Zariski decomposition of curves on algebraic varieties

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Abstract

We introduce a Zariski decomposition for curve classes and use it to develop the theory of the volume function for curves defined by the second named author. For toric varieties and for hyperkähler manifolds the Zariski decomposition admits an interesting geometric interpretation. With the decomposition, we prove some fundamental positivity results for curve classes, such as a Morse-type inequality. We compare the volume of a curve class with its mobility, yielding some surprising results about asymptotic point counts. Finally, we give a number of applications to birational geometry, including a refined structure theorem for the movable cone of curves.

Contents

1 Introduction

In [Zar62] Zariski introduced a fundamental tool for studying linear series on a surface now known as a Zariski decomposition. Over the past 50 years the Zariski decomposition and its generalizations to divisors in higher dimensions have played a central role in birational geometry. We introduce an analogous decomposition for curve classes on varieties of arbitrary dimension. Our decomposition is defined for big curve classes – elements of the interior of the pseudo-effective cone of curves $\overline{\mathrm{Eff}}_1(X)$. Throughout we work over $\mathbb{C}$, but the main results also hold over an algebraically closed field or in the Kähler setting (see Section 1.5).

Definition 1.1. Let X be a projective variety of dimension n and let $\alpha \in N_1(X)$ be a big curve class. Then a Zariski decomposition for α is a decomposition

$$\alpha = B^{n-1} + \gamma$$

where B is a big and nef $\mathbb{R}$-Cartier divisor class, γ is pseudo-effective, and $B \cdot \gamma = 0$. We call B^{n-1} the "positive part" and γ the "negative part" of the decomposition.

This definition directly generalizes Zariski's original definition, which (for big classes) is given by similar intersection criteria. It also generalizes the σ-decomposition of [Nak04], and mirrors the Zariski decomposition of [FL13], in the following sense. The basic feature of a Zariski decomposition is that the positive part should retain all the "positivity" of the original class. In our setting, we will measure the positivity of a curve class using an interesting new volume-type function defined in [Xia15].

Definition 1.2. (see [Xia15, Definition 1.1]) Let X be a projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then the volume of α is defined to be

$$\widehat{\mathrm{vol}}(\alpha) = \inf_{A \text{ big and nef divisor class}} \left(\frac{A \cdot \alpha}{\mathrm{vol}(A)^{1/n}} \right)^{\frac{n}{n-1}}.$$

We say that a big and nef divisor class A computes $\widehat{\mathrm{vol}}(\alpha)$ if this infimum is achieved by A. When α is a curve class that is not pseudo-effective, we set $\widehat{\mathrm{vol}}(\alpha) = 0$.

This is a kind of polar transformation of the volume function for divisors. It is motivated by the realization that the volume of a divisor has a similar intersection-theoretic description against curves as in [Xia15, Theorem 2.1]. [Xia15] proves that $\widehat{\mathrm{vol}}$ satisfies many of the desirable analytic features of the volume for divisors.

By [FL13, Proposition 5.3], we know that the σ-decomposition $L = P_\sigma(L) + N_\sigma(L)$ is the unique decomposition of L into a movable piece and a pseudo-effective piece such that $\mathrm{vol}(L) = \mathrm{vol}(P_\sigma(L))$. In the same way, the decomposition of Definition 1.1 is compatible with the volume function for curves:

Theorem 1.3. *Let X be a projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)^\circ$ be a big curve class. Then α admits a unique Zariski decomposition $\alpha = B^{n-1} + \gamma$. Furthermore,*

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(B^{n-1}) = \mathrm{vol}(B)$$

and B is the unique big and nef divisor class with this property satisfying $B^{n-1} \preceq \alpha$. Any big and nef divisor class computing $\widehat{\mathrm{vol}}(\alpha)$ is proportional to B.

We define the complete intersection cone $\mathrm{CI}_1(X)$ to be the closure of the set of classes of the form A^{n-1} for an ample divisor A on X. The positive part of the Zariski decomposition takes values in $\mathrm{CI}_1(X)$.

