VOLUME-TYPE FUNCTIONS FOR NUMERICAL CYCLE CLASSES

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Abstract. A numerical equivalence class of k-cycles is said to be big if it lies in the interior of the closed cone generated by effective classes. We construct analogues for arbitrary cycle classes of the volume function for divisors which distinguishes big classes from boundary classes.

1. Introduction

Let X be an integral projective variety over an algebraically closed field. We will let $N_k(X)_{\mathbb{Z}}$ denote the group of numerical classes of k-cycles on X and $N_k(X) := N_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$. The pseudo-effective cone $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$ is defined to be the closure of the cone generated by all effective k-cycles; this cone encodes the homology of all k-dimensional subvarieties of X and is an important tool in higher dimensional geometry. While the pseudo-effective cone has been thoroughly studied for divisors and curves, much less is known about cycles in general.

The most basic problem concerning $\overline{\mathrm{Eff}}_k(X)$ is to find geometric criteria that distinguish classes on the interior of the cone – known as big classes – from boundary classes. This question has interesting links to a number of other geometric problems (see for example [Voi10, Theorem 0.8], [DJV13, Remark 6.4], and Section 6.2). Our goal is to give several geometric characterizations of big cycles similar to well-known criteria for divisors.

Example 1.1. One might expect that a subvariety with "positive" normal bundle will have a big numerical class. However, [Voi10, Example 2.4] shows that even a subvariety with an ample normal bundle need not be big, indicating the need for a different geometric approach.

An important tool for understanding big divisor classes is the volume function. The volume of a Cartier divisor L is the asymptotic rate of growth of dimensions of sections of L. More precisely, if X has dimension n,

$$\mathrm{vol}(L) := \limsup_{m\to\infty} \frac{\dim H^0(X, \mathcal{O}_X(mL))}{m^n/n!}.$$

It turns out that the volume is an invariant of the numerical class of L and satisfies many advantageous geometric properties. On a smooth variety X,

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divisors with positive volume are precisely the divisors with big numerical class.

Our first generalization of the volume function is geometric in nature. One can interpret the volume of a divisor L as an asymptotic measurement of the number of general points contained in members of $|mL|$ as m increases. [DELV11] suggests studying a similar notion for arbitrary cycles. Given a class $\alpha \in N_k(X)_{\mathbb{Z}}$, we define

$$\mathrm{mc}(\alpha) = \max \left\{ b \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c} \text{any } b \text{ general points of } X \text{ are contained} \\ \text{in an effective cycle of class } \alpha \end{array} \right\}.$$

We would then like to understand the asymptotic behavior of $\mathrm{mc}(m\alpha)$ as m increases. The "expected" growth behavior can be predicted by considering complete intersections in $\mathbb{P}^n$: an intersection of $n-k$ general elements of $|\mathcal{O}(d)|$ has dimension k, degree d^{n-k}, and will contain about $\frac{1}{n!}d^n$ general points. Thus one expects (and can easily verify) that $\mathrm{mc}(m\alpha) \sim Cm^{\frac{n}{n-k}}$ for some constant C. The mobility function identifies the best possible constant C. (See Definition 5.3 for a more precise formulation.)

Definition 1.2. Let X be an integral projective variety of dimension n and suppose $\alpha \in N_k(X)_{\mathbb{Z}}$ for $0 \leq k < n$. The mobility of α is

$$\mathrm{mob}(\alpha) = \limsup_{m \to \infty} \frac{\mathrm{mc}(m\alpha)}{m^{\frac{n}{n-k}}/n!}$$

The mobility function shares many of the important properties of the volume function for divisors. Our first theorem shows that bigness is characterized by positive mobility, confirming [DELV11, Conjecture 6.5].

Theorem 1.3. *Let X be an integral projective variety. Then* mob *extends uniquely from $N_k(X)_{\mathbb{Z}}$ to a continuous homogeneous function on $N_k(X)$. In particular, $\alpha \in N_k(X)$ is big if and only if* $\mathrm{mob}(\alpha) > 0$.

Remark 1.4. Theorem 1.3 has analogues in the setting of other equivalence relations on cycles. The main step in the proof of Theorem 1.3 is to show that if $\mathrm{mob}(\alpha) > 0$ then α is big; the proof does not use any special feature of $N_k(X)$ besides the ability to intersect against Cartier divisors. To prove the converse implication, one needs to work with an equivalence relation whose classes form a finitely generated group.

For example, suppose X is an integral projective variety over $\mathbb{C}$. The statement of Theorem 1.3 holds for the subspace $N'_k(X) \subset H_{2k}(X, \mathbb{R})$ spanned by classes of cycles and for the homological analogue of the mobility function.

The following examples illustrate how the mobility captures basic geometric information about a class. They also show that the mobility is difficult to compute; however, see Question 1.14 for a conjectural intersection-theoretic description.

Example 1.5. If X is a smooth projective variety and L is a Cartier divisor then $\mathrm{mob}([L]) = \mathrm{vol}(L)$ as shown in Example 5.7. However, the definition of mobility makes sense for a numerical Weil divisor class on any integral projective variety. [FL14] verifies that if X admits a resolution of singularities $\phi : X' \to X$, then the mobility of a divisor class α is

$$\mathrm{mob}(\alpha) = \sup_{\beta \in N_k(X'),\, \phi_*\beta = \alpha} \mathrm{vol}(\beta)$$

Example 1.6. Let ℓ denote the class of a line on $\mathbb{P}^3$. The mobility of ℓ is determined by an enumerative question: what is the minimal degree of a curve in $\mathbb{P}^3$ going through b general points?

It turns out that the answer to this question is not known (even asymptotically as the degree increases). [Per87] conjectures that the "optimal" curves are complete intersections of two divisors of equal degree, which would imply that $\mathrm{mob}(\ell) = 1$. We discuss this interesting question in more depth in Section 6.1.

Example 1.7. We define the rational mobility of a class $\alpha \in N_k(X)_{\mathbb{Z}}$ in a similar way by counting the number of general points lying on cycles in a fixed rational equivalence class inside of α (see Definition 5.3). Rational mobility is interesting even for 0-cycles.

Let $A_0(X)$ denote the set of rational equivalence classes of 0-cycles on X. Recall that $A_0(X)$ is said to be representable if the addition map $X^{(r)} \to A_0(X)_{\deg(r)}$ is surjective for some $r > 0$. In Section 6.2 we show for normal varieties over $\mathbb{C}$ that $A_0(X)$ is representable if and only if the rational mobility of the class of a point is the maximal possible value $(\dim X)!$.

Example 1.8. Suppose that $\alpha \in \overline{\mathrm{Eff}}_k(X)_{\mathbb{Z}}$ lies on the boundary of the pseudo-effective cone. Theorem 1.3 implies that the growth of $\mathrm{mc}(m\alpha)$ is bounded above by $Cm^{r/n-k}$ for some constant $r < n$. It is natural to ask: what is the optimal value of r? (This is the analogue of the Iitaka dimension for divisors.)

The first case to consider is when X is a Grassmannian and α is the class of a Schubert cycle on X. It turns out that often the optimal exponent r can be calculated in this case. A closely related property known as Schur rigidity has been extensively studied – see for example [Wal97], [Bry05], [Hon05], [Hon07], [RT12], and [Rob13]. Schur rigidity of α implies that the optimal constant is $r = 1$.

Example 1.9. In contrast to the situation for divisors, it is possible for a subvariety V to have big numerical class even if no multiple of V moves in an algebraic family. For example, [FL82] constructs a surface S with ample normal bundle in a fourfold X such that no multiple of S moves in X. [Pet09, Example 4.10] and a calculation of Fulger verify that $[S] \in N_2(X)$ is big. This highlights the importance of working with an appropriate equivalence relation.

Our next generalization of the volume function comes from intersection theory. Recall that for an ample divisor A we have $\mathrm{vol}(A) = A^n$; more generally, the volume of a big divisor can be computed as an intersection of an ample divisor on a birational model via a Fujita approximation. Thus the volume of a divisor is in a sense an "intersection theoretic" quantity.

Recently [Xia15] has defined an interesting positivity function for curves: for an integral projective variety X of dimension n and for $\alpha \in \overline{\mathrm{Eff}}_1(X)$ set

$$\widehat{\mathrm{vol}}(\alpha) = \inf_{A \text{ big and nef } \mathbb{R}\text{-divisor}} \left(\frac{A \cdot \alpha}{\mathrm{vol}(A)^{1/n}} \right)^{n/n-1}.$$

The function $\widehat{\mathrm{vol}}$ satisfies all the expected properties of a volume function and gives a rich theory for curves parallel to the divisor case. The two definitions can be unified in the following way. For $\alpha, \beta \in N_k(X)$, we write $\alpha \preceq \beta$ if $\beta - \alpha \in \overline{\mathrm{Eff}}_k(X)$.

Definition 1.10. Let X be an integral projective variety of dimension n and suppose $\alpha \in N_k(X)$ for $0 \leq k < n$. Define

$$\widehat{\mathrm{vol}}(\alpha) := \sup_{\phi, A} \{A^n\}$$

as $\phi : Y \to X$ varies over all birational models of X and as A varies over all big and nef $\mathbb{R}$-Cartier divisors on Y such that $\phi_*[A^{n-k}] \preceq \alpha$.

When α is not in the interior of the pseudo-effective cone, the set of suitable divisors A is empty and this expression should be interpreted as returning 0.

Example 1.11. When X is smooth and L is a Cartier divisor, the theory of Fujita approximations (extended by [Tak07] to arbitrary characteristic) shows that $\widehat{\mathrm{vol}}([L]) = \mathrm{vol}(L)$.

Example 1.12. Suppose that B is a big and nef $\mathbb{R}$-Cartier divisor and set $\alpha = [B^{n-k}]$. Then Example 7.3 shows that $\widehat{\mathrm{vol}}(\alpha) = \mathrm{vol}(B)$. Many more examples for curve classes are computed in [LX15].

$\widehat{\mathrm{vol}}$ satisfies the most basic properties of a volume-type function.

Theorem 1.13. *Let X be an integral projective variety. Then $\widehat{\mathrm{vol}}$ is a continuous homogeneous function on $N_k(X)$. In particular, $\alpha \in N_k(X)$ is big if and only if $\widehat{\mathrm{vol}}(\alpha) > 0$.*

While this function is easier to compute than the mobility, it is unclear how it relates to the geometry of the cycles of class proportional to α. This interpretation is provided by our main question.

Question 1.14. Let X be an integral projective variety of dimension n and suppose $\alpha \in \overline{\mathrm{Eff}}_k(X)$ for some $0 < k < n$. Then is

$$\widehat{\mathrm{vol}}(\alpha) = \mathrm{mob}(\alpha)?$$

Theorem 7.6 proves the inequality $\leq$. Question 1.14 is known for divisor classes (even in the singular case), and [LX15] shows that for curves it suffices to prove the special case when $\alpha = A^{n-1}$ for an ample divisor A. For classes of intermediate codimension, one may need to broaden the definition of $\widehat{\mathrm{vol}}$ from ample divisors to other kinds of positive classes to obtain an equality.

Our final generalization of the volume function extrapolates between the two previous ones: in some examples it can be computed using intersection theory but it retains the flavor of the mobility. The key idea is that singular points of cycles should contribute more to the mobility count. This convention better reflects the intersection theory on the blow-up of the points, as the strict transform of a cycle which is singular at a point will be have larger intersection against the exceptional divisor than the strict transform of a smooth cycle.

Following a suggestion of R. Lazarsfeld, we define the weighted mobility count of a class $\alpha \in N_k(X)_{\mathbb{Z}}$ as:

$$\mathrm{wmc}(\alpha) = \max \left\{ b \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c} \text{there is a } \mu \in \mathbb{Z}_{>0} \text{ and an effective cycle of} \\ \text{class } \mu\alpha \text{ through any } b \text{ points of } X \text{ with} \\ \text{multiplicity at least } \mu \text{ at each point} \end{array} \right\}.$$

This definition has the effect of counting singular points with a higher "weight". It is designed to be compatible with the calculation of multipoint Seshadri constants – see Section 8 for details. Just as with our other constants, the expected growth rate on a variety of dimension n is $\mathrm{wmc}(m\alpha) \sim Cm^{n/n-k}$, suggesting the following definition.

Definition 1.15. Let X be an integral projective variety of dimension n and suppose $\alpha \in N_k(X)_{\mathbb{Z}}$ for $0 \leq k < n$. The weighted mobility of α is

$$\mathrm{wmob}(\alpha) = \limsup_{m\to\infty} \frac{\mathrm{wmc}(m\alpha)}{m^{\frac{n}{n-k}}}$$

The rescaling factor $n!$ is now omitted to ensure that the hyperplane class on $\mathbb{P}^n$ has weighted mobility 1.

Theorem 1.16. *Let X be an integral projective variety. Then* wmob *is a continuous homogeneous function on $N_k(X)$. In particular, $\alpha \in N_k(X)$ is big if and only if* $\mathrm{wmob}(\alpha) > 0$.

Example 1.17. Suppose that X is smooth over an uncountable algebraically closed field and that L is a Cartier divisor on X. Then Example 8.23 shows that $\mathrm{wmob}([L]) = \mathrm{vol}(L)$.

Example 1.18. Suppose that X is an integral projective variety over an uncountable algebraically closed field and that $\alpha = H^{n-k}$ where H is a big and nef $\mathbb{R}$-divisor. Then Example 8.22 shows that $\mathrm{wmob}(\alpha) = \mathrm{vol}(H)$.

[LX15] shows that there is an equality $\mathrm{wmob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$ for curve classes.

1.1. **Organization.** Section 2 reviews background material on cycles. Section 3 describes several geometric constructions for families of cycles. Section 4 analyzes the geometric properties of the mobility count. Section 5 defines mobility and proves Theorem 1.3. Section 6 discusses some examples of the mobility. Section 7 defines and analyzes $\widehat{\mathrm{vol}}$. Finally, Section 8 analyzes the weighted mobility function.

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2. Preliminaries

Throughout we work over a fixed algebraically closed field K. A variety will mean a quasiprojective scheme of finite type over K (which may be reducible and non-reduced). We will often use the following special case of [RG71, Théorème 5.2.2].

Theorem 2.1 ([RG71], Théorème 5.2.2)**.** *Let $f : X \to S$ be a projective morphism of varieties such that some component of X dominates S. There is a birational morphism $\pi : S' \to S$ such that the morphism $f' : X' \to S'$ is flat, where $X' \subset X \times_S S'$ is the closed subscheme defined by the ideal of sections whose support does not dominate S'.*

2.1. **Cycles.** Suppose that X is a projective variety. A k-cycle on X is a finite formal sum $\sum a_i V_i$ where the a_i are integers and each V_i is an integral closed subvariety of X of dimension k. The support of the cycle is the union of the V_i (with the reduced structure). The cycle is said to be effective if each $a_i \geq 0$. For a k-dimensional closed subscheme V of X, the fundamental cycle of V is $\sum m_i V_i$ where the V_i are the k-dimensional components of the reduced scheme underlying V and the m_i are the lengths of the corresponding Artinian local rings $\mathcal{O}_{V,V_i}$.

The group of k-cycles is denoted $Z_k(X)$ and the group of k-cycles up to rational equivalence is denoted $A_k(X)$. We will follow the conventions of [Ful84] in the use of various intersection products on $A_k(X)$.

[Ful84, Chapter 19] defines a k-cycle on X to be numerically trivial if its rational equivalence class has vanishing intersection with every weighted homogeneous degree-k polynomial in Chern classes of vector bundles on X. Two cycles are numerically equivalent if their difference is numerically trivial. We let $N_k(X)_{\mathbb{Z}}$ denote the abelian group of numerical equivalence classes of k-cycles on X. By [Ful84, Example 19.1.4] $N_k(X)_{\mathbb{Z}}$ is a finitely generated free abelian group.

We also define

$$N_k(X)_{\mathbb{Q}} := N_k(X)_{\mathbb{Z}} \otimes \mathbb{Q}$$
$$N_k(X) := N_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$$

Thus $N_k(X)$ is a finitely generated $\mathbb{R}$-vector space and there are natural injections $N_k(X)_{\mathbb{Z}} \hookrightarrow N_k(X)_{\mathbb{Q}} \hookrightarrow N_k(X)$. We denote the dual group of $N_k(X)_{\mathbb{Z}}$ by $N^k(X)_{\mathbb{Z}}$ and the dual vector spaces of $N_k(X)_{\mathbb{Q}}$ and $N_k(X)$ by $N^k(X)_{\mathbb{Q}}$ and $N^k(X)$ respectively.

