X+\Delta$-negative contractions.

Note that if (X, Δ) is klt and $K_X+\Delta$ is big, then a perturbation argument shows that there are only finitely many $K_X+\Delta$-negative minimal rays.

As mentioned before, analyzing outcomes of the MMP (up to isomorphism) is essentially the same as counting regions in the pseudo-effective cone corresponding to the pullback of the nef cone on the various results of the program. Thus it will still be useful to phrase results in terms of the structure of cones.

We now discuss the final outcomes of the MMP in the three cases (i), (ii), (iii). Often one can interpret the set of outcomes via a "cone theorem" describing the structure of the cone of curves. When $K_X+\Delta$ is big, the finiteness noted above persists through the entire MMP process:

Theorem 5.14 ([BCHM10], Corollary 1.1.5)**.** *Let (X, Δ) be a klt pair with $K_X+\Delta$ big. Let T be a subcone of $\overline{\mathrm{Eff}}^1(X)$ over a compact set. Suppose that every ray in T is generated by a pair (X, Δ) satisfying Condition 5.4. Then there are only finitely many birational contractions defined by running the MMP for the corresponding divisors.*

In other words, the region in $\overline{\mathrm{Eff}}^1(X)$ consisting of divisors which are proportional to a pair as in Condition 5.4 looks "Mori Dream Space-like", in the sense that it admits a chamber decomposition satisfying the same properties as a Mori Dream Space. (See [HK00] or [CL13] for a discussion of this viewpoint.) This is particularly useful for log Fano varieties, since it shows that they are Mori Dream Spaces.

If we pass to the situation when $K_X + \Delta$ lies on the pseudo-effective boundary the picture becomes much more complicated. [Les15] gives an example of a non-uniruled terminal threefold with infinitely many K_X-negative extremal rays; any resolution of this variety will admit an infinite set of minimal model outcomes. Nevertheless the set of MMP outcomes conjecturally has a rich structure. Given an effective divisor Δ on X, we denote by

- $\mathrm{Aut}(X, \Delta)$ the automorphisms of X which preserve Δ,
- $\mathrm{PsAut}(X, \Delta)$ the birational maps $\psi : X \dashrightarrow X$ which are an isomorphism in codimension 1 and which preserve Δ.

We will only phrase the conjecture in the case when $K_X + \Delta$ is numerically trivial; for the essentially identical relative statement which addresses all case (ii) morphisms, see [Tot08].

Conjecture 5.15 (Kawamata-Morrison Cone Conjectures)**.** *Let* (X, Δ) *be a klt pair such that* $K_X + \Delta$ *is numerically trivial.*

(1) There is a finite rational polyhedral cone Π *which is a fundamental domain for the action of* $\mathrm{Aut}(X, \Delta)$ *on the effective nef cone (that is, the intersection of* $\mathrm{Nef}^1(X)$ *with the cone generated by all effective divisors).*
(2) There is a finite rational polyhedral cone Π' *which is a fundamental domain for the action of* $\mathrm{PsAut}(X, \Delta)$ *on the effective movable cone (that is, the intersection of* $\mathrm{Mov}^1(X)$ *with the cone generated by all effective divisors).*

These conjectures also predict that there are only finitely many equivalence classes under the group action of faces of the cones corresponding to actual morphisms or marked small $\mathbb{Q}$-factorial modifications.

Warning 5.16. These conjectures are false for log canonical pairs. [Tot08] gives the example where X is $\mathbb{P}^2$ blown up at 9 very general points and Δ is the strict transform of the elliptic curve through the points. In this case the cone is not rational polyhedral but the automorphism group of X is trivial.

The Kawamata-Morrison Cone Conjectures are known for abelian varieties ([PS12b]) and (essentially) for hyperkähler manifolds ([AV14] and [MY14] for 1, [Mar11] for 2). They are also known completely in dimension ≤ 2 ([Ste85], [Nam85], [Tot10]) and the relative versions are known in dimension 3 over a positive dimensional base ([Kaw97b]). Many additional special cases have been proved in [PS12a], [CPS14], [Ogu01], [Ogu14], [CO11], [LP13], [Bor91], [Wil94], [Ueh04], [Sze99], [Zha14], etc.