Our goal is to develop the theory of Zariski decompositions of curves and the theory of $\widehat{\mathrm{vol}}$. Due to their close relationship, we will see that is very fruitful to develop the two theories in parallel. In particular, we recover Zariski's original intuition that asymptotic point counts coincide with numerical invariants for curves.

Example 1.4. If X is an algebraic surface, then the Zariski decomposition provided by Theorem 1.3 coincides (for big classes) with the numerical version of the classical definition of [Zar62]. Indeed, using Proposition 5.14 one sees that the negative part γ is represented by an effective curve N. The self-intersection matrix of N must be negative-definite by the Hodge Index Theorem. (See e.g. [Nak04] for another perspective focusing on the volume function.)

Example 1.5. An important feature of Zariski decompositions and $\widehat{\mathrm{vol}}$ for curves is that they can be calculated via intersection theory directly on X once one has identified the nef cone of divisors. (In contrast, the analogous divisor constructions may require passing to birational models of X to admit an interpretation via intersection theory.) This is illustrated by Example 5.5 where we calculate the Zariski decomposition of any curve class on the projective bundle over $\mathbb{P}^1$ defined by $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)$.

Example 1.6. If X is a Mori Dream Space, then the movable cone of divisors admits a chamber structure defined via the ample cones on small $\mathbb{Q}$-factorial modifications. This chamber structure behaves compatibly with the σ-decomposition and the volume function for divisors.

For curves we obtain a complementary picture. The movable cone of curves admits a "chamber structure" defined via the complete intersection cones on small $\mathbb{Q}$-factorial modifications. However, the Zariski decomposition and volume of curves are no longer invariant under small $\mathbb{Q}$-factorial modifications but instead exactly reflect the changing structure of the pseudo-effective cone of curves. Thus the Zariski decomposition is the right tool to understand the birational geometry of movable curves on X. See Example 7.5 for more details.

It turns out that most of the important properties of the volume function for divisors have analogues in the curve case. First of all, Zariski decompositions are continuous and satisfy a linearity condition (Theorems 5.3 and 5.6). While the negative part of a Zariski decomposition need not be represented by an effective curve, Proposition 5.14 proves a "rigidity" result which is a suitable analogue of the familiar statement for divisors. Zariski decompositions and $\widehat{\mathrm{vol}}$ exhibit very nice birational behavior, discussed in Section 5.6.

Other important properties include a Morse inequality (Corollary 5.19), the strict log concavity of $\widehat{\mathrm{vol}}$ (Theorem 5.10), and the following description of the derivative which mirrors the results of [BFJ09] and [LM09].

Theorem 1.7. *Let X be a projective variety of dimension n. Then the function $\widehat{\mathrm{vol}}$ is $\mathcal{C}^1$ on the big cone of curves. More precisely, let α be a big curve class on X and write $\alpha = B^{n-1} + \gamma$*

for its Zariski decomposition. For any curve class β, we have

$$\frac{d}{dt}\bigg|_{t=0} \widehat{\mathrm{vol}}(\alpha + t\beta) = \frac{n}{n-1} B \cdot \beta.$$

1.1 Examples

The Zariski decomposition is particularly striking for varieties with a rich geometric structure. We discuss two examples: toric varieties and hyperkähler manifolds.

First, suppose that X is a simplicial projective toric variety of dimension n defined by a fan Σ. A class α in the interior of the movable cone of curves corresponds to a positive Minkowski weight on the rays of Σ. A fundamental theorem of Minkowski attaches to such a weight a polytope P_α whose facet normals are the rays of Σ and whose facet volumes are determined by the weights.

Theorem 1.8. *The complete intersection cone of X is the closure of the positive Minkowski weights α whose corresponding polytope P_α has normal fan Σ. For such classes we have $\widehat{\mathrm{vol}}(\alpha) = n!\,\mathrm{vol}(P_\alpha)$.*

In fact, for any positive Minkowski weight the normal fan of the polytope P_α constructed by Minkowski's Theorem describes the birational model associated to α as in Example 1.6.