[Ful84] defines the Chern class of a vector bundle $c_i(E)$ as an operation $A_k(X) \to A_{k-i}(X)$. It follows formally from the definition that Chern classes descend to maps $N_k(X) \to N_{k-i}(X)$.

Suppose that $f : Z \to X$ is an l.c.i. morphism of codimension d. Then [Ful84, Example 19.2.3] shows that the Gysin homomorphism $f^* : A_k(X) \to A_{k-d}(Z)$ descends to numerical equivalence classes. We will often use this fact when Z is a Cartier divisor on X to obtain maps $f^* : N_k(X) \to N_{k-1}(Z)$.

Convention 2.2. When we discuss k-cycles on an integral projective variety X, we will always implicitly assume that $0 \leq k < \dim X$. This allows us to focus on the interesting range of behaviors without repeating hypotheses.

For a cycle Z on X, we let $[Z]$ denote the numerical class of Z, which can be naturally thought of as an element in $N_k(X)_{\mathbb{Z}}$, $N_k(X)_{\mathbb{Q}}$, or $N_k(X)$. If α is the class of an effective cycle Z, we say that α is an effective class.

Definition 2.3. Let X be a projective variety. The pseudo-effective cone $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$ is the closure of the cone generated by all classes of effective k-cycles. $\overline{\mathrm{Eff}}_k(X)$ is a full-dimensional salient cone by [FL13, Theorem 0.2]. The big cone is the interior of the pseudo-effective cone. The cone in $N^k(X)$ dual to the pseudo-effective cone is known as the nef cone and denoted $\mathrm{Nef}^k(X)$.

We say that $\alpha \in N_k(X)$ is pseudo-effective (resp. big) if it lies in the pseudo-effective cone (resp. big cone), and $\beta \in N^k(X)$ is nef if it lies in the nef cone. For $\alpha, \alpha' \in N_k(X)$ we write $\alpha \preceq \alpha'$ when $\alpha' - \alpha$ is pseudo-effective.

For any morphism of projective varieties $f : X \to Y$, there is a pushforward map $f_* : N_k(X) \to N_k(Y)$. It is clear that $f_*(\overline{\mathrm{Eff}}_k(X)) \subset \overline{\mathrm{Eff}}_k(Y)$. There is also a formal dual $f^* : N^k(Y) \to N^k(X)$ that preserves nefness.

The following lemmas record some basic properties of pseudo-effective cycles.

Lemma 2.4 ([FL13], Corollary 3.20)**.** *Let $f : X \to Y$ be a surjective morphism of integral projective varieties. Then $f_* \overline{\mathrm{Eff}}_k(X) = \overline{\mathrm{Eff}}_k(Y)$.*

Lemma 2.5. *Let X be an integral projective variety.*

(1) *If A is a nef Cartier divisor then $\cdot A : N_k(X) \to N_{k-1}(X)$ takes $\overline{\mathrm{Eff}}_k(X)$ into $\overline{\mathrm{Eff}}_{k-1}(X)$.*

(2) *If $\alpha \in N_k(X)$ is a big class and A is an ample Cartier divisor then $\alpha \cdot A \in N_{k-1}(X)$ is also big.*
(3) *Let D be the support of an effective big j-cycle Z with injection $i : D \to X$. If $\alpha \in N_k(D)$ is big (for $0 \leq k \leq j$) then $i_*\alpha \in N_k(X)$ is big.*

Proof. By continuity and homogeneity it suffices to prove (1) when A is very ample. Let Z be an integral k-dimensional subvariety; for sufficiently general elements $H \in |A|$, the cycle underlying $H|_Z$ is an effective cycle of class $[Z] \cdot A$, proving (1). To see (2), write $\alpha = \alpha' + cA^{n-k}$ for some pseudo-effective class α' and some small $c > 0$. Applying (1), it suffices to note that A^{n-k+1} is a big class by [FL13, Theorem 0.2]. Similarly, to show (3) fix an ample Cartier divisor A on X and consider the class $\beta := (i^*A)^{j-k}$ in $N_k(D)$. Choose m large enough so that $m[D] \succeq [Z]$. By the projection formula $i_*(m\beta) \succeq [A^{j-k} \cdot Z]$, so that $i_*\beta$ is big on X by (2). Writing $\alpha = \alpha' + c\beta$ for a sufficiently small c, the claim follows from the fact that i_* preserves pseudo-effectiveness. $\square$

2.2. **Analytic lemmas.** We are interested in invariants constructed as asymptotic limits of functions on $N_k(X)_{\mathbb{Z}}$. The following lemmas will allow us to conclude several important properties of these functions directly from some easily verified conditions.

Lemma 2.6 ([Laz04] Lemma 2.2.38)**.** *Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function. Suppose that for any $r, s \in \mathbb{N}$ with $f(r) > 0$ we have that $f(r+s) \geq f(s)$. Then for any $k \in \mathbb{R}_{>0}$ the function $g : \mathbb{N} \to \mathbb{R} \cup \{\infty\}$ defined by*

$$g(r) := \limsup_{m \to \infty} \frac{f(mr)}{m^k}$$

satisfies $g(cr) = c^k g(r)$ for any $c, r \in \mathbb{N}$.

Remark 2.7. Although [Laz04, Lemma 2.2.38] only explicitly address the volume function, the essential content of the proof is the more general statement above. In particular k does not need to be an integer.

Lemma 2.8. *Let V be a finite dimensional $\mathbb{Q}$-vector space and let $C \subset V$ be a salient full-dimensional closed convex cone. Suppose that $f : V \to \mathbb{R}_{\geq 0}$ is a function satisfying*

(1) *$f(e) > 0$ for any $e \in C^{int}$,*
(2) *there is some constant $c > 0$ so that $f(me) = m^c f(e)$ for any $m \in \mathbb{Q}_{>0}$ and $e \in C$, and*
(3) *for every $v \in C^{int}$ and $e \in C^{int}$ we have $f(v + e) \geq f(v)$.*

Then f is locally uniformly continuous on C^{int}.

Proof. Endow V with the Euclidean metric for some fixed basis. Let $T \subset C^{int}$ be any bounded set such that

$$\inf_{p \in T, q \notin C} \|p - q\| > 0.$$

We show that f is uniformly continuous on T. Let $\mathcal{T}$ be the cone over T. There is some constant $\xi > 0$ such that if $v \in \mathcal{T}$ satisfies $\|v\| = \mu$ then the open ball $B_{\xi\mu}(v)$ satisfies $B_{\xi\mu}(v) \subset C^{int}$.

Let $M = \sup_{w \in T} f(w)$; since there is some element $x \in C^{int}$ such that $T \subset x - C^{int}$, we see that M is a positive real number.

Fix $\epsilon > 0$ and let $v \in T$. Note that the set $\left(1 - \frac{\epsilon}{M}\right)^{1/c} v + C^{int}$ contains the open ball $B_{r_v}(v)$, where

$$r_v = \xi \left(1 - \left(1 - \frac{\epsilon}{M}\right)^{1/c}\right) \|v\|$$

Every $e \in B_{r_v}(v)$ satisfies $f(e) \geq f(v) - \epsilon$. Similarly, the set $\left(1 + \frac{\epsilon}{M}\right)^{1/c} v + (-C^{int})$ contains the open ball $B_{s_v}(v)$ where

$$s_v = \xi \left(\left(1 + \frac{\epsilon}{M}\right)^{1/c} - 1\right) \|v\|$$

Every $e \in B_{s_v}(v)$ satisfies $f(e) \leq f(v) + \epsilon$.

As we vary $v \in T$, the length $\|v\|$ has a positive lower bound (since by assumption T avoids a sufficiently small neighborhood of the origin). Thus, there is some $\delta > 0$ such that $\delta < \inf_{v \in T} \min\{s_v, r_v\}$. Then $|f(v') - f(v)| \leq \epsilon$ for every v and v' in T satisfying $\|v' - v\| < \delta$, showing uniform continuity on T. By varying T, we obtain local uniform continuity on C^{int}. □

3. Families of cycles

Although there are several different notions of a family of cycles in the literature, the theory we will develop is somewhat insensitive to the precise choices. It will be most convenient to use a simple geometric definition.

Definition 3.1. Let X be a projective variety. A family of k-cycles on X consists of an integral variety W, a reduced closed subscheme $U \subset W \times X$, and an integer a_i for each component U_i of U, such that for each component U_i of U the first projection map $p : U_i \to W$ is flat dominant of relative dimension k. If each $a_i \geq 0$ we say that we have a family of effective cycles. We say that $\sum a_i U_i$ is the cycle underlying the family.

In this situation $p : U \to W$ will denote the first projection map and $s : U \to X$ will denote the second projection map unless otherwise specified. We will usually denote a family of k-cycles using the notation $p : U \to W$, with the rest of the data implicit.

For a closed point $w \in W$, the base change $w \times_W U_i$ is a subscheme of X of pure dimension k and thus defines a fundamental k-cycle Z_i on X. The cycle-theoretic fiber of $p : U \to W$ over w is defined to be the cycle $\sum a_i Z_i$ on X. We will also call these cycles the members of the family p.

Definition 3.2. Let X be a projective variety. We say that a family of k-cycles $p : U \to W$ on X is a rational family if every cycle-theoretic fiber lies in the same rational equivalence class.

Remark 3.3. Definition 3.1 has a number of deficiencies. For example, many intuitive constructions of families of cycles fail to meet the criteria: the map $\mathbb{A}^2 \times \mathbb{A}^2 \to \mathrm{Sym}^2\mathbb{A}^2$ is not flat over a characteristic 2 field as pointed out in [Kol96]. Since we are primarily interested in the "generic" behavior of families of cycles, these shortcomings are not important for us. On the other hand, the geometric flexibility of Definition 3.1 will be very useful.

For any projective variety X, [Kol96] constructs a Chow variety $\mathrm{Chow}(X)$. Any family of cycles in the sense of Definition 3.1 is also a family of cycles in the refined sense of [Kol96]; this is an immediate consequence of [Kol96, I.3.14 Lemma] and [Kol96, I.3.15 Corollary]. Thus, if $p : U \to W$ is a family of cycles, there is an induced map $\mathrm{ch} : W \dashrightarrow \mathrm{Chow}(X)$ defined on the open locus where W is normal. For a more in-depth discussion of the relationship between Definition 3.1 and [Kol96] we refer to [Leh14].

The following constructions show how to construct families of cycles from subsets $U \subset W \times X$.

Construction 3.4 (Cycle version)**.** Let X be a projective variety and let W be an integral variety. Suppose that $Z = \sum a_i V_i$ is a $(k + \dim W)$-cycle on $W \times X$ such that the first projection maps each V_i dominantly onto W. Let $W^0 \subset W$ be the (non-empty) open locus over which every projection $p : V_i \to W$ is flat and let $U \subset \mathrm{Supp}(Z)$ denote the preimage of W^0. Then the map $p : U \to W^0$ defines a family of cycles where we assign the coefficient a_i to the component $V_i \cap U$ of U.

Construction 3.5 (Subscheme version)**.** Suppose that Y is a reduced variety and that X is a projective variety. Let $\tilde{U} \subset Y \times X$ be a closed subscheme such that the fibers of the projection $p : \tilde{U} \to Y$ are equidimensional of dimension k. There is a natural way to construct a finite collection of families of effective cycles associated to the subscheme $\tilde{U}$.

Consider the image $p(\tilde{U})$ (with its reduced induced structure). Let $\{\tilde{W}_j\}$ denote the irreducible components of $p(\tilde{U})$. For each there is a non-empty open subset $W_j \subset \tilde{W}_j$ such that the restriction of p to each component of $p^{-1}(W_j)_{red}$ is flat. Since furthermore p has equidimensional fibers, we obtain a family of effective k-cycles $p_j : U_j \to W_j$ where $U_j = p^{-1}(W_j)_{red}$ and we assign coefficients so that the cycle underlying the family p_j coincides with the fundamental cycle of $p^{-1}(W_j)$. We can then replace $\tilde{U}$ by the closed subscheme obtained by taking the base change to $p(\tilde{U}) - \cup_j W_j$ and repeat. The end result is a collection of families $p_i : U_i \to W_i$ parametrizing the cycles contained in $\tilde{U}$.

If $p(\tilde{U})$ is irreducible and we are interested only in the generic behavior of the cycles in $\tilde{U}$, we can stop after the first step to obtain a single family of cycles.

It will often be helpful to replace a family $p : U \to W$ by a slightly modified version.

Lemma 3.6. *Let X be a projective variety and let $p : U \to W$ be a family of effective cycles on X. Then there is a normal projective variety W' that is birational to W and a family of cycles $p' : U' \to W'$ such that p and p' agree over an open subset of the base.*

Proof. Let $\tilde{W}$ be any projective closure of W and let $\tilde{U}$ be the closure of U in $\tilde{W} \times X$. Let $\phi : W' \to \tilde{W}$ be the normalization of a simultaneous flattening of the morphisms $\tilde{p} : \tilde{U}_i \to \tilde{W}$ for the components $\tilde{U}_i$ of $\tilde{U}$. Let U' denote the reduced subscheme of $W' \times X$ defined by the components of $\tilde{U} \times_{\tilde{W}} W'$ that dominate W'. Since the components of U' are in bijection with the components of U, we can assign to each component of U' the coefficient of the corresponding component of U. Then $p' : U' \to W'$ is a family of effective k cycles. □

Remark 3.7. It is also important to know whether a rational family $p : U \to W$ can be extended to a *rational* family over a projective closure of W (although we will not need such statements below). The arguments of [Sam56, Theorem 3] show that the subset of $\mathrm{Chow}(X)$ parametrizing cycles in a fixed rational equivalence class is a countable union of closed subvarieties. Thus we can extend families in this way when working over an uncountable algebraically closed field K.

3.1. **Geometry of families.**

Definition 3.8. Let X be a projective variety and let $p : U \to W$ be a family of effective cycles on X. We say that p is an irreducible family if U only has one component. For any component U_i of U, we have an associated irreducible family $p_i : U_i \to W$ (with coefficient a_i).

We will also need several geometric constructions.

Construction 3.9 (Flat pullback families)**.** Let $g : Y \to X$ be a flat morphism of projective varieties of relative dimension d. Suppose that $p : U \to W$ is a family of effective k-cycles on X with underlying cycle V. The flat pullback cycle g^*V on $W \times Y$ is effective and has relative dimension $(d + k)$ over W. We define the flat pullback family $g^*p : U' \to W^0$ of effective $(d + k)$-cycles on Y over an open subset $W^0 \subset W$ by applying Construction 3.4 to g^*V.

Construction 3.10 (Pushforward families)**.** Let $f : X \to Y$ be a morphism of projective varieties. Suppose that $p : U \to W$ is a family of effective k-cycles on X with underlying cycle V. Consider the cycle pushforward f_*V on $W \times Y$ and assume $f_*V \neq 0$. Construction 3.4 yields a family of k-cycles $f_*p : \tilde{U} \to W^0$ over an open subset of W. We call f_*p the pushforward family. Note that this operation is compatible with the pushforward on cycle-theoretic fibers over W^0 by [Kol96, I.3.2 Proposition].

Construction 3.11 (Restriction families)**.** Let X be a projective variety and let $p : U \to W$ be a family of effective k-cycles on X. Let $W' \subset W$ be an integral subvariety. For each component U_i of U, the restriction $U_i \times_W W'$ is flat over W' of relative dimension k. Consider the cycle V on $W' \times X$ defined as the sum $V = \sum_i a_i V_i$ where V_i is the fundamental cycle of U_i restricted to W'. We define the restriction of the family p to W' over an open subset $W'^0 \subset W'$ by applying Construction 3.4 to V. Note that this operation leaves the cycle-theoretic fibers unchanged over W'^0. Note also that if $W' \subset W$ is open, then we may take $W^0 = W'$ and the family p is simply the base-change to W^0.

Construction 3.12 (Family sum)**.** Let X be a projective variety and let $p : U \to W$ and $q : S \to T$ be two families of effective k-cycles on X. We construct the family sum of p and q over an open subset of $W \times T$ as follows. Let V_p and V_q denote the underlying cycles for p and q on $W \times X$ and $T \times X$ respectively. The family sum of p and q is the family defined by applying Construction 3.4 to the sum of the flat pullbacks of V_p and V_q to $W \times T \times X$.

Construction 3.13 (Strict transform families)**.** Let X be an integral projective variety and let $p : U \to W$ be a family of effective k-cycles on X. Suppose that $\phi : X \dashrightarrow Y$ is a birational map. We define the strict transform family of effective k-cycles on Y as follows.