Finally, in the case when $K_X + \Delta$ is not pseudo-effective, we should instead look for all possible birational Mori fiber space structures. Again, this can be interpreted as a structure theorem for a suitable cone of curves. Building on [Bat92], [Ara10] we have:

Theorem 5.17 ([Leh12])**.** *Let (X, Δ) be a dlt pair. There are countably many $(K_X + \Delta)$-negative movable curves C_i such that*

$$\overline{\mathrm{Eff}}_1(X)_{K_X+\Delta\geq 0} + \mathrm{Nef}_1(X) = \overline{\mathrm{Eff}}_1(X)_{K_X+\Delta\geq 0} + \overline{\sum \mathbb{R}_{\geq 0}[C_i]}.$$

The rays $\mathbb{R}_{\geq 0}[C_i]$ only accumulate along hyperplanes that support both $\mathrm{Nef}_1(X)$ and $\overline{\mathrm{Eff}}_1(X)_{K_X+\Delta\geq 0}$.

The birational equivalence classes of birational Mori fiber space structures are in bijection with the faces of this cone which admit a supporting hyperplane not intersecting $\overline{\mathrm{Eff}}_1(X)_{K_X+\Delta\geq 0}$.

The presence of the term $\overline{\mathrm{Eff}}_1(X)_{K_X+\Delta\geq 0}$ on both sides has the effect of "rounding out the cone" and can not be removed. However, Batyrev conjectures a stronger statement: the accumulation of rays only occurs along the hyperplane $(K_X + \Delta)^{\perp}$.

Remark 5.18. Again, one can pose a harder question: if we fix a smooth projective variety X, what is the set of all minimal models X' birationally equivalent to X equipped with a birational contraction $X \dashrightarrow X'$? We identify $\psi : X \dashrightarrow X'$ and $\phi : X \dashrightarrow X''$ only if the rational map $\psi^{-1} \circ \phi$ extends to an isomorphism. This is the same as counting the number of distinct subcones $\psi^{-1}\mathrm{Amp}^1(X') \subset \overline{\mathrm{Eff}}^1(X)$ defined by birational contractions to minimal models.

This set will usually be larger than the set of runs of the MMP, since we now allow flops as well as K_X-negative contractions. An important example of [Rei83] shows that the number of minimal models marked with a rational map can be infinite (and a minimal model can admit an infinite sequence of flops). Note however that the Kawamata-Morrison Cone Conjectures also predict a nice structure for this set.

5.2.3. *Structure of MMP outcomes.* Given two outcomes of the $(K_X + \Delta)$-MMP

$$\phi_1 : X \dashrightarrow Y_1 \qquad \text{and} \qquad \phi_2 : X \dashrightarrow Y_2,$$

it is natural to ask for a relationship between Y_1 and Y_2. Since both are constructed from X by a sequence of simple steps, it is natural to wonder whether the induced rational map $Y_1 \dashrightarrow Y_2$ can also be factored as a sequence of simple steps. As always we will need to discuss the various cases separately.

First suppose we are in case (ii) or (iii), so the final outcome of the MMP is a minimal model. Given two minimal models $\phi_1 : X \dashrightarrow Y_1$ and $\phi_2 : X \dashrightarrow Y_2$, we would like to factor the induced map $\psi : Y_1 \dashrightarrow Y_2$ into a series of simple steps.

Theorem 5.19 ([Kaw08] Theorem 1)**.** *Let (X, Δ) and (X', Δ') be two terminal $\mathbb{Q}$-factorial pairs where Δ, Δ' are $\mathbb{Q}$-divisors such that there is a birational map $\phi : X \dashrightarrow X'$ sending $\phi_*\Delta = \Delta'$. Suppose that $K_X + \Delta$ and $K_{X'} + \Delta'$ are nef. Then ϕ decomposes into a sequence of flops.*

Now suppose that $K_X + \Delta$ is not pseudo-effective and that we are given two Mori fiber space outcomes of the $(K_X + \Delta)$-MMP:

$$\begin{array}{lll} \phi_1 : X \dashrightarrow Y_1 & \text{with contraction} & \pi_1 : Y_1 \to Z_1 \\ \phi_2 : X \dashrightarrow Y_2 & \text{with contraction} & \pi_2 : Y_2 \to Z_2 \end{array}$$

As discussed above, one would like to factor the rational map $\psi : Y_1 \dashrightarrow Y_2$ into a number of "basic steps." We should of course also keep track of the Mori fibrations π_1, π_2. This picture is modeled on dimension 2, where any two minimal ruled surfaces over a curve are connected by a series of elementary transformations. In this case, the study of these "basic steps" is known as the Sarkisov program. The main goal of the program was accomplished in [HM13], which decomposes any two outcomes of the MMP as above into a finite sequence of Sarkisov links. These links come in four types, where each step is characterized by outcomes of the two-ray game. We refer to [HM13] for the precise statement. See also [Kal13] for a discussion of the relations in the Sariskov program.