We next discuss the Zariski decomposition and volume of a positive Minkowski weight α. In this setting, the calculation of the volume is the solution of an isoperimetric problem: fixing P_α, amongst all polytopes whose normal fan refines Σ there is a unique Q (up to homothety) minimizing the mixed volume calculation

$$\frac{V(P_\alpha^{n-1}, Q)}{\mathrm{vol}(Q)^{1/n}}.$$

If we let Q vary over all polytopes then the Brunn-Minkowski inequality shows that the minimum is given by $Q = cP_\alpha$, but the normal fan condition on Q yields a new version of this classical problem.

From this viewpoint, the compatibility with the Zariski decomposition corresponds to the fact that the solution of an isoperimetric problem should be given by a condition on the derivative. We show in Section 8 that this isoperimetric problem can be solved (with no minimization necessary) using the Zariski decomposition.

We next turn to hyperkähler manifolds. The results of [Bou04, Section 4] show that the volume and σ-decomposition of divisors satisfy a natural compatibility with the Beauville-Bogomolov form. We prove the analogous properties for curve classes. The following theorem is phrased in the Kähler setting, although the analogous statements in the projective setting are also true.

Theorem 1.9. *Let X be a hyperkähler manifold of dimension n and let q denote the bilinear form on $H^{n-1,n-1}(X)$ induced via duality from the Beauville-Bogomolov form on $H^{1,1}(X)$.*

1. *The cone of complete intersection $(n-1, n-1)$-classes is q-dual to the cone of pseudo-effective $(n-1, n-1)$-classes.*

2. *If α is a complete intersection $(n-1,n-1)$-class then* $\widehat{\mathrm{vol}}(\alpha) = q(\alpha,\alpha)^{n/2(n-1)}$.

3. *Suppose α lies in the interior of the cone of pseudo-effective $(n-1,n-1)$-classes and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. Then $q(B^{n-1},\gamma) = 0$ and if γ is non-zero then $q(\gamma,\gamma) < 0$.*

1.2 Volume and mobility

The main feature of the Zariski decomposition for surfaces is that it clarifies the relationship between the asymptotic sectional properties of a divisor and its intersection-theoretic properties. By analogy with the work of [Zar62], it is natural to wonder how the volume function $\widehat{\mathrm{vol}}$ of a curve class is related to the asymptotic geometry of the curves represented by the class. We will analyze this question by comparing $\widehat{\mathrm{vol}}$ with two "volume-type" functions for curves: the mobility function and the weighted mobility function of [Leh13b]. This will also allow us to contrast our definition of Zariski decompositions with the notion from [FL13].

The definition of the mobility is a close parallel to the definition of the volume of a divisor via asymptotic growth of sections.

Definition 1.10. (see [Leh13b, Definition 1.1]) Let X be a projective variety of dimension n and let $\alpha \in N_1(X)$ be a curve class with integer coefficients. The mobility of α is defined to be

$$\mathrm{mob}(\alpha) := \limsup_{m\to\infty} \frac{\max\left\{ b \in \mathbb{Z}_{\geqslant 0} \;\middle|\; \begin{array}{c}\text{Any } b \text{ general points are contained} \\ \text{in an effective curve of class } m\alpha\end{array} \right\}}{m^{\frac{n}{n-1}}/n!}.$$

In [Leh13b], the first named author shows that the mobility extends to a continuous homogeneous function on all of $N_1(X)$. The following theorem continues a project begun by the second named author (see [Xia15, Conjecture 3.1 and Theorem 3.2]). Proposition 1.22 below gives a related statement.

Theorem 1.11. *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then:*

1. $\widehat{\mathrm{vol}}(\alpha) \leqslant \mathrm{mob}(\alpha) \leqslant n!\widehat{\mathrm{vol}}(\alpha)$.

2. *Assume Conjecture 1.12 below. Then* $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$.