First, modify U by removing all irreducible components whose image in X is contained in the locus where ϕ is not an isomorphism. Then define the cycle U' on $W \times Y$ by taking the strict transform of the remaining components of U. We define the strict transform family by applying Construction 3.4 to U' over W.

Note that when ϕ is a morphism, the strict transform family may differ from the pushforward family due to the removed components.

Construction 3.14 (Intersecting against divisors)**.** Let X be a projective variety and let $p : U \to W$ be a family of effective k-cycles on X. Let D be an effective Cartier divisor on X. If every cycle in our family has a component contained in $\mathrm{Supp}(D)$, we say that the intersection family of p and D is empty.

Otherwise, let $s : U \to X$ denote the projection map. By assumption the effective Cartier divisor s^*D does not contain any component of U, so we may take a cycle-theoretic intersection of s^*D with the cycle underlying the family p to obtain a $(k-1+\dim W)$-cycle V on $W \times \mathrm{Supp}(D)$. We then apply Construction 3.4 to obtain a family of cycles on $\mathrm{Supp}(D)$ over an open subset of W which we denote by $p \cdot D$. We can also consider the intersection as a family of cycles on X by pushing forward.

Finally, suppose that we have a linear series $|L|$. We define the intersection of $|L|$ with a family $p : U \to W$ as follows. Consider the flat pullback family $q : U' \to W^0$ on $\mathbb{P}(|L|) \times X$. Then intersect the family q against the pullback of the universal divisor on $\mathbb{P}(|L|) \times X$ to obtain a family of cycles on $\mathbb{P}(|L|) \times X$. The underlying cycle has dimension $k-1+\dim W + \dim(\mathbb{P}(|L|))$; by using Construction 3.4, we can convert this cycle to a family of effective $(k-1)$-cycles on X over an open subset of $W \times \mathbb{P}(|L|)$.

4. Mobility count

The mobility count of a family of effective cycles can be thought of informally as a count of how many general points of X are contained in members of the family. Although we are mainly interested in families of cycles, it will be helpful to set up a more general framework.

Definition 4.1. Let X be an integral projective variety and let W be a reduced variety. Suppose that $U \subset W \times X$ is a subscheme and let $p : U \to W$ and $s : U \to X$ denote the projection maps. The mobility count $\mathrm{mc}(p)$ of the morphism p is the maximum non-negative integer b such that the map

$$U \times_W U \times_W \ldots \times_W U \xrightarrow{s \times s \times \ldots \times s} X \times X \times \ldots \times X$$

is dominant, where we have b terms in the product on each side. (If the map is dominant for every positive integer b, we set $\mathrm{mc}(p) = \infty$.)

For $\alpha \in N_k(X)_{\mathbb{Z}}$, the mobility count of α, denoted $\mathrm{mc}(\alpha)$, is defined to be the largest mobility count of any family of effective cycles representing α. We define the rational mobility count $\mathrm{rmc}(\alpha)$ in the analogous way by restricting our attention to rational families.

Example 4.2. Let X be an integral projective variety and let $p : U \to W$ be a family of effective k-cycles on X. Then $\mathrm{mc}(p) \leq (\dim W)/(\dim X - k)$. Indeed, if the map of Definition 4.1 is dominant then dimension considerations show that $\mathrm{mc}(p)k + \dim W \geq \mathrm{mc}(p) \dim X$.

The mobility count should be seen as an analogue of the dimension of $H^0(X, \mathcal{O}_X(L))$ for a Cartier divisor L, as demonstrated by the following example. This analogue is surprisingly robust and seems a good indication of what behavior to expect for higher codimension cycles.

Example 4.3. Let X be an normal projective variety and let L be a Cartier divisor on X. Let $p : U \to W$ denote the family of effective divisors on X defined by the complete linear series for L. Then

$$\mathrm{mc}(p) = \dim H^0(X, \mathcal{O}_X(L)) - 1.$$

Indeed, it is easy to see that the set of divisors in our family which contain a general point of X corresponds to a codimension 1 linear subspace of $|L|$. Furthermore, an easy induction argument shows that the collection of b sufficiently general points corresponds to a collection of b hyperplanes which intersect transversally. Thus, the maximum number of general points contained in a member of $|L|$ is exactly $\dim \mathbb{P}(|L|)$, and using the incidence correspondence one identifies this number as $\mathrm{mc}(p)$ as well.

Lemma 4.4. *Let X be an integral projective variety. Let W be a reduced variety and let $p : U \to W$ denote a closed subscheme of $W \times X$. Suppose that T is another reduced variety and $q : S \to T$ is a closed subscheme of $T \times X$ such that every fiber of p over a closed point of W is contained in a fiber of q over some closed point of T (as subsets of X). Then* $\mathrm{mc}(p) \leq \mathrm{mc}(q)$.

Proof. The conditions imply that for any $b > 0$, the s^b-image of any fiber of $p^b : U^{\times_W b} \to W$ over a closed point of W is set-theoretically contained in the image of a fiber of $q^b : S^{\times_T b} \to T$ over a closed point of T (as subsets of $X^{\times b}$). The statement follows. □

Proposition 4.5. *Let X be an integral projective variety.*

(1) *Let W be an integral variety and let $U \subset W \times X$ be a closed subscheme such that $p : U \to W$ is flat. For an open subvariety $W^0 \subset W$ let $p^0 : U^0 \to W^0$ be the base change to W^0. Then* $\mathrm{mc}(p) = \mathrm{mc}(p^0)$.

(2) *Let $p : U \to W$ be a family of effective cycles on X. For an open subvariety $W^0 \subset W$ let $p^0 : U^0 \to W^0$ be the restriction family. Then* $\mathrm{mc}(p) = \mathrm{mc}(p^0)$.

(3) *Let W be a normal integral variety and let $U \subset W \times X$ be a closed subscheme such that:*
   - *Every fiber of the first projection map $p : U \to W$ has the same dimension.*
   - *Every component of U dominates W under p.*

   Let $W^0 \subset W$ be an open subset and $p^0 : U^0 \to W^0$ be the preimage of W^0. Then $\mathrm{mc}(p) = \mathrm{mc}(p^0)$.

Remark 4.6. Proposition 4.5 indicates that the mobility count is insensitive to the choice of definition of a family of effective cycles. It also shows that the mobility count only depends on general members of the family.

Proof. (1) The map $p^b : U^{\times_W b} \to W$ is proper flat, so that every component of $U^{\times_W b}$ dominates W. Then $(U^0)^{\times_{W^0} b}$ is dense in $U^{\times_W b}$ for any b. Thus $\mathrm{mc}(p^0) = \mathrm{mc}(p)$.

(2) Let $\{U_i\}$ denote the irreducible components of U. Every irreducible component of $U^{\times_W b}$ is contained in a product of the U_i over W. Since each $p|_{U_i} : U_i \to W$ is flat, we can apply the same argument as in (1).

(3) The inequality $\mathrm{mc}(p) \geq \mathrm{mc}(p^0)$ is clear. To show the converse inequality, we may suppose that U is reduced. We may also shrink W^0 and assume that p^0 is flat.

Let $p' : U' \to W'$ be a flattening of p via the birational morphism $\phi : W' \to W$. We may ensure that ϕ is an isomorphism over W^0. Choose a closed point $w \in W$ and let $T \subset W'$ be the set-theoretic preimage. Since W is normal T is connected.

Choose a closed point $w' \in T$. By construction the fiber $U'_{w'}$ is set theoretically contained in U_w (as subsets of X). Since they have the same dimension, $U'_{w'}$ is a union of components of U_w. Since p' is flat over T and T is connected, in fact $U'_{w'}$ and U_w have the same number of components and thus are set-theoretically equal. Applying Lemma 4.4 and part (1) we see that $\mathrm{mc}(p) \leq \mathrm{mc}(p') = \mathrm{mc}(p^0)$. □

We can now describe how the mobility count changes under certain geometric constructions of families of cycles.

Lemma 4.7. *Let X be an integral projective variety and let $p : U \to W$ be a family of effective k-cycles. Suppose that U has a component U_i whose image in X is contained in a proper subvariety. Then $\mathrm{mc}(p) = \mathrm{mc}(p')$ where p' is the family defined by removing U_i from U.*

Proof. This is immediate from the definition. □

Lemma 4.8. *Let $\psi : X \dashrightarrow Y$ be a birational morphism of integral projective varieties. Let $p : U \to W$ be a family of effective k-cycles on X and let p' denote the strict transform family on Y. Then $\mathrm{mc}(p) = \mathrm{mc}(p')$.*

Proof. By Lemma 4.7 we may assume that every component of U dominates X. Using Proposition 4.5 (2), we may replace p by the restricted family $p^0 : U^0 \to W^0$, where W^0 is the locus of definition of the strict transform family $p' : U' \to W^0$. The statement is then clear using the fact that the morphisms $(U^0)^{\times_{W^0} b} \to X^{\times b}$ and $(U')^{\times_{W^0} b} \to Y^{\times b}$ are birationally equivalent for every b. □

Lemma 4.9. *Let X be an integral projective variety. Suppose that W and T are reduced varieties and that $p_1 : U \to W$ and $p_2 : S \to T$ are closed subschemes of $W \times X$ and $T \times X$ respectively. Let $q : V \to W \times T$ denote the subscheme*

$$U \times T \cup W \times S \subset W \times T \times X.$$

Then $\mathrm{mc}(q) = \mathrm{mc}(p_1) + \mathrm{mc}(p_2)$.

In particular, if p_1 and p_2 are families of effective k-cycles, then the mobility count of the family sum is the sum of the mobility counts.

Proof. Set $b_1 = \mathrm{mc}(p_1)$ and $b_2 = \mathrm{mc}(p_2)$. There is a dominant projection map

$$\left(U^{\times_W b_1} \times T\right) \times_{W \times T} \left(W \times S^{\times_T b_2}\right) \to X^{\times (b_1 + b_2)}.$$

Since the domain is naturally a subscheme of $V^{\times_{W\times T} b_1+b_2}$, we obtain $\mathrm{mc}(q) \geq \mathrm{mc}(p_1) + \mathrm{mc}(p_2)$.

Conversely, any irreducible component of $V^{\times_{W \times T} c}$ is (up to reordering the terms) a subscheme of

$$\left(U^{\times_W c_1} \times T\right) \times_{W \times T} \left(W \times S^{\times_T c_2}\right)$$

for some non-negative integers c_1 and c_2 with $c = c_1 + c_2$ where the map to $X^{\times b}$ is component-wise. This yields the reverse inequality.

To extend the lemma to the family sum, first replace p_1 and p_2 by their restrictions to the normal locus of W and T respectively; this does not change the mobility count by Proposition 4.5 (2). Then by Proposition 4.5 (3) the mobility count of the family sum is the same as the mobility count of the subscheme $U \times T \cup W \times S$ as defined in Construction 3.12. □

Corollary 4.10. *Let X be an integral projective variety and let $p : U \to W$ be a family of effective k-cycles. Let $p_i : U_i \to W$ denote the irreducible components of U. Then $\mathrm{mc}(p) \leq \sum_i \mathrm{mc}(p_i)$.*

Proof. Let $q : S \to T$ denote the family sum of the p_i. By Lemma 4.4 we have $\mathrm{mc}(p) \leq \mathrm{mc}(q)$; by Lemma 4.9 $\mathrm{mc}(q) = \sum_i \mathrm{mc}(p_i)$. $\square$

4.1. **Families of divisors.** We next analyze families of effective divisors. The goal is to find bounds on the mobility count that depend only on the numerical class of the divisor. The key result is Corollary 4.12, which is the base case of inductive arguments used in the following sections.

Proposition 4.11. *Let X be an integral projective variety of dimension n. Let $\alpha \in N_{n-1}(X)_{\mathbb{Z}}$ and suppose that A is a very ample divisor such that $\alpha - [A]$ is not pseudo-effective. Then for a general element $H \in |A|$*

$$\mathrm{mc}(\alpha) \leq \mathrm{mc}_H(\alpha \cdot H).$$

Proof. For a general $H \in |A|$ we have that H is integral. Let $p : U \to W$ be a family of effective $(n-1)$-cycles representing α. By Lemma 3.6 we may suppose that $s : U \to X$ is projective and W is normal. Set $b = \mathrm{mc}(p)$ so that

$$U^{\times_W b} \to X^{\times b}$$

is surjective. Since surjectivity is preserved by base change, we see that the base change of s to H is still surjective, yielding a closed subset $U_H \subset W \times H$ with mobility count $\geq b$. Note that since $\alpha - [A]$ is not pseudo-effective, no divisor in the family p can contain H in its support, so that U_H has pure relative dimension $n-2$ over W.

Since no divisor in the family p contains H in its support, we have a well-defined intersection family $p \cdot H$ on H in the sense of Construction 3.14. Over an open subset W^0 of the parameter space, this family coincides set-theoretically with the closed subset U_H. By Proposition 4.5.(3), the mobility count of U_H does not change upon restriction to the open subset of the base, so that $\mathrm{mc}_H(p \cdot H) \geq b$. The result follows by varying p over all families representing α. $\square$

Corollary 4.12. *Let X be an integral projective variety of dimension n and let $\alpha \in N_{n-1}(X)_{\mathbb{Z}}$.*

(1) *Suppose that A is a very ample Cartier divisor on X and s is a positive integer such that $\alpha \cdot A^{n-1} < sA^n$. Then*

$$\mathrm{mc}(\alpha) < s^n A^n.$$

(2) *Suppose $n \geq 2$. Let A and H be very ample divisors and let s be a positive integer such that $\alpha - [H]$ is not pseudo-effective and $\alpha \cdot A^{n-2} \cdot H < sA^{n-1} \cdot H$. Then*

$$\mathrm{mc}(\alpha) < s^{n-1} A^{n-1} \cdot H.$$

Proof. (1) The proof is by induction on the dimension of X. The statement is clear if X is a curve, since an effective family of points of degree d can vary in at most a dimension d family. In general, note that $\alpha - s[A]$ is not pseudo-effective since it has negative intersection against the class $[A^{n-1}]$. Note also that the numerical condition on α, A, and s is preserved by restriction to a general element $A' \in |sA|$. Thus by Proposition 4.11 and the induction hypothesis

$$\mathrm{mc}(\alpha) < s^{n-1}(A|_{A'})^{n-1} = s^n A^n.$$

(2) Apply Proposition 4.11 to H, then apply (1) to α and A restricted to H. □

Finally, we prove a similar statement for sections of line bundles.

Lemma 4.13. *Let X be an equidimensional projective variety of dimension n and let A be a very ample Cartier divisor on X. Then*

$$h^0(X, \mathcal{O}_X(A)) \leq (n+1)A^n.$$

Proof. Computing the cohomology of the exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0$$

shows by induction on dimension that $h^0(X, \mathcal{O}_X) \leq A^n$. Computing the cohomology of the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(A) \to \mathcal{O}_A(A) \to 0$$

gives the desired statement by induction on dimension. □

5. The mobility function

As suggested by [DELV11], we will define the mobility of a class $\alpha \in N_k(X)_{\mathbb{Z}}$ to be the asymptotic growth rate of the number of general points contained in cycles representing multiples of α. We prove that big classes are precisely those with positive mobility, confirming [DELV11, Conjecture 6.5].

Recall that by Convention 2.2 we only consider k-cycles for $0 \leq k < \dim X$. The first step is:

Proposition 5.1. *Let X be an integral projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Z}}$. Fix a very ample divisor A and choose a positive constant $c < 1$ so that $h^0(X, mA) \geq \lfloor cm^n \rfloor$ for every positive integer m. Then any family $p : U \to W$ representing α has*

$$\mathrm{mc}(p) \leq (n+1)2^n \left(\frac{2(k+1)}{c} \right)^{\frac{n}{n-k}} (\alpha \cdot A^k)^{\frac{n}{n-k}} A^n.$$

In particular, there is some constant C so that $\mathrm{mc}(m\alpha) \leq Cm^{\frac{n}{n-k}}$.

We will develop a bound that does not depend on the constant c in Theorem 5.16.