5.3. **Relating the geometry of (X, Δ) to the canonical model.** Given a canonical model Z of (X, Δ), it is natural to ask how the geometry of (X, Δ) is related to the geometry of Z. Even when $\Delta = 0$ and X and Z are smooth, it is too much to hope for the positivity of K_X and K_Z to be directly related due to the presence of singularities of the canonical map. Instead, one must account for the discriminant locus of Z by including a boundary divisor Δ_Z.

For simplicity, we will assume right away that $\pi : X \to Z$ is a morphism with connected fibers such that $K_X + \Delta$ is trivial along the fibers (so the canonical model map is a special example). To obtain a clean statement one first must pass to a birational model of π to ensure that all the singularities of π are detected in codimension 1 on the base.

The following statement builds on Kodaira's canonical bundle formula for elliptic fibrations and is due to [Kaw97b], [Kaw98], [FM00], [Amb04], [Amb05], [FG14b], [Fuj15]. See [Fuj15, Section 3] for an extensive discussion of this result and the inputs of various authors. The log canonical version is due to [FG14b] and generalizes results of Ambro in the klt case.

Theorem 5.20 ([FG14b])**.** *Suppose that (X, Δ) is an lc pair and that $\pi : X \to Z$ is a morphism with connected fibers such that $(K_X + \Delta)|_F \sim_{\mathbb{Q}} 0$ for a general fiber F of π. Then there exists:*

- *a log smooth model (X', Δ') of (X, Δ), where we have $\mu : X' \to X$ birational and $K_{X'} + \Delta' = \mu^*(K_X + \Delta) + E$ for an effective μ-exceptional divisor E,*
- *a smooth variety Z' and an effective divisor $\Delta_{Z'}$*
- *a morphism $\pi' : X' \to Z'$ birationally equivalent to π,*
- *a $\mathbb{Q}$-Cartier divisor B on X' which we express as the difference $B = B^+ - B^-$ of effective divisors with no common components*

such that

(1) $K_{X'} + \Delta' = \pi'^*(K_{Z'} + \Delta_{Z'}) + B$,
(2) *there is a positive integer* b *(clearing denominators) such that*

$$H^0(X', mb(K_{X'} + \Delta')) = H^0(Z', mb(K_{Z'} + \Delta_{Z'}))$$

for any positive integer m,
(3) B^- *is* f'*-exceptional and* μ*-exceptional,*
(4) $\pi'_*\mathcal{O}_{X'}(\lfloor lB^+ \rfloor) = \mathcal{O}_{Z'}$ *for every positive integer* l.

Furthermore, the divisor $\Delta_{Z'}$ *admits a decomposition* $\Delta_{Z'} \sim_{\mathbb{Q}} D+M$, *where*

- D *is the "discriminant part": it is effective and is explicitly determined as in* [FM00] *via the log canonical thresholds of* $K_{X'}+\Delta'$ *over prime divisors in* Z'. *The pair* (Z', D) *is lc, and is klt if* (X, Δ) *is klt.*
- M *is the "moduli part": it is nef.*

If for a general fiber F *of* π *the pair* $(F, \Delta|_F)$ *has a good minimal model, then by choosing* Z' *appropriately we may ensure that* M *is nef and abundant. In this case, if* (X, Δ) *is klt then we may ensure* $(Z', \Delta_{Z'})$ *is klt (by choosing* $\Delta_{Z'}$ *appropriately in its* $\mathbb{Q}$*-linear equivalence class).*

Some remarks are in order. There are two ways in which a divisor can be "trivial" with respect to a map: it can be contracted to a locus of codimension ≥ 2, or the pushforward of the corresponding sheaf can be trivial. In our situation the negative B^- satisfies the stronger first property and can thus essentially be ignored when comparing sections of divisors on X' and Z'. The positive part B^+ satisfies the weaker second property and thus does not affect the comparison of sections. So the conclusion (1) expresses a very tight relationship between the pair upstairs and the pair downstairs.