The driving force behind Theorem 1.11 is a comparison of the Zariski decomposition for mob constructed in [FL13] with the Zariski decomposition for $\widehat{\mathrm{vol}}$ defined above. The second part of this theorem relies on the following (difficult) conjectural description of the mobility of a complete intersection class:

Conjecture 1.12. *(see [Leh13b, Question 7.1]) Let X be a smooth projective variety of dimension n and let A be an ample divisor on X. Then*

$$\mathrm{mob}(A^{n-1}) = A^n.$$

Theorem 1.11 is quite surprising: it suggests that the mobility count of *any* curve class is optimized by complete intersection curves.

Example 1.13. Let α denote the class of a line on $\mathbb{P}^3$. The mobility count of α is determined by the following enumerative question: what is the minimal degree of a curve through b general points of $\mathbb{P}^3$? The answer is unknown, even in an asymptotic sense.

[Per87] conjectures that the "optimal" curves (which maximize the number of points relative to their degree to the 3/2) are complete intersections of two divisors of the same degree. Theorem 1.11 supports a vast generalization of Perrin's conjecture to all big curve classes on all smooth projective varieties.

While the weighted mobility of [Leh13b] is slightly more complicated, it allows us to prove an unconditional statement. The weighted mobility is similar to the mobility, but it counts singular points of the cycle with a higher "weight"; we give the precise definition in Section 10.1.

Theorem 1.14. *Let X be a smooth projective variety and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then $\widehat{\mathrm{vol}}(\alpha) = \mathrm{wmob}(\alpha)$.*

Thus $\widehat{\mathrm{vol}}$ captures some fundamental aspects of the asymptotic geometric behavior of curves.

1.3 Formal Zariski decompositions

According to the philosophy of [FL13], one should interpret the Zariski decomposition (or the σ-decomposition for divisors) as capturing the failure of strict log concavity of the volume function. This suggests that one should use the tools of convex analysis – in particular some version of the Legendre-Fenchel transform – to analyze Zariski decompositions. We will show that many of the basic analytic properties of $\widehat{\mathrm{vol}}$ and Zariski decompositions can in fact be deduced from a much more general duality framework for arbitrary concave functions. From this perspective, the most surprising feature of $\widehat{\mathrm{vol}}$ is that it captures actual geometric information about curves representing the corresponding class.

Let $\mathcal{C}$ be a full dimensional closed proper convex cone in a finite dimensional vector space. For any $s > 1$, let $\mathrm{HConc}_s(\mathcal{C})$ denote the collection of functions $f : \mathcal{C} \to \mathbb{R}$ that are upper-semicontinuous, homogeneous of weight $s > 1$, strictly positive on the interior of $\mathcal{C}$, and which are s-concave in the sense that

$$f(v)^{1/s} + f(x)^{1/s} \leqslant f(x+v)^{1/s}$$

for any $v, x \in \mathcal{C}$. In this context, the correct analogue of the Legendre-Fenchel transform is the (concave homogeneous) polar transform. For any $f \in \mathrm{HConc}_s(\mathcal{C})$, the polar $\mathcal{H}f$ is an element of $\mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$ for the dual cone $\mathcal{C}^*$ defined as

$$\mathcal{H}f(w^*) = \inf_{v \in \mathcal{C}^\circ} \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/s-1} \qquad \forall w^* \in \mathcal{C}^*.$$

We define what it means for $f \in \mathrm{HConc}_s(\mathcal{C})$ to have a "Zariski decomposition structure" and show that it follows from the differentiability of $\mathcal{H}f$. This is the analogue in our situation of how the Legendre-Fenchel transform relates differentiability and strict convexity. Furthermore, this structure allows one to systematically transform geometric inequalities from one setting to the other. Many of the basic geometric inequalities in algebraic geometry – and hence for polytopes or convex bodies via toric varieties (as in [Tei82] and [Kho89] and the references therein) – can be understood in this framework.