Proof. By Lemma 4.13, the support Z of any effective cycle representing α satisfies

$$h^0(X, \mathcal{I}_Z(dA)) \geq \lfloor cd^n \rfloor - (k+1)d^k(\alpha \cdot A^k)$$

for any positive integer d. Thus an effective cycle representing α is set-theoretically contained in an element of $|\lceil d\rceil A|$ as soon as d is sufficiently large to make the right hand side greater than 1, and in particular, for

$$d = \left(\frac{2(k+1)}{c}\right)^{\frac{1}{n-k}} (\alpha \cdot A^k)^{\frac{1}{n-k}}.$$

Let $q : \tilde{U} \to \mathbb{P}(|\lceil d\rceil A|)$ denote the family of divisors defined by the linear series. By Lemma 4.4 we have $\mathrm{mc}(p) \leq \mathrm{mc}(q)$. Since $c < 1$, $d \geq 1$ so that $\lceil d\rceil < 2d$. Applying Lemma 4.13 again, Example 4.3 indicates that

$$\mathrm{mc}(p) \leq h^0(X, \lceil d\rceil A) - 1 < (n+1)2^n \left(\frac{2(k+1)}{c}\right)^{\frac{n}{n-k}} (\alpha \cdot A^k)^{\frac{n}{n-k}} A^n.$$

□

Furthermore, the growth rate of $Cm^{\frac{n}{n-k}}$ is always achieved by some big class as demonstrated by the next example.

Example 5.2. Let X be an integral projective variety and let A be a very ample divisor such that $h^i(X, \mathcal{O}_X(mA)) = 0$ for every $i > 0, m > 0$ and $h^0(X, \mathcal{O}_X(mA)) > 0$. Choose a positive constant c such that we have $h^0(X, \mathcal{O}_X(mA)) > cm^n$ for every positive integer m.

For any positive integer s, one can construct by induction a collection of s distinct reduced closed points $P_s \subset X$ with

$$h^1(X, I_{P_s}(A)) = \max\{0, s - h^0(X, \mathcal{O}_X(A))\}.$$

Furthermore this is the maximal value of $h^1(X, I_P(A))$ for any collection of s distinct closed points P. Thus the ideal sheaf I_P is m-regular as soon as $c(m-n)^n > s$. In particular, for large m we can find a complete intersection of k elements of $|mA|$ that has dimension $n-k$ and contains any $\lfloor c(m-n)^n \rfloor$ closed points of X. Setting $\alpha = A^{n-k}$, it is then clear that

$$\mathrm{rmc}(m^{n-k}\alpha) \geq \lfloor c(m-n)^n \rfloor.$$

Proposition 5.1 and Example 5.2 indicate that we should make the following definition.

Definition 5.3. Let X be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Z}}$. The mobility of α is

$$\mathrm{mob}_X(\alpha) = \limsup_{m \to \infty} \frac{\mathrm{mc}(m\alpha)}{m^{n/n-k}/n!}.$$

We will omit the subscript X when the ambient variety is clear from the context. We define the rational mobility $\mathrm{ratmob}(\alpha)$ in an analogous way using rmc.

The coefficient $n!$ is justified by Section 6.1. We verify in Example 5.7 that the mobility agrees with the volume function for Cartier divisors on a smooth integral projective variety.

Remark 5.4. There is another way to generalize the volume function for divisors. If one interprets $h^0(X, \mathcal{O}_X(mL))$ as measuring "how much mL deforms", it is natural to consider the asymptotic behavior of the dimension of $\mathrm{Chow}(X, m\alpha)$, the space parametrizing all effective cycles on X with numerical class $m\alpha$. This alternative is studied in [Leh14].

Example 5.5. Let X be an integral projective variety of dimension n and let $\alpha \in N_0(X)$ be the class of a point. Then $\mathrm{mc}(m\alpha) = m$, so that the mobility of the point class is $n!$. Rational mobility is more interesting; we will analyze it in more detail in Section 6.2.

5.1. **Basic properties.** We now turn to the basic properties of the mobility function.

Lemma 5.6. *Let X be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Z}}$. Fix a positive integer a. Then $\mathrm{mob}(a\alpha) = a^{\frac{n}{n-k}} \mathrm{mob}(\alpha)$ (and similarly for* ratmob*).*

Proof. If $\mathrm{mc}(r\alpha) > 0$ then $r\alpha$ is represented by a family of effective cycles. Thus $\mathrm{mc}((r+s)\alpha) \geq \mathrm{mc}(s\alpha)$ for any positive integer s by the additivity of mobility count under family sums as in Lemma 4.9. Conclude by Lemma 2.6. ☐

Lemma 5.6 allows us to extend the definition of mobility to any $\mathbb{Q}$-class by homogeneity, obtaining a function $\mathrm{mob} : N_k(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$.

Example 5.7. Let X be a smooth projective integral variety of dimension n. For any Cartier divisor L on X we have

$$\mathrm{ratmob}([L]) = \mathrm{mob}([L]) = \mathrm{vol}(L).$$

To prove this, note first that by Example 4.2 we have

$$\mathrm{rmc}(m[L]) \leq \mathrm{mc}(m[L]) \leq \dim \mathrm{Chow}(X, m[L])$$

where $\mathrm{Chow}(X, m[L])$ is the locus of $\mathrm{Chow}(X)$ parametrizing divisors of class $m[L]$. We can estimate the latter using

$$\begin{aligned} h^0(X, \mathcal{O}_X(mL)) - 1 &\leq \dim \mathrm{Chow}(X, m[L]) \\ &\leq \dim \mathrm{Pic}^0(X) + \max_{D \equiv mL} h^0(X, \mathcal{O}_X(D)) - 1. \end{aligned}$$

While the rightmost term may be greater than $h^0(X, \mathcal{O}_X(mL)) - 1$, the difference is bounded by a polynomial of degree $n-1$ in m (see the proof of [Laz04, Proposition 2.2.43]). By taking asymptotics, we find that $\mathrm{ratmob}([L]) \leq \mathrm{mob}([L]) \leq \mathrm{vol}(L)$. In particular, if L is not big then we must have equalities everywhere.

Conversely, by Example 4.3 we have $\mathrm{rmc}(m[L]) \geq h^0(X, \mathcal{O}_X(mL)) - 1$, so that taking asymptotics $\mathrm{ratmob}([L]) \geq \mathrm{vol}(L)$.

Lemma 5.8. *Let X be an integral projective variety. Suppose that $\alpha, \beta \in N_k(X)_{\mathbb{Q}}$ are classes such that some positive multiple of each is represented* *by an effective cycle. Then* $\mathrm{mob}(\alpha+\beta) \geq \mathrm{mob}(\alpha) + \mathrm{mob}(\beta)$ *(and similarly for* ratmob*).*

Proof. We may verify the inequality after rescaling α and β by the same positive constant c. Thus we may suppose that every multiple of each is represented by an effective cycle. Using the additivity of mobility counts under family sums as in Lemma 4.9, we see that

$$\mathrm{mc}(m(\alpha+\beta)) \geq \mathrm{mc}(m\alpha) + \mathrm{mc}(m\beta)$$

and the conclusion follows. □

By Example 5.2, we find:

Corollary 5.9. *Let X be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$ be a big class. Then* $\mathrm{mob}(\alpha) > 0$ *(and similarly for* ratmob*).*

Theorem 5.10. *Let X be an integral projective variety. The function* $\mathrm{mob} : N_k(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ *is locally uniformly continuous on the interior of* $\overline{\mathrm{Eff}}_k(X)_{\mathbb{Q}}$ *(and similarly for* ratmob*).*

Theorem 5.20 extends this result to prove that mob is continuous on all of $N_k(X)$.

Proof. The conditions (1)-(3) in Lemma 2.8 are verified by Corollary 5.9, Lemma 5.6, and Lemma 5.8. □

The mobility should also have good concavity properties. Here is a strong conjecture in this direction:

Conjecture 5.11. *Let X be an integral projective variety. Then* mob *is a log-concave function on $\overline{\mathrm{Eff}}_k(X)$: for any classes $\alpha, \beta \in \overline{\mathrm{Eff}}_k(X)$ we have*

$$\mathrm{mob}(\alpha+\beta)^{\frac{n-k}{n}} \geq \mathrm{mob}(\alpha)^{\frac{n-k}{n}} + \mathrm{mob}(\beta)^{\frac{n-k}{n}}$$

We note one other basic property:

Proposition 5.12. *Let $\pi : X \to Y$ be a dominant generically finite morphism of integral projective varieties. For any $\alpha \in N_k(X)_{\mathbb{Q}}$ we have* $\mathrm{mob}(\pi_*\alpha) \geq \mathrm{mob}(\alpha)$*.*

Proof. For sufficiently divisible m set p_m to be a family of effective k-cycles representing $m\alpha$ of maximal mobility count. The pushforward family $\pi_* p_m$ clearly has the same mobility count as p_m and represents $\pi_*(m\alpha)$, giving the result. □

5.2. **Mobility and bigness.** We now show that big cycles are precisely those with positive mobility:

Theorem 5.13. *Let X be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$. The following statements are equivalent:*

1. *α is big.*
2. $\mathrm{ratmob}(\alpha) > 0$*.*

(3) $\mathrm{mob}(\alpha) > 0$.

The implication (1) $\implies$ (2) follows from Corollary 5.9 and (2) $\implies$ (3) is obvious. The implication (3) $\implies$ (1) is a consequence of the more precise statement in Corollary 5.17.

Example 5.14. Let X be a smooth projective variety. By Example 5.7, Theorem 5.13 is equivalent to the usual characterization of big divisors using the volume function.

Example 5.15. Let α be a curve class on an integral projective complex variety X. [BCE$^+$02, Theorem 2.4] shows that if two general points of X can be connected by an effective curve whose class is proportional to α, then α is big. Theorem 5.13 is a somewhat weaker statement in this situation.

For positive integers n and for $0 \leq k < n$, define $\epsilon_{n,k}$ inductively by setting $\epsilon_{n,n-1} = 1$ and

$$\epsilon_{n,k} = \frac{\frac{n-k-1}{n-k}\epsilon_{n-1,k}}{\frac{n-1}{n-k-1} - \epsilon_{n-1,k}}.$$

For positive integers n and for $0 \leq k < n$, define $\tau_{n,k}$ inductively by setting $\tau_{n,n-1} = 1$ and

$$\tau_{n,k} = \min\left\{ \frac{n-k-1}{n-1}\tau_{n-1,k}, \frac{\frac{n-k-1}{n-k}\tau_{n-1,k}}{\frac{n-1}{n-k-1} - \tau_{n-1,k}} \right\}.$$

It is easy to verify that $0 < \tau_{n,k} \leq \epsilon_{n,k} \leq \frac{1}{n-k}$ and that the last inequality is strict as soon as $n-k > 1$.

Theorem 5.16. *Let X be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Z}}$. Let A be a very ample divisor and let s be a positive integer such that $\alpha \cdot A^k < sA^n$. Then*

(1)
$$\mathrm{mc}(\alpha) < 2^{kn+3n} s^{\frac{n}{n-k}} A^n.$$

(2) *Suppose furthermore that $\alpha - [A]^{n-k}$ is not pseudo-effective. Then*
$$\mathrm{mc}(\alpha) < 2^{kn+3n} s^{\frac{n}{n-k} - \epsilon_{n,k}} A^n.$$

(3) *Suppose that t is a positive integer such that $t \leq s$ and $\alpha - t[A]^{n-k}$ is not pseudo-effective. Then*
$$\mathrm{mc}(\alpha) < 2^{kn+3n} s^{\frac{n}{n-k} - \tau_{n,k}} t^{\tau_{n,k}} A^n.$$

Proof. We prove (1) by induction on the dimension n of X. The induction step may reduce the codimension $n-k$ of our cycle class by at most 1. Thus, for the base case it suffices to consider when $k = 0$ or when n is arbitrary and $n-k = 1$. These cases are proved by Example 5.5 and Corollary 4.12 (1) respectively.

Let $p : U \to W$ be a family of effective k-cycles representing α. By Proposition 4.5 we may modify p by Lemma 3.6 to assume W is projective without changing the mobility count of p. Choose a general divisor H in the very ample linear series $|\lceil s^{\frac{1}{n-k}} \rceil A|$ on X such that H is integral and does not contain the image of any component of U. We can associate several families of subschemes to p and to H.

- Consider the base change $U \times_X H$. We can view this as a subscheme $U_H \subset W \times H$ with projection map $\pi : U_H \to W$.
- We can intersect the family p with the divisor H to obtain a family of effective $(k-1)$ cycles $q : S \to T$ on H as in Construction 3.14. Note that T is an open subset of W; we may shrink T so that it is normal. By Lemma 3.6 we may extend q over a projective closure of T which we continue to denote by $q : S \to T$.
- Let $V \subset U_H$ be the reduced closed subset consisting of points whose local fiber dimension for π attains the maximal possible value k. The map $\pi|_V : V \to W$ has equidimensional fibers of dimension k. Thus we can associate to $\pi|_V$ a collection of families $p_i : V_i \to W_i$ as in Construction 3.5. We will think of these as families of effective k-cycles on H.

It will be useful to combine q and the p_i as follows. Let $\widetilde{W}$ denote the disjoint union of the irreducible varieties $T \times W_i$ as we vary over all i. This yields the subscheme $S \times (\sqcup_i W_i) \cup T \times (\sqcup_i V_i)$ of $\widetilde{W} \times X$. We denote the first projection map by $\tilde{p}$.

Using the universal property, one can see that

$$(U \times_X H)^{\times_W b} \cong (U^{\times_W b}) \times_{X^{\times b}} H^{\times b}.$$

Since the base change of a surjective map is surjective, we see that $\mathrm{mc}_H(\pi) \geq \mathrm{mc}_X(p)$. Furthermore, by Krull's principal ideal theorem every component of a fiber of π over a closed point of W has dimension k or $k-1$. In particular, any member of the family π is set theoretically contained in a member of the family $\tilde{p}$. Applying in order the inequality from the start of this paragraph, Lemma 4.4, and Lemma 4.9, we obtain

$$\mathrm{mc}_X(p) \leq \mathrm{mc}_H(\pi) \leq \mathrm{mc}_H(\tilde{p}) = \mathrm{mc}_H(q) + \sup_i \mathrm{mc}_H(p_i).$$

We will use induction to bound the two terms on the right, giving us our overall bound for $\mathrm{mc}_X(p)$.

The family q of effective $(k-1)$-cycles on H has class $\alpha \cdot H$. Note that $(\alpha \cdot H) \cdot A|_H^{k-1} < sA|_H^{n-1}$. By induction on the dimension of the ambient variety,

$$\begin{aligned} \mathrm{mc}_H(q) &< 2^{(k-1)(n-1)+3(n-1)} s^{\frac{n-1}{n-k}} (A|_H^{n-1}) \\ &\leq 2^{(k-1)(n-1)+3(n-1)} s^{\frac{n-1}{n-k}} (2s^{\frac{1}{n-k}} A^n) \\ &\leq 2^{(k-1)(n-1)+3(n-1)+1} s^{\frac{n}{n-k}} A^n. \end{aligned}$$

Next consider a family p_i of effective k-cycles on H. Let α_i denote the corresponding class. Let $j : H \to X$ be the inclusion; by construction, it is clear that $\alpha - j_*\alpha_i$ is the class of an effective cycle. In particular

$$\alpha_i \cdot A|_H^k \leq \alpha \cdot A^k < \lceil s^{\frac{n-k-1}{n-k}} \rceil A|_H^{n-1}.$$

By induction on the dimension of the ambient variety,

$$\begin{aligned} \mathrm{mc}_H(p_i) &< 2^{(n-1)k+3(n-1)} \lceil s^{\frac{n-k-1}{n-k}} \rceil^{\frac{n-1}{n-k-1}} (A|_H^{n-1}) \\ &\leq 2^{(n-1)k+3(n-1)} 2^{\frac{n-1}{n-k-1}} s^{\frac{n-1}{n-k}} (2s^{\frac{1}{n-k}} A^n) \\ &\leq 2^{(n-1)k+3(n-1)+k+2} s^{\frac{n}{n-k}} A^n. \end{aligned}$$

By adding these contributions, we see that

$$\mathrm{mc}_X(p) \leq 2^{kn+3n} s^{\frac{n}{n-k}} A^n.$$

(2) is proved in a similar way. The argument is by induction on the codimension $n-k$ of α. The base case – when n is arbitrary and $n-k=1$ – is a consequence of Corollary 4.12 (2) (applied with $H = A$).