The main point of the theorem is to understand the positivity of M. In Kodaira's original formula, the moduli part of the elliptic fibration was pulled back from the moduli space of pointed elliptic curves. Morally speaking, in general the moduli part M on the base Z' should be the pullback of an ample divisor over a rational morphism $g : Z' \dashrightarrow \mathcal{H}$ to a "universal parameter space" $\mathcal{H}$ for the fibers of the map. So in fact, one expects the moduli part to be birationally semiample instead of just nef (or nef and abundant). It is an important problem to clarify this potential link to the geometry of the map.

A closely related approach is given by Campana's theory of orbifolds; see [Cam11] and the references therein.

6. Conditional results

6.1. **Termination of flips.** [HMX14] establishes an inductive procedure for establishing termination of flips (and hence the existence of minimal models). Building on [Bir07], [dFEM10] [dFEM11], the paper proves:

Theorem 6.1 ([HMX14] Corollary 1.2)**.** *Assume termination of flips in dimension $\leq n-1$. Then termination of flips holds for any klt pair (X,Δ) of dimension n such that $K_X+\Delta$ is numerically equivalent to an effective divisor.*

Many similar but weaker statements have been proven by [Bir10a], [Bir11], [Bir12b]; for example, [Bir11, Corollary 1.7] proves the analogous statement with "termination of flips" replaced by "existence of minimal models".

Note that the condition on $K_X+\Delta$ in Theorem 6.1 would follow from various flavors of the Abundance Conjecture. (In particular Klt Termination of flips$_{n-1}$ + Klt Non-vanishing$_n$ $\implies$ Klt Termination of flips$_n$.) Thus Abundance is in some sense the only missing piece of the minimal model program, and we focus on this conjecture henceforth.

6.2. **Abundance conjecture.** One way to try to prove the main conjectures of the MMP by induction is to choose a divisor on X and to try to "lift sections" from this divisor to all of X using vanishing theorems. This approach has been very successful under certain conditions. Since the big case is reasonably well-understood, we only mention statements which are potentially applicable to the remaining case (ii). These results are far too varied to summarize here, so we only give a couple easily-stated results which illustrate recent developments. The first is the key to Siu's proof of invariance of plurigenera.

Theorem 6.2 ([Pău07] Theorem 1)**.** *Let $f: X \to Y$ be a smooth morphism with connected fibers. Let L be a divisor on X and let h_L be a singular hermitian metric on $\mathcal{O}_X(L)$ with positive curvature. Suppose that the restriction of h_L to a fiber X_0 is well-defined. Then for any positive integer m, the restriction map*

$$H^0(X,\mathcal{O}_X(mK_X+L)) \to H^0(X_0,\mathcal{O}_{X_0}(mK_X+L))$$

surjects onto sections of $\mathcal{O}_{X_0}(mK_X+L)\otimes\mathcal{I}(h_L|_{X_0})$.

Theorem 6.3 ([DHP13] Corollary 1.8)**.** *Let $(X,S+B)$ be a plt pair such that K_X+S+B is nef and is $\mathbb{Q}$-linearly equivalent to an effective divisor D satisfying $S\subset \mathrm{Supp}(D)\subset \mathrm{Supp}(S+B)$. Then the restriction map*

$$H^0(X,\mathcal{O}_X(m(K_X+S+B))) \to H^0(S,\mathcal{O}_S(m(K_X+S+B)))$$

is surjective for all sufficiently divisible integers m.

Another way to prove the main conjectures by induction is to try to "lift positivity" from the base of a morphism. There are several such theorems in the literature. Usually, one requires that the fibers of the morphism be trivial in some way.

The first such statement concerns the Iitaka fibration of $K_X+\Delta$. For this map the restriction of $K_X+\Delta$ to the fibers is "sectionally trivial." While the following theorem of [Lai10] is only stated in the terminal case, it is also true for klt pairs.

Theorem 6.4 ([Lai10] Theorem 4.4)**.** *Let (X, Δ) be a klt pair. Suppose that $\kappa(X, \Delta) \geq 0$ and (for simplicity) the Iitaka fibration for $K_X + \Delta$ is a morphism $f : X \to Y$. If $(F, \Delta|_F)$ has a good minimal model for the general fiber of f, then (X, Δ) has a good minimal model.*

Another version assumes that the restriction of $K_X + \Delta$ to the fibers of the morphism are "numerically trivial." Statements of this kind appear in [Amb04], [Amb05]. Building on these, we have:

Theorem 6.5 ([GL13] Theorem 1.3)**.** *Let (X, Δ) be a klt pair. Suppose that $f : X \to Y$ is a surjective morphism with connected fibers such that $\nu((K_X + \Delta)|_F) = 0$ for a general fiber F. Then there is a birational model Y' of Y and a klt pair $(Y', \Delta_{Y'})$ such that (X, Δ) has a good minimal model if and only if $(Y', \Delta_{Y'})$ has a good minimal model.*

An important advantage is that there is no assumption on the existence of log pluricanonical sections or on termination of flips.