Example 1.15. Let q be a bilinear form on a vector space V of signature $(1, \dim V - 1)$ and set $f(v) = q(v, v)$. Suppose $\mathcal{C}$ is a closed full-dimensional convex cone on which f is non-negative. Identifying V with V^* under q, we see that $\mathcal{C} \subset \mathcal{C}^*$ and that $\mathcal{H}f|_{\mathcal{C}} = f$ by the Hodge inequality. Then $\mathcal{H}f$ on the entire cone $\mathcal{C}^*$ is controlled by a "Zariski decomposition" projecting onto $\mathcal{C}$. This is of course the familiar picture for surfaces, where f is the self-intersection on the nef cone and $\mathcal{H}f$ is the volume on the pseudo-effective cone.

Example 1.16. Fix a spanning set of vectors $\mathcal{Q}$ in $\mathbb{R}^n$. Then the set of all polytopes whose facet normals are (up to rescaling) a subset of $\mathcal{Q}$ are naturally parametrized by a cone $\mathcal{C}$ in a vector space V. The volume function defines a homogeneous non-negative function on $\mathcal{C}$.

The dual space V^* is the set of Minkowski weights on $\mathcal{Q}$. A classical theorem of Minkowski shows that each strictly positive Minkowski weight α defines a polytope P_α whose facet normals are given by $\mathcal{Q}$ and whose facet areas are controlled by the values of α. Such weights define a cone $\mathcal{M}$ contained in $\mathcal{C}^*$. Then the polar of the function vol restricted to $\mathcal{M}$ is again just the volume (after normalizing properly). This is proved in Section 8. It would be interesting to see a version which applies to arbitrary convex bodies.

1.4 Other applications

Finally, we discuss some connections with other areas of birational geometry.

An important ancillary goal of the paper is to prove some new results concerning the volume function of divisors and the movable cone of curves. The key tool is another intersection-theoretic invariant $\mathfrak{M}$ of nef curve classes from [Xia15, Definition 2.2]. Since the results seem likely to be of independent interest, we recall some of them here.

First of all, we give a refined version of a theorem of [BDPP13] describing the movable cone of curves. In [BDPP13], it is proved that the movable cone $\mathrm{Mov}_1(X)$ is *generated* by $(n-1)$-self positive products of big divisors. We show that the interior points in $\mathrm{Mov}_1(X)$ are *exactly* the set of $(n-1)$-self positive products of big divisors on the interior of $\mathrm{Mov}^1(X)$.

Theorem 1.17. *Let X be a smooth projective variety of dimension n and let α be an interior point of $\mathrm{Mov}_1(X)$. Then there is a unique big movable divisor class L_α lying in the interior of $\mathrm{Mov}^1(X)$ and depending continuously on α such that $\langle L_\alpha^{n-1} \rangle = \alpha$.*

Example 1.18. This result shows that the map $\langle -^{n-1} \rangle$ is a homeomorphism from the interior of the movable cone of divisors to the interior of the movable cone of curves. Thus, any chamber decomposition of the movable cone of curves naturally induces a decomposition of the movable cone of divisors and vice versa. This relationship could be useful in the study of geometric stability conditions (as in [Neu10]).

As an interesting corollary, we obtain:

Corollary 1.19. *Let X be a projective variety of dimension n. Then the rays over classes of irreducible curves which deform to dominate X are dense in* $\mathrm{Mov}_1(X)$.

We can describe the boundary of $\mathrm{Mov}_1(X)$.

Theorem 1.20. *Let X be a smooth projective variety and let α be a curve class lying on the boundary of* $\mathrm{Mov}_1(X)$. *Then exactly one of the following alternatives holds:*

- $\alpha = \langle L^{n-1} \rangle$ *for a big movable divisor class L on the boundary of* $\mathrm{Mov}^1(X)$.
- $\alpha \cdot M = 0$ *for a movable divisor class M.*

The homeomorphism from $\mathrm{Mov}^1(X)^\circ \to \mathrm{Mov}_1(X)^\circ$ *extends to map the big movable divisor classes on the boundary of* $\mathrm{Mov}^1(X)$ *bijectively to the classes of the first type.*

We also extend [BFJ09, Theorem D] to a wider class of divisors.