Let $p : U \to W$ be a family of effective k-cycles representing α. Set $c := \frac{1}{n-k} - \epsilon_{n,k}$. Let H be an integral element of $|\lceil s^c \rceil A|$ that does not contain the image of any component of U. We construct the families $q : S \to T$ and $p_i : V_i \to W_i$ just as in (1). The same argument shows that

$$\mathrm{mc}_X(p) \leq \mathrm{mc}_H(q) + \sup_i \mathrm{mc}_H(p_i).$$

The family q of effective $(k-1)$-cycles on H has class $\alpha \cdot H$. Note that $(\alpha \cdot H) \cdot A|_H^{k-1} < sA|_H^{n-1}$. By (1), we have

$$\begin{aligned} \mathrm{mc}_H(q) &< 2^{(k-1)(n-1)+3(n-1)} s^{\frac{n-1}{n-k}} (A|_H^{n-1}) \\ &\leq 2^{(k-1)(n-1)+3(n-1)} s^{\frac{n-1}{n-k}} (2s^c A^n) \\ &\leq 2^{(k-1)(n-1)+3(n-1)+1} s^{\frac{n}{n-k} - \epsilon_{n,k}} A^n. \end{aligned}$$

Next consider the family p_i of effective k-cycles on H. Let α_i denote the class of the family p_i on H. Let $j : H \to X$ be the inclusion; by construction, it is clear that $\alpha - j_*\alpha_i$ is the class of an effective cycle. In particular

$$\alpha_i \cdot A|_H^k \leq \alpha \cdot A^k < \lceil s^{1-c} \rceil A|_H^{n-1}.$$

Note furthermore that $\alpha_i - [A|_H]^{n-1-k}$ is not pseudo-effective; otherwise it would push forward to a pseudo-effective class on X, contradicting the fact that $\alpha - [A]^{n-k}$ is not pseudo-effective. By induction on the codimension of the cycle,

$$\begin{aligned} \mathrm{mc}_H(p_i) &< 2^{k(n-1)+3(n-1)} \lceil s^{1-c} \rceil^{\frac{n-1}{n-k-1} - \epsilon_{n-1,k}} (A|_H^{n-1}) \\ &\leq 2^{k(n-1)+3(n-1)} 2^{\frac{n-1}{n-k-1}} s^{\frac{(1-c)(n-1)}{n-k-1} - (1-c)\epsilon_{n-1,k}} (2s^c A^n) \\ &\leq 2^{k(n-1)+3(n-1)+k+2} s^{\frac{n}{n-k} - \epsilon_{n,k}} A^n. \end{aligned}$$

Adding the two contributions proves the statement as before.

The proof of (3) is also very similar. The argument is by induction on the codimension $n-k$ of α. The base case – when n is arbitrary and $n-k=1$ – is a consequence of Corollary 4.12 (2) (applied with $H=tA$).

Let $p : U \to W$ be a family of effective k-cycles representing α. Set $c := \frac{1}{n-k} - \tau_{n,k}$. Let H be an integral element of $|\lceil s^{c}t^{\tau_{n,k}}\rceil A|$ that does not contain the image of any component of U. We construct the families $q : S \to T$ and $p_i : V_i \to W_i$ just as in (1). The same argument shows that

$$\mathrm{mc}_X(p) \leq \mathrm{mc}_H(q) + \sup_i \mathrm{mc}_H(p_i).$$

The family q of effective $(k-1)$-cycles on H has class $\alpha \cdot H$. Note that $(\alpha \cdot H) \cdot A|_H^{k-1} < sA|_H^{n-1}$. By (1), we have

$$\begin{aligned}
\mathrm{mc}_H(q) &< 2^{(k-1)(n-1)+3(n-1)} s^{\frac{n-1}{n-k}} (A|_H^{n-1}) \\
&\leq 2^{(k-1)(n-1)+3(n-1)} s^{\frac{n-1}{n-k}} (2s^{c}t^{\tau_{n,k}}A^n) \\
&\leq 2^{(k-1)(n-1)+3(n-1)+1} s^{\frac{n}{n-k}-\tau_{n,k}} t^{\tau_{n,k}} A^n.
\end{aligned}$$

Next consider the family p_i of effective k-cycles on H. Let α_i denote the class of the family p_i on H. Let $j : H \to X$ be the inclusion; by construction, it is clear that $\alpha - j_*\alpha_i$ is the class of an effective cycle. In particular

$$\alpha_i \cdot A|_H^k \leq \alpha \cdot A^k < \lceil s^{1-c}t^{-\tau_{n,k}} \rceil A|_H^{n-1}.$$

Also, we have that

$$\alpha_i - \lceil t^{1-\tau_{n,k}}s^{-c}\rceil [A|_H]^{n-k-1}$$

is not pseudo-effective, since the difference between $\alpha - t[A]^{n-k}$ and the push forward of this class to X is pseudo-effective. Finally, note that $\lceil s^{1-c}t^{-\tau_{n,k}}\rceil \geq \lceil t^{1-\tau_{n,k}}s^{-c}\rceil$ so that we may apply (3) inductively to the family p_i with the constants $s' = \lceil s^{1-c}t^{-\tau_{n,k}}\rceil$ and $t' = \lceil t^{1-\tau_{n,k}}s^{-c}\rceil$.

There are two cases to consider. First suppose that $t^{1-\tau_{n,k}}s^{-c} \geq 1$. Then by induction on the codimension of the cycle,

$$\begin{aligned}
\mathrm{mc}_H(p_i) &< 2^{k(n-1)+3(n-1)} \lceil s^{1-c}t^{-\tau_{n,k}}\rceil^{\frac{n-1}{n-k-1}-\tau_{n-1,k}} \\
&\qquad \lceil t^{1-\tau_{n,k}}s^{-c}\rceil^{\tau_{n-1,k}} (A|_H^{n-1}) \\
&\leq 2^{k(n-1)+3(n-1)} 2^{\frac{n-1}{n-k-1}} s^{\frac{(1-c)(n-1)}{n-k-1}-\tau_{n-1,k}} \\
&\qquad t^{\tau_{n-1,k}-\frac{n-1}{n-k-1}\tau_{n,k}} (2s^{c}t^{\tau_{n,k}}A^n) \\
&\leq 2^{k(n-1)+3(n-1)+k+2} s^{\frac{n}{n-k}-\tau_{n,k}+\left(\frac{n-1}{n-k-1}\tau_{n,k}-\tau_{n-1,k}\right)} \\
&\qquad t^{\tau_{n,k}+\left(\tau_{n-1,k}-\frac{n-1}{n-k-1}\tau_{n,k}\right)} A^n.
\end{aligned}$$

Since $\tau_{n-1,k} \geq \frac{n-1}{n-k-1}\tau_{n,k}$, the part of the exponents in parentheses is non-positive for s and non-negative for t. By assumption $s \geq t$, so

$$\mathrm{mc}_H(p_i) < 2^{kn+3(n-1)+2} s^{\frac{n}{n-k}-\tau_{n,k}} t^{\tau_{n,k}} A^n$$

Next suppose that $t^{1-\tau_{n,k}} s^{-c} < 1$. Then by (2) we find

$$\begin{aligned}
\mathrm{mc}_H(p_i) &< 2^{k(n-1)+3(n-1)} \lceil s^{1-c} t^{-\tau_{n,k}} \rceil^{\frac{n-1}{n-k-1} - \epsilon_{n-1,k}} (A|_H^{n-1}) \\
&\leq 2^{k(n-1)+3(n-1)} \lceil s^{1-c} \rceil^{\frac{n-1}{n-k-1} - \tau_{n-1,k}} (A|_H^{n-1}) \\
&\leq 2^{k(n-1)+3(n-1)} 2^{\frac{n-1}{n-k-1}} s^{\frac{(1-c)(n-1)}{n-k-1} - (1-c)\tau_{n-1,k}} (2 s^c t^{\tau_{n,k}} A^n) \\
&\leq 2^{kn+3(n-1)+2} s^{\frac{n}{n-k} - \tau_{n,k}} t^{\tau_{n,k}} A^n.
\end{aligned}$$

This upper bound for the two cases is the same; by adding it to the upper bound for $\mathrm{mc}_H(q)$ we obtain the desired upper bound for $\mathrm{mc}(p)$. $\square$

We can apply Theorem 5.16 (2) to any class α in $\partial \overline{\mathrm{Eff}}_k(X) \cap N_k(X)_{\mathbb{Z}}$ to obtain the following corollary.

Corollary 5.17. *Let X be an integral projective variety and suppose that $\alpha \in N_k(X)_{\mathbb{Z}}$ is not big. Let A be a very ample divisor and let s be a positive integer such that $\alpha \cdot A^k < sA^n$. Then*

$$\mathrm{mc}(\alpha) < 2^{kn+3n}(k+1) s^{\frac{n}{n-k} - \epsilon_{n,k}} A^n.$$

Remark 5.18. The exponent $\frac{n}{n-k} - \epsilon_{n,k}$ in Corollary 5.17 is not optimal in general. For example, [BCE$^+$02, Theorem 2.4] shows that for a curve class α that is not big there is a positive constant C such that $\mathrm{mc}(m\alpha) < Cm$.

Example 5.19. Let $f : X \to Z$ be a surjective morphism from a smooth integral projective variety of dimension n to a smooth integral projective variety of dimension k for some $1 < k < n$. Fix ample divisors A on X and H on Z and define $\alpha = [A]^{n-k-1} \cdot [f^*H]$. α is not big since $\alpha \cdot [f^*H]^k = 0$. By taking the complete intersection of $(n-k-1)$ elements of $H^0(X, \mathcal{O}_X(\lfloor m^{\frac{k}{(n-k)(k+1)}} \rfloor A))$ with an element of $H^0(X, \mathcal{O}_X(\lfloor m^{\frac{n}{(n-k)(k+1)}} \rfloor f^*H))$ we see

$$\mathrm{mc}(m\alpha) \geq C m^{\frac{nk}{(n-k)(k+1)}}$$

for some positive constant C. Rewriting

$$\frac{nk}{(n-k)(k+1)} = \frac{n}{n-k} - \frac{n}{(n-k)(k+1)}$$

shows that the optimal value of $\epsilon_{n,k}$ is at most $\frac{n}{(n-k)(k+1)}$.

5.3. **Continuity of mobility.** Theorem 5.16 also allows us to prove the continuity of the mobility function.

Theorem 5.20. *Let X be an integral projective variety. Then the mobility function* mob : $N_k(X)_{\mathbb{Q}} \to \mathbb{R}$ *can be extended to a continuous function on $N_k(X)$.*

Proof. Note that mob can be extended to a continuous function on the interior of $\overline{\mathrm{Eff}}_k(X)$ by Theorem 5.10. Furthermore mob is identically 0 on every element in $N_k(X)_{\mathbb{Q}}$ not contained in $\overline{\mathrm{Eff}}_k(X)$. Thus it suffices to show that mob approaches 0 for classes approaching the boundary of $\overline{\mathrm{Eff}}_k(X)$.

Let α be a point on the boundary of $\overline{\mathrm{Eff}}_k(X)$. Fix $\mu > 0$; we show that there exists a neighborhood U of α such that $\mathrm{mob}(\beta) < \mu$ for any class $\beta \in U \cap N_k(X)_{\mathbb{Q}}$.

Fix a very ample divisor A and a positive integer s such that $\alpha \cdot A^k < \frac{s}{2} A^n$. Choose δ sufficiently small so that

$$n! 2^{kn+3n+1} s^{\frac{n}{n-k}} A^n \delta^{\tau_{n,k}} < \mu.$$

Let U be a sufficiently small neighborhood of α so that:

- $\beta \cdot A^k < sA^n$ for every $\beta \in U$, and
- $\beta - \delta s [A]^{n-k}$ is not pseudo-effective for every $\beta \in U$.

Suppose now that $\beta \in U \cap N_k(X)_{\mathbb{Q}}$ and that m is any positive integer such that $m\beta \in N_k(X)_{\mathbb{Z}}$. Then:

- $m\beta \cdot A^k < smA^n$ and
- $m\beta - \lceil \delta ms \rceil [A]^{n-k}$ is not pseudo-effective.

Theorem 5.16 shows that

$$\mathrm{mc}(m\beta) < 2^{kn+3n} (ms)^{\frac{n}{n-k} - \tau_{n,k}} (\lceil \delta ms \rceil)^{\tau_{n,k}} A^n.$$

When m is sufficiently large, $\lceil \delta ms \rceil \leq 2\delta ms$, so we obtain for such m

$$\mathrm{mc}(m\beta) < 2^{kn+3n+1} m^{\frac{n}{n-k}} s^{\frac{n}{n-k}} A^n \delta^{\tau_{n,k}} < \frac{\mu}{n!} m^{\frac{n}{n-k}}$$

showing that $\mathrm{mob}(\beta) < \mu$ as desired. □

Remark 5.21. It is sometimes easier to compute the mobility count of families satisfying extra conditions (for example, smoothness or ACMness of each component of a general fiber). If one formulates the condition so that it is preserved by

- closure of families,
- family sums,
- intersections against general very ample divisors, and
- pushforwards from subvarieties

then by an analogous construction one can define a "mobility-type" function which only considers families with this restriction. These functions share many of the same formal properties as the mobility. However, any improvements in computability seem more than offset by the loss of theoretical flexibility.

6. Examples of mobility

The mobility seems difficult to calculate explicitly. By analogy with the volume, one wonders whether the mobility is related to intersection numbers for "sufficiently positive" classes (just as the volume of an ample divisor is a self-intersection product). In particular, we ask:

Question 6.1. Let X be an integral projective variety and let H be an ample Cartier divisor. For $0 < k < n$, is

$$\operatorname{mob}(H^{n-k}) = \operatorname{vol}(H)?$$

An affirmative answer would imply that the "optimal cycles" with respect to the mobility count for H^{n-k} are complete intersections of $(n-k)$ general elements of $|dH|$. This question is vastly generalized by Question 1.14, but already this particular case is very interesting.

Remark 6.2. Note that the statement in Question 6.1 does not hold for point classes: for an ample divisor H we have $\operatorname{vol}(H) = H^n$ but $\operatorname{mob}(H^n) = n!H^n$.

Example 6.3. It is not hard to show that $\operatorname{mob}(H^{n-k}) \geq H^n$ for an ample divisor H. Indeed, by homogeneity we may suppose that H is very ample and that all the higher cohomology of multiples of H vanishes. Arguing as in Example 5.2, we see that we can find a basepoint free family of divisors of class $m[H]$ containing $h^0(X, mH) - 1$ general points of X. By taking complete intersections, we obtain a family of class $m^{n-k}[H^{n-k}]$ going through $h^0(X, mH) - 1 - n + k$ general points of X. Taking a limit shows the inequality.

In the remainder of this section we discuss two examples in detail. We will work over the base field $\mathbb{C}$ to cohere with the cited references.

6.1. **Curves on $\mathbb{P}^3$.** Let ℓ denote the class of a line on $\mathbb{P}^3$ over $\mathbb{C}$. The mobility of ℓ is determined by the following enumerative question: what is the minimal degree of a curve in $\mathbb{P}^3$ going through b very general points? The answer is unknown (even in the asymptotic sense).

[Per87] conjectures that the "optimal" curves are the complete intersections of two hypersurfaces of degree d. Indeed, among all curves not contained in a hypersurface of degree $(d-1)$, [GP78] shows that these complete intersections have the largest possible arithmetic genus, and thus conjecturally the corresponding Hilbert scheme has the largest possible dimension.

Complete intersections of two hypersurfaces of degree d have degree d^2 and pass through $\approx \frac{1}{6}d^3$ general points. Letting d go to infinity, we find the lower bound

$$1 \leq \operatorname{mob}(\ell)$$

and conjecturally equality holds.

Theorem 6.4. *Let ℓ be the class of a line on $\mathbb{P}^3$. Then*

$$1 \leq \operatorname{mob}(\ell) < 3.54.$$

In the proof we simply repeat the argument of Theorem 5.16 with more careful constructions of families and better estimates.