7. Additional results

In this subsection we briefly outline results which do not "fit into the narrative" but are absolutely essential for understanding the modern MMP. I will focus on directions of current research.

7.1. **Boundedness results.** There are many kinds of boundedness statements arising from the minimal model program. Such statements are closely related to the boundedness of the moduli functor for stable pairs. We refer to [HM10b] for an informative discussion of such results and the recent results of Hacon, M^cKernan, and Xu for the precise statements.

7.2. **Moduli of stable pairs.** The modern approach to constructing moduli of stable pairs relies heavily on the minimal model program. We refer to the excellent survey paper [Kov09] and to the upcoming book of Kollár for more technical details.

7.3. **Foliations.** Surprisingly, foliations seem to exhibit a beautiful structure analogous to the structure for varieties provided by MMP. There are two kinds of results in this direction.

First, Brunella, McQuillan, and Mendes have proved some MMP-type structure results for foliations on surfaces. The next cases are under active investigation and there are many interesting open questions. We refer to [Bru15] for an overview of this area.

Second, one can deduce the existence of rational curves from the presence of "negative" foliations. We refer to [BM01] and [KSCT07] and the subsequent literature for this important circle of ideas.

7.4. **Characteristic p.** It is interesting to develop the MMP over arbitrary fields. Recent work has focused on algebraically closed fields of positive characteristic.

The first step is to understand how to generalize the technical results underlying the MMP – namely, vanishing theorems. As is well-known, even the Kodaira vanishing theorem can fail in characteristic p. Nevertheless, there is a large body of work which successfully establishes some good analogues in positive characteristic. This area is too broad to summarize here; we instead refer to the recent survey paper [ST12] and the references therein.

The second step is to apply these results to obtain a MMP theory. We refer to recent work of Birkar, Hacon, Tanaka, Xu, and their collaborators for such applications.

7.5. **Existence of rational curves.** Mori's celebrated bend-and-break theorem relates the negativity of K_X against curves with the existence of rational curves.

Theorem 7.1 ([MM86] Theorem 5)**.** *Let X be a smooth variety and let A be an ample Cartier divisor on X. Suppose that C is an irreducible curve on X such that $K_X \cdot C < 0$. Then through every point of C there is a rational curve T on X satisfying $A \cdot T \leq 2\dim(X)\frac{A\cdot C}{-K_X\cdot C}$.*

It is important to understand the behavior of rational curves on singular varieties as well. In this direction, we have:

Theorem 7.2. [Kaw91, Theorem 1] *Let (X, Δ) be a klt pair. Every $K_X+\Delta$-negative extremal ray of $\overline{\mathrm{Eff}}_1(X)$ is spanned by the class of a rational curve C satisfying $0 < -(K_X + \Delta) \cdot C \leq 2\dim X$.*

Since many birational maps can be interpreted using the minimal model program, this shows the existence of rational curves in a wide variety of situations. There are related results due to [HM07], [Tak08], [BBP13].

An important open problem is:

Question 7.3. Suppose that (X, Δ) is a klt pair such that $K_X + \Delta$ is not pseudo-effective. Is there a rational curve C through a very general point of X such that $(K_X + \Delta) \cdot C < 0$?

By running the MMP to obtain a variety X' with a Mori fiber space structure, it is clear that X' is covered by rational curves C satisfying $(K_{X'}+\Delta') \cdot C < 0$. However, it is not at all clear whether the preimage of these curves on X satisfy the desired condition. Even the surface case, which was settled by [KM99], is quite difficult and requires many new techniques.

7.6. **MMP for quasi-projective varieties.** Suppose that U is a quasi-projective variety. By Hironaka's resolution of singularities, U admits a projective completion X such that the complement of U is a simple normal crossing divisor. An important idea of Iitaka ("Iitaka's philosophy") is that we can understand the geometry of U by studying the log pair (X, Δ). Often (but not always!) a theorem for projective varieties has an analogue for quasi-projective varieties: we replace projective invariants – plurigenera, rational curves, the cotangent bundle – by their log versions – log plurigenera, log rational curves, the log cotangent bundle.

Recently, there has been new progress towards establishing Iitaka's philosophy. We will mention only the most recent developments. For understanding log rational curves, log degenerations have been a key tool: see [KM99], [CZ14], [CZ15]. Another approach is to study the log cotangent bundle: see Campana's theory of orbifolds (for example [Cam11]) and [Zhu15]. Finally, [GKKP11] and [GKP14] discuss when forms extend from an open subset to the compactification in the klt setting.

7.7. **Running the MMP for moduli spaces.** Suppose that X is a moduli space (for example, the moduli space of stable curves or a Hilbert scheme of points on a surface). By running the MMP, we obtain a special sequence of birational models of X connected by divisorial contractions and flips. If we construct X using GIT, one can obtain a sequence of birational models by varying the linearization. Surprisingly, these birational models often seem to admit interpretations as moduli spaces as well. An interesting and exciting field, initiated by Hasset and Keel for $\overline{M}_{g,n}$, is to explicitly construct the moduli problems for the birational models constructed abstractly by the MMP. Another important setting is moduli spaces of sheaves arising from stability conditions in the derived category.

The literature is far too vast to survey here; we direct readers to the survey paper [FS13] for moduli spaces of curves and [ABCH13] and [BM14] for the use of stability conditions in studying Hilbert schemes of points on a surface.

7.8. **MMP and derived categories.** An interesting idea originating from the work of Bondal and Orlov ([BO]) is that the birational structure of the MMP should be reflected on the level of derived categories. More precisely, the contractions constructed by the cone theorem should naturally yield semi-orthogonal decompositions of the derived category, and flops should yield a derived equivalence of some kind. Furthermore, the different models should correspond to the variation of a stability condition. [Kaw09] gives a good introduction to the questions and technical difficulties of the area.

In addition to a number of special examples, progress has been made for surfaces (see for example [Tod13], [Tod14]), threefolds (see [Bri02] and the many subsequent generalizations, and [Tod13]) and for toric varieties (see [Kaw06], [Kaw13a]). There are also interesting connections to noncommutative algebras; see for example [IW14].

7.9. **Singularities, the dual complex, and Berkovich spaces.** Suppose that $0 \in X$ is a singularity and that $\phi : Y \to X$ is a birational map such that the preimage of 0 is a simple normal crossing divisor E. One can associate to E its dual complex $D(E)$ encoding how the components of E intersect. While the dual complex depends on the choice of resolution, certain topological features of the dual complex (such as the homotopy type) turn out to be independent of the resolution chosen (see [Pay13]). Interestingly, certain algebraic features of the singularity are captured by the topological properties of $D(E)$. This relationship has been studied for a long time for the dual graphs of resolutions of surface singularities.

New advances in the MMP have opened up the study of higher dimension varieties. By using the MMP to systematically contract components of E, [dFKX12] (following up on [Kol13a], [Kol14], [KK14]) identifies an "optimal" dual complex which is well-defined up to piecewise linear homomorphism. Building on this viewpoint, [NX13] uses the MMP to study Berkovich spaces (which exhibit structure similar to a limit of dual complexes). This circle of ideas has found further applications in [NX14] and [KX15].

7.10. **Rational points over number fields.** Suppose that X is a smooth projective variety over a number field. As discussed in Section 2, conjecturally the behavior of rational points on X is constrained by the Kodaira dimension of X. Thus the minimal model program should be an essential tool for analyzing rational points. However, there are currently not many number-theoretic results using the full strength of the MMP, mainly due to the difficulty of the area. Even for surfaces (for which the MMP is comparatively easy), the behavior of rational points is still quite far from being understood.

Recently the MMP has found interesting applications to Manin's Conjecture. Suppose that X is a Fano variety and that $\mathcal{O}_X(L)$ is an adelically metrized ample line bundle inducing a height function H on the points of X. Manin's Conjecture predicts that (after a finite base change) the number of rational points on X of bounded H-height is controlled by certain geometric invariants associated to X and L. Building on [BT98] and [HTT15], [LTT15] uses the MMP to systematically analyze these geometric invariants. The results indicate that often one will need to remove a thin set of points, rather than the points in a closed subset, for Manin's Conjecture to hold.

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