Proof. Fix a degree d. Let $s = \left\lceil \sqrt{\frac{9-\sqrt{69}}{2}}d \right\rceil$ and let S be a Noether-Lefschetz general hypersurface of degree s. Then every curve on S is the restriction of a hypersurface on $\mathbb{P}^3$. In particular, $\mathrm{Pic}(S) \cong \mathbb{Z}$, and if H denotes the hyperplane class on $\mathbb{P}^3$ then the mobility count of $cH|_S$ is

$$\binom{c+3}{3} - \binom{c-s+3}{3}$$

(where we use the convention that the rightmost term is 0 when $c < s$). Let $p : U \to W$ be a family of degree d curves on $\mathbb{P}^3$. Consider the base change $p' : U \times_{\mathbb{P}^3} S \to W$. Every component of a fiber of p' has dimension 1 or 0. We can stratify W by locally closed subsets W_i based on the degree d' of the components of the fibers of dimension 1; the components of dimension 0 then have degree $(d-d')s$. Applying Construction 3.5 to construct families of cycles as in the proof of Theorem 5.16, Lemma 4.4 implies that

$$\begin{aligned} \mathrm{mc}(d\ell) &\leq \max_{0 \leq d' \leq d} \mathrm{mc}_S((d-d')\ell \cdot S) + \mathrm{mc}_S\left(\left\lceil \frac{d'}{s} \right\rceil H|_S \right) \\ &\leq \max_{0 \leq d' \leq d} (d-d')s + \binom{\lceil \frac{d'}{s} \rceil + 3}{3} - \binom{\lceil \frac{d'}{s} \rceil - s + 3}{3}. \end{aligned}$$

A straightforward computation shows that for $\lceil d'/s \rceil \geq s$ the maximum value is achieved when $d' = d$ and for $\lceil d'/s \rceil < s$ the maximum value is achieved when $d' = 0$. In either case, the asymptotic value of the computation above is

$$\mathrm{mc}(d\ell) \leq \sqrt{\frac{9-\sqrt{69}}{2}} d^{3/2} + O(d)$$

yielding the desired bound. □

Remark 6.5. Suppose that $p : U \to W$ is a family of smooth curves on $\mathbb{P}^3$ of degree d. Then [Per87, Proposition 6.29] proves the stronger result

$$\mathrm{mc}(p) \leq \frac{1}{2} d^{3/2} + O(d)$$

To prove a statement of this kind, note that one may assume that the general curve in the family is not contained on a surface of degree $< \sqrt{d}$. Using the Gruson-Peskine bounds on genus in [GP78], one can estimate the dimension of the normal bundle of the curve, and hence the dimension of the Hilbert scheme parametrizing such curves. The result follows easily.

6.2. **Rational mobility of points.** In this section we relate rational mobility with the theory of rational equivalence of 0-cycles. In order to cohere with the cited references, we work only with normal integral varieties X over $\mathbb{C}$ (although the results easily extend to a more general setting). Recall that $A_0(X)$ denotes the group of rational-equivalence classes of 0-cycles on X. We will denote the rth symmetric power of X by $X^{(r)}$; by [Kol96, I.3.22 Exercise] this is the component of $\mathrm{Chow}(X)$ parametrizing 0-cycles of degree r.

Remark 6.6. The universal family of 0-cycles of degree r (in the sense of Definition 3.1) is not $u : X^{\times r} \to X^{(r)}$ but a flattening of this map. However, note that the rational mobility computations are the same whether we work with u or a flattening by Lemma 4.5. For simplicity we will work with u and $X^{(r)}$ despite the slight incongruity with Definition 3.1.

We start by recalling the results of [Roĭ72] concerning $A_0(X)$. Consider the map $\gamma_{m,n} : X^{(m)} \times X^{(n)} \to A_0(X)$ sending $(p,q) \mapsto p-q$. [Roĭ72, Lemma 1] shows that the fibers of $\gamma_{m,n}$ are countable unions of closed subvarieties.

A subset $V \subset A_0(X)$ is said to be irreducible closed if it is the $\gamma_{m,n}$-image of an irreducible closed subset Y of $X^{(m)} \times X^{(n)}$ for some m and n. The dimension of such a subset V is defined to be the dimension of Y minus the minimal dimension of a component of a fiber of $\gamma_{m,n}|_Y$. [Roĭ72, Lemma 9] shows that the dimension is independent of the choice of Y, m, and n.

Lemma 6.7. *Let $V, W \subset A_0(X)$ be irreducible closed subsets with $V \subsetneq W$. Then $\dim(V) < \dim(W)$.*

Proof. Let $Z \subset X^{(m)} \times X^{(n)}$ be an irreducible closed subset whose $\gamma_{m,n}$-image is W. Let $Y \subset Z$ denote the preimage of V; [Roĭ72, Lemma 5] shows that Y is a countable union of closed subsets. By [Roĭ72, Lemma 6] some component $Y' \subset Y$ dominates V. But then $\dim(Y') < \dim(Z)$, proving the statement. $\square$

We are mainly interested in when $A_0(X)$ is an irreducible closed set. This is equivalent to the following notion:

Definition 6.8. $A_0(X)$ is said to be representable if there is a positive integer r such that the addition map $a_r : X^{(r)} \to A_0(X)_{\deg r}$ is surjective.

We now relate these notions to the rational mobility of 0-cycles on X.

Proposition 6.9. *Let X be a normal integral projective variety over $\mathbb{C}$ and let α denote the class of a point in $N_0(X)$. Then the following are equivalent:*

1. *$A_0(X)$ is representable.*
2. *$\mathrm{ratmob}(\alpha) = n!$.*
3. *$\mathrm{ratmob}(\alpha) > n!/2$.*

Proof. (1) $\implies$ (2). Suppose that $A_0(X)$ is representable. There is some positive integer r such that the addition map $a_r : X^{(r)} \to A_0(X)_{\deg r}$ is surjective. Fix $m > 0$ and choose some class $\tau \in A_0(X)_{\deg(m+r)}$. For any effective 0-cycle Z of degree m, there is an effective 0-cycle T_Z of degree r such that $T_Z + Z \in \tau$. As $Z \in X^{(m)}$ varies, the effective cycles $Z + T_Z$ are rationally equivalent, showing that $\mathrm{rmc}((m+r)\alpha) \geq m$ and $\mathrm{ratmob}(\alpha) = n!$.

(3) $\implies$ (1). Suppose that $A_0(X)$ is not representable. Note that non-representability implies that for every m there is some closed point p such that $a_m(X^{(m)}) + p \subsetneq a_{m+1}(X^{(m+1)})$ for every m: if we had equality for some m and every p, we would also have equality for every $m' > m$ and

the map $a_m : X^{(m)} \to A_0(X)_{\deg m}$ would be surjective. Thus by Lemma 6.7 $\dim(a_m(X^{(m)}))$ strictly increases in m.

Suppose that $\mathrm{rmc}(m\alpha) = b$. This implies that there is some rational equivalence class τ of degree m so that for any $p \in X^{(b)}$, there is an element $q \in X^{(m-b)}$ such that $p + q \in \tau$. In particular, the subset $\tau - X^{(b)} \subset A_0(X)_{\deg m-b}$ is contained in $a_{m-b}(X^{(m-b)})$. By Lemma 6.7, $\dim(a_b(X^{(b)})) \leq \dim(a_{m-b}(X^{(m-b)}))$. But since these dimensions are strictly increasing in m we must have $m \geq 2b$. Thus we see that $\mathrm{ratmob}(\alpha) \leq n!/2$, proving the statement. $\square$

Example 6.10. Let X be an integral projective variety of dimension n and let α be the class of a point in $N_0(X)$. Let A be a very ample divisor on X; for sufficiently large m we have $h^0(X, \mathcal{O}_X(mA)) \approx \frac{1}{n!}A^n$. By taking complete intersections of n elements of $|mA|$, we see that $\mathrm{ratmob}(\alpha) \geq 1$.

Example 6.11. Let X be a smooth surface over $\mathbb{C}$ and let α be the class of a point. By combining Example 6.10 with Proposition 6.9 we see that there are two possibilites:

- $A_0(X)$ is representable and $\mathrm{ratmob}(\alpha) = 2$.
- $A_0(X)$ is not representable and $\mathrm{ratmob}(\alpha) = 1$.

7. Intersection-theoretic volume function

In this section we study the function $\widehat{\mathrm{vol}}$ defined in the introduction.

Definition 7.1. Let X be an integral projective variety of dimension n and suppose $\alpha \in N_k(X)$ for $0 \leq k < n$. Define the volume of α to be

$$\widehat{\mathrm{vol}}(\alpha) := \sup_{\phi, A}\{A^n\}$$

as $\phi : Y \to X$ varies over all birational models of X and as A varies over all big and nef $\mathbb{R}$-Cartier divisors on Y such that $\phi_* A^{n-k} \preceq \alpha$. If the set of appropriate data ϕ, A is empty, then we interpret this expression as returning 0.

In contrast to the mobility, one can readily compute this function in simple examples.

Example 7.2. If $\alpha \in N_0(X)$, then $\widehat{\mathrm{vol}}(\alpha)$ is exactly the degree of α.

Example 7.3. Suppose that B is a big and nef $\mathbb{R}$-Cartier divisor and that $\alpha = [B^{n-k}]$. We claim that $\widehat{\mathrm{vol}}(\alpha) = \mathrm{vol}(B)$.

Indeed, suppose that $\phi : Y \to X$ is a birational map and A is a big and nef $\mathbb{R}$-Cartier divisor such that $\phi_*[A^{n-k}] \preceq \alpha$. Recall that by the Khovanskii-Teissier inequalities (see for example [Laz04, Corollary 1.6.3 and Remark 1.6.5])

$$A^{n-k} \cdot \phi^* B^k \geq \mathrm{vol}(A)^{n-k/n} \, \mathrm{vol}(B)^{k/n}.$$

Then we have

$$\begin{aligned}\mathrm{vol}(A) &\leq \left(\frac{A^{n-k} \cdot \phi^* B^k}{\mathrm{vol}(B)^{k/n}}\right)^{n/n-k} \\ &\leq \left(\frac{\alpha \cdot B^k}{\mathrm{vol}(B)^{k/n}}\right)^{n/n-k} \\ &= \mathrm{vol}(B).\end{aligned}$$

Since clearly $\widehat{\mathrm{vol}}(\alpha) \geq \mathrm{vol}(B)$, we obtain the claimed equality.

Example 7.4. Suppose that X is smooth and that $\alpha \in \overline{\mathrm{Eff}}_1(X)$ is a curve class. Then [LX15] shows that $\widehat{\mathrm{vol}}(\alpha)$ agrees with the expression in [Xia15] and so can be computed once one knows the nef cone of X:

$$\widehat{\mathrm{vol}}(\alpha) = \inf_{A \text{ big and nef } \mathbb{R}\text{-divisor}} \left(\frac{A \cdot \alpha}{\mathrm{vol}(A)^{1/n}}\right)^{n/n-1}.$$

In fact, for curve classes the supremum in Definition 7.1 is actually achieved by a divisor on X – there is no need to pass to a birational model. This leads to a robust notion of a Zariski decomposition for curve classes (see [LX15] for more details).

Lemma 7.5. *Let X be an integral projective variety of dimension n and suppose $\alpha \in N_k(X)$ for $0 \leq k < n$.*

1. *For any positive constant c we have $\widehat{\mathrm{vol}}(c\alpha) = c^{n/n-k}\widehat{\mathrm{vol}}(\alpha)$.*
2. *If α is big then $\widehat{\mathrm{vol}}(\alpha) > 0$.*
3. *If α is pseudo-effective, then for any class $\beta \in N_k(X)$ we have $\widehat{\mathrm{vol}}(\alpha + \beta) \geq \widehat{\mathrm{vol}}(\beta)$.*

Proof. To see (1), note that for any birational map $\phi : Y \to X$ and for any big and nef $\mathbb{R}$-Cartier divisor A on Y we have $\phi_*[A^{n-k}] \preceq \alpha$ if and only if $\phi_*[(c^{1/n-k}A)^{n-k}] \preceq c\alpha$.

For (2), if α is big then there is some ample divisor A on X such that $[A^{n-k}] \preceq \alpha$. So certainly $\widehat{\mathrm{vol}}(\alpha) > 0$.

To see (3), note that if $\phi_*[A^{n-k}] \preceq \beta$ then certainly $\phi_*[A^{n-k}] \preceq \alpha$ as well. ☐

The main point is to understand the behavior of $\widehat{\mathrm{vol}}$ as we approach the boundary. This is controlled by the following theorem.

Theorem 7.6. *Let X be an integral projective variety of dimension n. For any $\alpha \in \overline{\mathrm{Eff}}_k(X)$ we have*

$$\widehat{\mathrm{vol}}(\alpha) \leq \mathrm{mob}(\alpha).$$

Proof. Consider varying all birational morphisms $\phi : Y \to X$ and all big and nef divisors A satisfying $\phi_*[A^{n-k}] \preceq \alpha$. By combining Lemma 5.8

with the continuity of mob, we see that $\mathrm{mob}(\alpha) \geq \mathrm{mob}(\phi_*[A^{n-k}])$ whenever $\alpha \succeq \phi_*[A^{n-k}]$. Thus

$$\begin{aligned} \mathrm{mob}(\alpha) &\geq \sup_{\phi,A} \mathrm{mob}(\phi_*[A^{n-k}]) \\ &\geq \sup_{\phi,A} \mathrm{mob}([A^{n-k}]) \text{ by Proposition 5.12.} \\ &\geq \sup_{\phi,A} \mathrm{vol}(A) \text{ by Example 6.3.} \end{aligned}$$

This latter quantity is $\widehat{\mathrm{vol}}(\alpha)$. □

Corollary 7.7. *Let X be an integral projective variety. Then $\widehat{\mathrm{vol}}$ is a continuous function on $N_k(X)$ for any $0 \leq k < n$.*

Proof. Lemma 7.5 verifies the hypotheses of Lemma 2.8, showing that $\widehat{\mathrm{vol}}$ is continuous on the interior of the big cone. Theorem 5.20 and Theorem 7.6 show that $\widehat{\mathrm{vol}}$ must limit to 0 as we approach the boundary of the pseudo-effective cone. □

8. Weighted mobility

One approach for calculating the mobility is to study the geometry of the blow-up of X through general points, but unfortunately this does not seem very effective (see Example 8.1). One can obtain a closer relationship using a "weighted" mobility count. The key idea is that the singular points of a family of cycles should contribute more to the mobility count – this better reflects the intersection theory on the blow-up, as the strict transform of a cycle which is singular at a point will have larger intersection against the exceptional divisor than the strict transform of a smooth cycle. This idea was suggested to me by R. Lazarsfeld and this section is based off his suggestion.

Example 8.1. One could try to compute the mobility of the line class ℓ on $\mathbb{P}^n$ as follows. Consider all curves of degree d^{n-1}. Let $\phi : Y \to \mathbb{P}^n$ be the blow-up of b general points with exceptional divisors E_i. Choose positive constants $\{a_i\}_{i=1}^b$ such that $D = d\phi^*H - \sum_i a_i E_i$ is nef. If C' is the strict transform of a degree d^{n-1} curve containing all b points then

$$0 \leq D \cdot C' \leq d^n - \sum_i a_i.$$

If we could find a nef divisor on Y satisfying $\sum_i a_i > d^n$, then we would know that $\mathrm{mc}(d^{n-1}\ell) < b$.

Unfortunately, it is impossible to find such a nef divisor for values of b near the conjectural mobility count $\approx \frac{d^n}{n!}$. Indeed, the condition on the a_i is then incompatible with the self-intersection condition $0 \leq D^n = d^n - \sum a_i^{n-1}$. In other words, the strict transform of a smooth degree d^{n-1} curve through $\approx \frac{d^n}{n!}$ points is necessarily in the interior of the pseudo-effective cone. While one can obtain upper bounds for $\mathrm{mc}(\ell)$ on $\mathbb{P}^3$ in this way, they are significantly worse than Theorem 6.4.

8.1. **Seshadri constants.** We start with a few reminders about Seshadri constants at general points; see [Laz04] or [BDRH$^+$09] for a more thorough introduction to the area.

Definition 8.2. Let X be an integral projective variety of dimension n and let A be an ample Cartier divisor on X. Fix distinct closed reduced points $\{x_i\}_{i=1}^b$ in the smooth locus of X. Set $\phi : Y \to X$ to be the blow-up of the x_i and let E denote the sum of all the exceptional divisors. The Seshadri constant of A along the $\{x_i\}$ is

$$\varepsilon(\{x_i\}, A) := \max \{ r \in \mathbb{R}_{\geq 0} \, | \phi^* A - rE \text{ is nef} \} .$$

We can identify the collection of sets of b distinct closed points in the smooth locus of X with an open subset of the symmetric power $X^{(b)}$. Using the openness of ampleness in families of divisors, one sees that for any such set of b points, there is an open neighborhood in the parameter space $X^{(b)}$ such that ε can only increase on the corresponding sets.

It is not hard to see that the values of ε are bounded above as we vary the set of b points. We define $\varepsilon_b(A)$ to be the supremum over all sets of b distinct closed points $\{x_i\}_{i=1}^b$ in the smooth locus of X of $\varepsilon(\{x_i\}, A)$. When we are working over an uncountable field there is actually a set of points achieving this supremum, but we will not need this fact.

It is an important but difficult problem to precisely establish the value of $\varepsilon_b(A)$. The following estimate gives a good asymptotic bound on $\varepsilon_b(A)$. It is certainly well-known to experts and the first variant for higher dimension varieties seems to have appeared in [Ang97].

Proposition 8.3. *Let X be an integral projective variety of dimension n over an uncountable algebraically closed field and let A be a very ample divisor on X. Suppose that $b \leq t^n A^n$ for a positive integer t. Fix general points $\{x_i\}_{i=1}^b$ on X. Then*

$$\frac{1}{t} \leq \varepsilon(\{x_i\}, A) \leq \frac{(A^n)^{1/n}}{b^{1/n}}.$$

In particular $\frac{1}{t} \leq \varepsilon_b(A) \leq \frac{(A^n)^{1/n}}{b^{1/n}}$.

Proof. We first prove the upper bound. Let $\phi : Y \to X$ be the blow-up of X at the b points $\{x_i\}$ and let E denote the sum of the exceptional divisors. Since $\phi^* A - \varepsilon(\{x_i\}, A)E$ is nef,

$$0 \leq (\phi^* A - \varepsilon(\{x_i\}, A)E)^n = A^n - b\varepsilon(\{x_i\}, A)^n.$$

We next prove the lower bound. Using openness of ampleness in families of divisors, it suffices to find a single blow-up of X at b points $x_1, \ldots, x_b$ such that $\varepsilon(\{x_i\}, A) \geq 1/t$. Set $r = t^n A^n - b$. Let $W \subset |tA|$ be a general $(n-1)$-dimensional linear system with basepoints $x_1, \ldots, x_b, q_1, \ldots, q_r$. Let $\phi : Y \to$

X be the blow-up of the points $x_1, \ldots, x_b$ with E the sum of the exceptional divisors, and let $\phi' : Y' \to X$ be the blow-up of the entire base locus with E' the sum of the exceptional divisors. The strict transform of an element $H \in W$ to Y' is clearly nef and has class $t\phi'^*A - E'$. Furthermore, the pushforward of the movable part of ϕ'^*W to Y only accumulates basepoints along a 0-dimensional set, so $t\phi^*A - E$ is also nef. Thus

$$\frac{1}{t} \leq \varepsilon(\{x_i\}, A).$$

The final statement about $\varepsilon_b(A)$ follows immediately since the upper bound does not depend on the choice of points. $\square$

8.2. **Loci of points with multiplicity.** If Z is an integral projective variety and $z \in Z$ is a reduced closed point, we denote by $\mathrm{mult}(Z, z)$ the multiplicity of Z at z (that is, the multiplicity of the maximal ideal $\mathfrak{m}_{Z,z}$ in the local ring $\mathcal{O}_{Z,z}$).

More generally, we define the multiplicity of a k-cycle $V = \sum_i a_i V_i$ at a reduced closed point $v \in \mathrm{Supp}(V)$ via the expression $\sum a_i \,\mathrm{mult}(V_i, v)$. Note that this definition is compatible with the blow-up definition of multiplicity: if $\phi : Y \to \mathrm{Supp}(V)$ denotes the blow-up of the reduced closed point v and V' denotes the strict transform of the cycle V, then $\mathrm{mult}(V, v) = (-1)^{k-1} E^k \cdot V'$ where E is the exceptional divisor.

Lemma 8.4. *Suppose that W is an integral variety. Let $p : U \to W$ be a family of effective k-cycles on X and let $U_{[w]}$ denote the cycle-theoretic fiber above w. The function on reduced closed points*

$$u \mapsto \mathrm{mult}(U_{[f(u)]}, u)$$

is upper semi-continuous.

Proof. Let Y denote the blow-up of the reduced diagonal Δ_{red} on $U \times U$ with exceptional divisor E. By composing the blow-down with $p \times id$ we obtain $g : Y \to W \times U$. Consider the flattening $\widetilde{g} : \widetilde{Y} \to \widetilde{W \times U}$ as in [RG71, Théorème 5.2.2] and let $h_W : \widetilde{W \times U} \to W \times U$ denote the birational morphism and $h_Y : \widetilde{Y} \to Y$ denote the restriction of the projection.

Note that the effective Cartier divisor $h_Y^* E$ defines a subscheme of $\widetilde{Y}$ such that the restriction of $\widetilde{g}$ to this subscheme has fibers of pure dimension k. For a reduced closed point $u \in U$, the multiplicity $\mathrm{mult}(U_{[f(u)]}, u)$ coincides with $(-1)^{k-1} h_Y^* E^k \cdot E_u$, where E_u denotes the $\widetilde{g}$-fiber of $h_Y^* E$ above any closed point $q \in \widetilde{W \times U}$ satisfying $h_W(q) = (p(u), u)$. Using generic flatness of $\widetilde{g}|_{h_Y^* E}$ one sees that the multiplicity is constant for an open subset of U due to the invariance of intersections in flat families. Repeating the argument and using Noetherian induction one sees that the multiplicity can only jump up in closed subsets due to the relative anti-ampleness of $h_Y^* E|_{h_Y^* E}$. $\square$

Definition 8.5. Let X be an integral projective variety and let $p : U \to W$ denote a family of effective k-cycles on X. Let $U_\mu \subset U$ denote the closed subset consisting of reduced closed points with multiplicity in the corresponding cycle-theoretic fiber at least μ. If $p_\mu : U_\mu \to W$ denotes the restriction of p, we define

$$\mathrm{mc}(p;\mu) := \mathrm{mc}(p_\mu).$$

If $\alpha \in N_k(X)_{\mathbb{Z}}$, we define

$$\mathrm{mc}(\alpha;\mu) := \sup_p \mathrm{mc}(p;\mu)$$

as we vary p over all families of effective k-cycles representing α. (If there are no such families, we set $\mathrm{mc}(\alpha;\mu) = 0$.)

We note the following easy properties.

Lemma 8.6. *Let X be an integral projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Z}}$. Then for any positive integer μ:*

1. *If β is represented by an effective cycle then* $\mathrm{mc}(\alpha+\beta;\mu) \geq \mathrm{mc}(\alpha;\mu)$.
2. *If both α and β are represented by effective cycles then* $\mathrm{mc}(\alpha+\beta;\mu) \geq \mathrm{mc}(\alpha;\mu) + \mathrm{mc}(\beta;\mu)$.
3. *For any $r \in \mathbb{Z}_{>0}$ we have* $\mathrm{mc}(\alpha;\mu) \leq \mathrm{mc}(r\alpha;r\mu)$.

Proof. The first two statements are obvious using the family sum construction since the multiplicity of a cycle at a point can only increase upon the addition of an effective cycle. For the third, if a family p represents α, then by rescaling the coefficients by r we obtain a family representing $r\alpha$, and it is clear the multiplicities go up by a factor of r. □

8.3. **Weighted mobility count.** By way of motivation, we start with a calculation. Let X be an integral projective variety of dimension n over an uncountable algebraically closed field and fix an ample divisor A on X. Choose b very general points on X and let $\phi : Y \to X$ be the blow-up of these points. Write $\{E_i\}_{i=1}^b$ for the exceptional divisors and E for their sum. Then

$$\alpha' = (\phi^*A - \epsilon_b(A)E)^{n-k}$$

is a pseudo-effective cycle class on Y satisfying $(-1)^{k-1}\alpha' \cdot E_i^k = \epsilon_b(A)^{n-k}$. It is unclear whether α' is represented by an effective cycle. However, it is approximated by complete intersections: if we perturb $\phi^*A - \epsilon_b(A)E$ slightly to a $\mathbb{Q}$-class in the ample cone, then for sufficiently large d the class $d^{n-k}(\alpha')$ is approximated by a complete intersection of very ample divisors.

Consider the pushforward $\alpha = \phi_*\alpha'$. The calculation above shows that $d^{n-k}\alpha$ is approximated arbitrarily closely by a k-dimensional cycle going through b points with multiplicity at least $\approx d^{n-k}\epsilon_b(A)^{n-k}$ at each. The following definition is designed to choose these families of cycles as the "optimal" ones representing α.

Definition 8.7. Let X be an integral projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Q}}$. Define the weighted mobility count of α to be

$$\mathrm{wmc}(\alpha) = \sup_\mu \mathrm{mc}(\mu\alpha;\mu).$$

where we vary μ over all positive integers such that $\mu\alpha \in N_k(X)_{\mathbb{Z}}$.

There are a couple features of this definition deserving comment. First, the wmc takes into account all multiples of α simultaneously but holds as constant the *ratio* of the coefficient of α to the multiplicity. This coheres with our calculation at the beginning of the section. The effect is to count a singularity of multiplicity μ "with weight μ".

Second, the wmc is calculated by insisting that each point contributing to the count have multiplicity $\geq \mu$. Instead, one could allow different multiplicities at every point and count with an appropriate weighting. By analogy, this calculation should correspond to understanding the nef thresholds of $\phi^*A - \sum a_i E_i$ where the a_i are allowed to vary. This is a much more subtle problem; our current phrasing has the advantage of a close relationship with Seshadri constants.

Remark 8.8. We expect that the "optimal" cycles for wmc are necessarily very singular. In other words, the sup in Definition 8.7 should rarely agree with the value for $\mu = 1$. Examples 8.1 and 8.22 demonstrate this principle for $\mathbb{P}^3$.

We next show that $\mathrm{wmc}(\alpha)$ is always finite. This implies that we can always choose a multiplicity μ maximizing the weighted mobility count, and in particular, the weighted mobility count can be computed by a double supremum as in the introduction (by combining Lemma 8.9 with Lemma 8.6). In fact, we prove a weak upper bound which also gives the correct asymptotic rate of growth.

Lemma 8.9. *Let X be an integral projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Q}}$. Fix a very ample Cartier divisor A on X. Then*

$$\mathrm{wmc}(\alpha) \leq \sup \left\{ A^n, \left(\frac{2}{(A^n)^{1/n}} \right)^{nk/n-k} (A^k \cdot \alpha)^{n/n-k} \right\}.$$

Proof. Fix b general points $\{x_i\}_{i=1}^b$ on X and let $\phi : Y \to X$ be the blow-up with total exceptional divisor E. Suppose that Z is an effective cycle of class $\mu\alpha$ with multiplicity $\geq \mu$ at each of the b points. Since $\phi^*A - \varepsilon(\{x_i\}, A)E$ is nef, the strict transform Z' of Z satisfies

$$0 \leq (\phi^*A - \varepsilon(\{x_i\}, A)E)^k \cdot Z' = A^k \cdot \mu\alpha - b\mu\varepsilon(\{x_i\}, A)^k.$$

Proposition 8.3 shows that either $b \leq A^n$ (when $t = 1$) or $\varepsilon(\{x_i\}, A) \geq \frac{1}{2}(A^n)^{1/n} b^{-1/n}$ (when $t > 1$ so that $\frac{t-1}{t} \geq \frac{1}{2}$). In the second case, for any family p_μ representing $\mu\alpha$, we obtain

$$\mathrm{mc}(p_\mu; \mu) \leq \left(\frac{2}{(A^n)^{1/n}} \right)^{nk/n-k} (A^k \cdot \alpha)^{n/n-k}.$$

Since this expression is independent of μ we obtain the proof. $\square$

Lemma 8.10. *Let X be an integral projective variety of dimension n and let $\alpha, \beta \in \overline{\mathrm{Eff}}_k(X)_{\mathbb{Q}}$. Suppose that some multiple of $\alpha - \beta$ is represented by an effective cycle. Then* $\mathrm{wmc}(\alpha) \geq \mathrm{wmc}(\beta)$.

Proof. Choose a sufficiently divisible integer μ so that $\mu(\alpha-\beta)$ is represented by an effective cycle, $\mathrm{wmc}(\alpha) = \mathrm{mc}(\mu\alpha; \mu)$, and $\mathrm{wmc}(\beta) = \mathrm{mc}(\mu\beta; \mu)$. Then $\mathrm{mc}(\mu\alpha; \mu) \geq \mathrm{mc}(\mu\beta; \mu)$ by Lemma 8.6. □

8.4. **Weighted mobility.** Lemma 8.9 indicates that we should take the following asymptotic definition.

Definition 8.11. Let X be an integral projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Q}}$. The weighted mobility of α is:

$$\mathrm{wmob}(\alpha) := \limsup_{m \to \infty} \frac{\mathrm{wmc}(m\alpha)}{m^{n/n-k}}.$$

Note that there is no longer a factor of $n!$. Lemma 8.9 shows that the weighted mobility is always finite.

Remark 8.12. One could also switch the limits and define

$$\widehat{\mathrm{wmob}}(\alpha) := \sup_{\mu \in \mathbb{Z}_{>0}} \limsup_{m \to \infty} \frac{\mathrm{mc}(m\mu\alpha; \mu)}{m^{n/n-k}}.$$

It is easy to see that $\widehat{\mathrm{wmob}}(\alpha) \leq \mathrm{wmob}(\alpha)$. (It is interesting, but subtle, to ask whether the two coincide.) Since the construction of wmob adheres more closely with the intuition developed for mob, we will work exclusively with this invariant.

The weighted mobility satisfies the same properties as mob with the same proofs; we make brief verifications when necessary.

Lemma 8.13. *Let X be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$. Fix a positive integer a. Then* $\mathrm{wmob}(a\alpha) = a^{\frac{n}{n-k}} \mathrm{wmob}(\alpha)$.

Proof. It suffices to consider the case when some multiple of α is effective. Then Lemma 8.10 shows that we can apply Lemma 2.6. □

Lemma 8.14. *Let X be an integral projective variety. Suppose that $\alpha, \beta \in N_k(X)_{\mathbb{Q}}$ are classes such that some positive multiple of each is represented by an effective cycle. Then* $\mathrm{wmob}(\alpha + \beta) \geq \mathrm{wmob}(\alpha) + \mathrm{wmob}(\beta)$.

Proof. It suffices to show that $\mathrm{wmc}(r\alpha + r\beta) \geq \mathrm{wmc}(r\alpha) + \mathrm{wmc}(r\beta)$ for any positive integer r. Choose μ_1, μ_2 to be multiplicities computing these two weighted multiplicity counts; then

$$\mathrm{wmc}(r\alpha + r\beta) \geq \mathrm{mc}(\mu_1\mu_2 r(\alpha + \beta); \mu_1\mu_2) \geq \mathrm{wmc}(r\alpha) + \mathrm{wmc}(r\beta)$$

by Lemma 8.6. □

It is not hard to show that a complete intersection of ample divisors always has positive wmob; a precise computation is done in Example 8.22. By Lemma 8.14 we obtain:

Corollary 8.15. *Let X be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$ be a big class. Then* $\mathrm{wmob}(\alpha) > 0$.

Then Lemma 2.8 shows:

Theorem 8.16. *Let X be an integral projective variety. The function* $\mathrm{wmob} : N_k(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ *is locally uniformly continuous on the interior of* $\overline{\mathrm{Eff}}_k(X)_{\mathbb{Q}}$.

Note that the multiplicity of a cycle at a general point can only increase upon taking a birational pushforward. Since mobility counts also can only increase upon pushforward, we have

Proposition 8.17. *Let $\pi : X \to Y$ be a surjective birational morphism of integral projective varieties. For any $\alpha \in N_k(X)_{\mathbb{Q}}$ we have* $\mathrm{wmob}(\pi_*\alpha) \geq \mathrm{wmob}(\alpha)$.

8.5. **Continuity of weighted mobility.** We analyze the continuity of the weighted mobility function using similar arguments as for the mobility. Again, the base case is in codimension 1.

Lemma 8.18. *Let X be an integral projective variety of dimension n. Let $\alpha \in N_{n-1}(X)_{\mathbb{Q}}$ and suppose that A is a very ample divisor and s is a positive integer such that $\alpha \cdot A^{n-1} < sA^n$. Fix a positive integer μ such that $\mu\alpha \in N_{n-1}(X)_{\mathbb{Z}}$. Fix a general element $H \in |sA|$. For $0 \leq i \leq \mu - 1$ there are positive integers k_i and collections of families of effective $(n-2)$-cycles $\{r_{i,j}\}_{j=1}^{k_i}$ on H such that $[r_{i,j}] \cdot A|_H^{n-2} < (\mu - i)sA|_H^{n-1}$ and*

$$\mathrm{mc}(\mu\alpha; \mu) \leq \sup_{i,j} \mathrm{mc}_H(r_{i,j}; \mu - i).$$

The $r_{i,j}$ are constructed by taking a family representing $\mu\alpha$, considering the subfamilies in which H occurs with multiplicity i, and then for each removing H (with the appropriate multiplicity) and restricting to H.

Proof. For a general $H \in |sA|$ we have that H is integral. Let $p : U \to W$ be a family of effective $(n-1)$-cycles representing $\mu\alpha$ realizing the weighted mobility count. By Lemma 3.6 we may suppose that $U \to X$ is projective and W is normal projective. Set $p_\mu : U_\mu \to W$ to be the closed subset of points whose multiplicity in the corresponding cycle-theoretic fiber is at least μ. Let $\widehat{U}_\mu$ denote the union of the components of U_μ which are not contained in the singular locus of H and denote by $\widehat{p}_\mu$ the restriction of p_μ; note that removing such components will not affect mobility counts. Set $b = \mathrm{mc}(p; \mu)$ so that

$$\widehat{U}_\mu^{\times_W b} \to X^{\times b}$$

is surjective.

Stratify W into locally closed subsets W_i such that H has multiplicity exactly i in every fiber of p over a closed point of W_i. Note that since $\mu\alpha - \mu H$ is not pseudo-effective, W_i is empty for $i \geq \mu$. Suppose that W_i has k_i irreducible components enumerated as $W_{i,j}$ and consider the restriction of the family p (in the sense of Construction 3.11) to $W_{i,j}$. This restricted family has one component corresponding to the constant divisor iH; after removing this divisor, we obtain a family $p_{i,j}$ of divisors on X not containing H in their support. Note that $[p_{i,j}] + is[A] = \mu\alpha$.

Replace $p_{i,j}$ by a projective normalized closure as in Lemma 3.6. For each family, let $p_{i,j,\mu-i}$ denote the closed locus of points whose multiplicities in the fibers is at least $\mu - i$. We claim that every fiber of $\widehat{p}_\mu$ over W is set theoretically contained in a fiber of some $p_{i,j,\mu-i}$. Indeed, since $\mu\alpha - \mu H$ is not pseudo-effective, we see that any point of multiplicity μ in a fiber of our original family p which is not contained in a singular point of H must have a contribution from components aside from H of multiplicity at least $\mu - i$. Thus, arguing as in Lemma 4.4 we obtain a finite collection of closed subvarieties $U_{i,j,\mu-i}$ of $W_{i,j} \times X$ such that

$$\bigcup_{i,j} U_{i,j,\mu-i}^{\times_{\cup W_{i,j}} b} \to X^{\times b}$$

is surjective. (In other words, away from the singular locus of H any point which has fiberwise multiplicity $\geq \mu$ in our original family must coincide with a point which has multiplicity $\geq \mu - i$ in one of our new families.) The base change of a surjective map is again surjective; in this way we obtain closed subsets $U^H_{i,j,\mu-i}$ such that the bth relative product over the base of the family maps surjectively onto $H^{\times b}$.

Recall that the support of divisors in the family $p_{i,j}$ never contains H. By intersecting the family $p_{i,j}$ with the divisor H we obtain a family $r_{i,j} : Q_{i,j} \to T_{i,j}$ of $(n-2)$-cycles on H satisfying

$$\begin{aligned}[r_{i,j}] \cdot A|_H^{n-2} &= [p_{i,j}] \cdot sA^{n-1} = s\mu\alpha \cdot A^{n-1} - is^2A^n \\ &< (\mu - i)s^2A^n = (\mu - i)sA|_H^{n-1}.\end{aligned}$$

Replace $r_{i,j}$ by a projective normalized closure as in Lemma 3.6, and note that every intersection of a member of $p_{i,j}$ with H is contained in the fiber of some $r_{i,j}$. Since the multiplicity of a cycle-theoretic fiber along a point in H can only increase upon intersection with H, we see that $Q_{i,j,\mu-i}$ set-theoretically contains the base change $U^H_{i,j,\mu-i}$. Again applying Lemma 4.4 we obtain the desired statement. □

Proposition 8.19. *Let X be an integral projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Q}}$.*

(1) *Suppose that A is a very ample Cartier divisor on X and s is a positive integer such that $\alpha \cdot A^{n-1} < sA^n$. Then*

$$\mathrm{wmc}(\alpha) < s^n A^n.$$

(2) *Suppose $n \geq 2$. Let A and H be very ample divisors and let s be a positive integer such that $\alpha - [H]$ is not pseudo-effective and $\alpha \cdot A^{n-2} \cdot H < sA^{n-1} \cdot H$. Then*

$$\mathrm{wmc}(\alpha) < s^{n-1}A^{n-1} \cdot H.$$

Proof. (1) The proof is by induction on the dimension of X. If X is a curve, then for any class $\beta \in N_0(X)_{\mathbb{Z}}$

$$\mathrm{mc}(\beta;\mu) = \left\lfloor \frac{\deg(\beta)}{\mu} \right\rfloor \leq \frac{\deg(\beta)}{\mu}.$$

Thus

$$\mathrm{wmc}(\alpha) = \sup_{\mu \text{ sufficiently divisible}} \mathrm{mc}(\mu\alpha;\mu) < s\deg(A).$$

In general, applying Lemma 8.18 to sA (and keeping the notation there) we find for any sufficiently divisible μ

$$\begin{aligned} \mathrm{mc}(\mu\alpha;\mu) &\leq \sup_{i,j} \mathrm{mc}_H(r_{i,j};\mu - i) \\ &\leq \sup_{i,j} \mathrm{wmc}_H([r_{i,j}]/\mu - i) \\ &< s^{n-1}A|_H^{n-1} = s^n A^n \end{aligned}$$

where the final inequality follows from induction.

(2) This follows by a similar argument by applying Lemma 8.18 to H, then applying (1) to α and A restricted to H. $\square$

Define the constants $\epsilon_{n,k}$, $\tau_{n,k}$ as in Theorem 5.16.

Theorem 8.20. *Let X be an integral projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$. Let A be a very ample divisor and let s be a positive integer such that $2^n\alpha \cdot A^k < sA^n$. Then*

(1)
$$\mathrm{wmc}(\alpha) < 2^{kn+3n} s^{\frac{n}{n-k}} A^n$$

(2) *Suppose furthermore that $2^n\alpha - [A]^{n-k}$ is not pseudo-effective. Then*
$$\mathrm{wmc}(\alpha) < 2^{kn+3n} s^{\frac{n}{n-k} - \epsilon_{n,k}} A^n.$$

(3) *Suppose that t is a positive integer such that $t \leq s$ and $2^n\alpha - t[A]^{n-k}$ is not pseudo-effective. Then*
$$\mathrm{wmc}(\alpha) < 2^{kn+3n} s^{\frac{n}{n-k} - \tau_{n,k}} t^{\tau_{n,k}} A^n.$$

The proof is essentially the same as the proof of Theorem 5.16. The key point is to understand how the weighted mobility count changes upon specializing our cycles into the hypersurface H; we will only highlight the necessary changes.

Proof. Choose a multiplicity μ and a family $p : U \to W$ representing $\mu\alpha$ such that $\mathrm{wmc}(\alpha) = \mathrm{mc}(p;\mu)$. Retain the constructions and notation of the proof of Theorem 5.16 for the family p and the divisor H. Thus we have a family of $(k-1)$-cycles $q : S \to T$ and families of k-cycles $p_i : V_i \to W_i$ which between them parametrize all the components of intersections of members of p with H.

Consider a fixed cycle $\sum_{i=1}^{r} a_i V_i$ in the family p. We may suppose that the first r' of these cycles are the components contained in H. Then for any point $x \in \mathrm{Supp}(V_i) \cap H$,

$$\begin{aligned}\mathrm{mult}(V,x) &= \sum_{i=1}^{r} a_i \,\mathrm{mult}(V_i,x) \\ &\leq \sum_{i=1}^{r'} a_i \,\mathrm{mult}(V_i,x) + \sum_{i=r'}^{r} a_i \,\mathrm{mult}(V_i \cdot H, x)\end{aligned}$$

as multiplicities can only increase upon intersection with a hyperplane. At least one of the terms on the right is $\geq \frac{1}{2}\,\mathrm{mult}(V,x)$. Thus, we see that every fiber of $p_\mu : U_\mu \to W$, where U_μ denotes the locus of points which have fiberwise multiplicity $\geq \mu$, is contained set theoretically in the union over all j of the loci in S of points which have fiberwise multiplicity at least $\mu/2$ and the loci in V_i of points which have fiberwise multiplicity at least $\mu/2$. Arguing in families, we have

$$\mathrm{mc}(\mu\alpha;\mu) \leq \mathrm{mc}(\mu\alpha \cdot H;\mu/2) + \sup_i \mathrm{mc}_H([p_i];\mu/2).$$

It is clear that $\mathrm{mc}(\mu\alpha \cdot H;\mu/2) \leq \mathrm{wmc}_H(2\alpha \cdot H)$ and $\sup_i \mathrm{mc}_H([p_i];\mu/2) \leq \sup_{i_*\beta \preceq \alpha} \mathrm{wmc}_H(2\beta)$.

At this point the proof of (1), (2), (3) proceeds exactly as in Theorem 5.16, but with some additional factors of 2:

- In the proof of (1), we must account for the halving of the multiplicity (or equivalently, the potential doubling of the integer s) at each step inductively. This is accomplished by the factor of 2^n in the inequality for $\alpha \cdot A^k$; the constant s is then preserved by the inductive step.
- In the proof of (2) we need to ensure inductively that while adding the coefficient of 2 to the families p_i the hypothesis of (2) still holds for the new families in the new ambient variety H. Again, the easiest way to do this is simply to ensure that $2^n\alpha - [A]$ is not pseudo-effective.
- The presence of the factor 2^n still exactly preserves the inductive structure of the argument for (3).

$\square$

Arguing just as in the proof of Theorem 5.20, we find:

Theorem 8.21. *Let X be an integral projective variety. Then the weighted mobility function* $\mathrm{wmob} : N_k(X)_{\mathbb{Q}} \to \mathbb{R}$ *can be extended to a continuous function on $N_k(X)$.*

8.6. **Computations of weighted mobility.** We now compute the weighted mobility in two special examples: for complete intersections of ample divisors and for big divisors on a smooth variety. For ease of notation we work over an uncountable algebraically closed field (although the computation would work equally well over any algebraically closed field using a slight perturbation of ε_b).

Example 8.22. Let X be an integral projective variety of dimension n over an uncountable algebraically closed field and let H be a big and nef $\mathbb{R}$-Cartier divisor. Set $\alpha = [H^{n-k}]$. We show that

$$\mathrm{wmob}(\alpha) = \mathrm{vol}(H).$$

Using continuity and homogeneity it suffices to consider the case when H is very ample.

We first show the inequality $\leq$. Suppose that Z is an effective $\mathbb{Z}$-cycle of class $m\mu\alpha$ which goes through b general points of X with multiplicity $\geq \mu$ at each. Let $\phi : Y \to X$ be the blow-up of b very general points and let E denote the sum of the exceptional divisors. Then the strict transform Z' satisfies

$$0 \leq Z' \cdot (\phi^*H - \varepsilon_b(H)E)^k \leq m\mu \,\mathrm{vol}(H) - b\mu\varepsilon_b(H)^k.$$

Choose a positive integer t such that $(t-1)^n \mathrm{vol}(H) < b \leq t^n \mathrm{vol}(H)$. Proposition 8.3 shows

$$\varepsilon_b(H) \geq \frac{1}{t}.$$

Combining the previous equations, we see that

$$\frac{b}{t^k} \leq m \,\mathrm{vol}(H).$$

If $t > 1$ then we have the relationship $\frac{1}{(t-1)^k} > (\frac{\mathrm{vol}(H)}{b})^{k/n}$, yielding

$$\left(\frac{t-1}{t}\right)^{kn/n-k} b \leq \mathrm{vol}(H) m^{n/n-k}$$

while if $t = 1$ then $b \leq \mathrm{vol}(H)$. For sufficiently large b, the left hand side approaches b. More precisely, for any $\delta > 0$ there is a constant b_0 such that $(1-\delta)\,\mathrm{wmc}(m\alpha) \leq \mathrm{vol}(H)m^{n/n-k}$ as soon as $\mathrm{wmc}(m\alpha)$ is at least b_0. Taking a limit, we find $\mathrm{wmob}(\alpha) \leq \mathrm{vol}(H)$.

To show the other inequality $\geq$, we need to construct complete intersection families on X. Fix a positive integer t and set $b = t^n \mathrm{vol}(H)$. Let $\phi : Y \to X$ be the blow-up of b very general points on X and let E be the sum of the exceptional divisors. By Proposition 8.3 we see that $\varepsilon_b(H) = 1/t$. Choose a sequence of rational numbers τ_i which limits to $\varepsilon_b(H)$ from beneath and choose integers c_i such that $c_i(\phi^*H - \tau_i E)$ is very ample. Take a complete intersection of elements of this very ample linear system and pushforward under ϕ. The result is an effective $\mathbb{Z}$-cycle of class $c_i^{n-k}\alpha$ which has multiplicity $\geq c_i^{n-k}\tau_i^{n-k}$ at each of the b points.

Set $m = t^{n-k}$ and $\mu = (c_i\tau_i)^{n-k}$. If we fix τ_i and let c_i vary over all sufficiently divisible integers, we find infinitely many values of μ for which there is a cycle of class $\frac{1}{\tau_i^{n-k}t^{n-k}}m\mu\alpha$ going through $m^{n/n-k}\mathrm{vol}(H)$ points with multiplicity $\geq \mu$ at each. Note that $\tau_i^{n-k}t^{n-k}$ can be made arbitrarily close to 1 for i sufficiently large. In other words, there is a sequence of positive rational numbers ϵ_i converging to 0 such that for each ϵ_i there are (infinitely many) values of μ satisfying

$$\mathrm{mc}(m\mu(1+\epsilon_i)\alpha;\mu) \geq m^{n/n-k}\,\mathrm{vol}(H).$$

This easily yields $\mathrm{wmob}(\alpha) \geq \mathrm{vol}(H)$.

Example 8.23. Suppose that X is a smooth projective variety of dimension n over an uncountable algebraically closed field and that L is a big $\mathbb{R}$-Cartier divisor on X. Then

$$\mathrm{wmob}([L]) = \mathrm{vol}(L).$$

By continuity on the big cone it suffices to consider the case when L is $\mathbb{Q}$-Cartier. Let $\phi : Y \to X$ and the ample $\mathbb{Q}$-Cartier divisor A on Y be an ϵ-Fujita approximation for X (constructed by [Tak07] in arbitrary characteristic). Example 8.22 shows that $\mathrm{wmob}([A]) = \mathrm{vol}(A)$, and pushing forward and applying Lemma 8.14 and Proposition 8.17 we see that $\mathrm{wmob}([L]) \geq \mathrm{vol}(L) - \epsilon$ for any positive $\epsilon > 0$.

Conversely, again fix an ϵ-Fujita approximation for L via a birational morphism $\phi : Y \to X$ and an ample $\mathbb{Q}$-Cartier divisor A on Y. Choose a fixed multiple rA that is very ample. Suppose we take a family of effective cycles $p_{m\mu}$ representing $m\mu[L]$. Consider the strict transform family $p'_{m\mu}$ on Y. Then we have $[p'_{m\mu}] \preceq m\mu\phi^*[L]$ and $\mathrm{mc}(p'_{m\mu};\mu) = \mathrm{mc}(p_{m\mu};\mu)$. Set b to be this common weighted mobility count; let $\phi' : Y' \to Y$ be the blow-up of b very general points and let E be the sum of the exceptional divisors. Let Z' be the strict transform of a cycle in the family $p'_{m\mu}$. Then

$$0 \leq Z' \cdot (\phi'^*A - \varepsilon_b(A)E)^{n-1} \leq m\mu\phi^*L \cdot A^{n-1} - b\mu\varepsilon_b(A)^{n-1}.$$

Choose a positive integer t so that $(t-1)^n r^n\,\mathrm{vol}(A) < b \leq t^n r^n\,\mathrm{vol}(A)$. Thus

$$\varepsilon_b(rA) = r\varepsilon_b(A) \geq \frac{1}{t}.$$

Repeating the calculation of Example 8.22 we see that if $t > 1$ then

$$\left(\frac{t-1}{t}\right)^{n(n-1)} b \leq m^n \frac{(\phi^*L \cdot A^{n-1})^n}{(A^n)^{n-1}} \leq m^n(1-\epsilon)^{1-n}\,\mathrm{vol}(L)$$

while if $t = 1$ then $b \leq r^n\,\mathrm{vol}(A)$. For any fixed $\delta > 0$, we see that the left hand side is at least $(1-\delta)b$ whenever b is sufficiently large. Thus for any fixed $\epsilon > 0$ and $\delta > 0$ we have $(1-\delta)\,\mathrm{wmc}(m[L]) \leq m^n(1-\epsilon)^{1-n}\,\mathrm{vol}(L)$ whenever the lefthand side is sufficiently large, yielding the result.

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