Theorem 1.21. *Let X be a smooth projective variety of dimension n. For any two big divisor classes L_1, L_2, we have*

$$\mathrm{vol}(L_1 + L_2)^{1/n} \geqslant \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n}$$

with equality if and only if the (numerical) positive parts $P_\sigma(L_1)$, $P_\sigma(L_2)$ are proportional. Thus the function $L \mapsto \mathrm{vol}(L)^{1/n}$ is strictly concave on the cone of big and movable divisors.

A basic technique in birational geometry is to bound the positivity of a divisor using its intersections against specified curves. These results can profitably be reinterpreted using the volume function of curves. For example:

Proposition 1.22. *Let X be a smooth projective variety of dimension n. Choose positive integers $\{k_i\}_{i=1}^r$. Suppose that $\alpha \in \mathrm{Mov}_1(X)$ is represented by a family of irreducible curves such that for any collection of general points $x_1, x_2, \ldots, x_r, y$ of X, there is a curve in our family which contains y and contains each x_i with multiplicity $\geqslant k_i$. Then*

$$\widehat{\mathrm{vol}}(\alpha)^{\frac{n-1}{n}} \geqslant \frac{\sum_i k_i}{r^{1/n}}.$$

We can thus apply volumes of curves to study Seshadri constants, bounds on volume of divisors, and other related topics. We defer a more in-depth discussion to Section 11, contenting ourselves with a fascinating example.

Example 1.23. If X is rationally connected, it is interesting to analyze the possible volumes for classes of special rational curves on X. When X is a Fano variety of Picard rank 1, these invariants will be closely related to classical invariants such as the length and degree.

For example, we say that $\alpha \in N_1(X)$ is a rationally connecting class if for any two general points of X there is a chain of rational curves of class α connecting the two points. Is there a uniform upper bound (depending only on the dimension) for the minimal volume of a rationally connecting class on a rationally connected X? [KMM92] and [Cam92] show that this is true for smooth Fano varieties. We discuss this question briefly in Section 11.2.

1.5 Outline of paper

In this paper we will work with projective varieties over $\mathbb{C}$ for simplicity of arguments and for compatibility with cited references. However, except for certain results in Section 6 through Section 11, all the results will extend to smooth varieties over arbitrary algebraically closed fields on the one hand and arbitrary compact Kähler manifolds on the other. We give a general framework for this extension in Sections 2.3 and 2.4 and then explain the details as we go.

In Section 2 we review the necessary background, and make several notes explaining how the proofs can be adjusted to arbitrary algebraically closed fields and compact Kähler manifolds. Sections 3 and 4 discuss polar transforms and formal Zariski decompositions for log concave functions. In Section 5 we construct the Zariski decomposition of curves and study its basic properties and its relationship with $\widehat{\mathrm{vol}}$. In Section 6, we give a refined structure of the movable cone of curves and generalize several results on big and nef divisors to big and movable divisors. Section 7 compares the complete intersection and movable cone of curves. Section 8 discusses toric varieties, and Section 9 is devoted to the study of hyperkähler manifolds. In Section 10 we compare the mobility function and $\widehat{\mathrm{vol}}$. Section 11 outlines some applications to birational geometry. Finally, Appendix A collects some "reverse" Khovanskii-Teissier type results in the analytic setting and a result related to the transcendental holomorphic Morse inequality, and Appendix B gives a toric example where the complete intersection cone of curves is not convex.

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2 Preliminaries

In this section, we first fix some notations over a projective variety X:

$N^1(X)$: the real vector space of numerical classes of divisors;

$N_1(X)$: the real vector space of numerical classes of curves;

$\overline{\mathrm{Eff}}^1(X)$: the cone of pseudo-effective divisor classes.

$\mathrm{Nef}^1(X)$: the cone of nef divisor classes;

$\overline{\mathrm{Eff}}_1(X)$: the cone of pseudo-effective curve classes;

$\mathrm{Mov}_1(X)$: the cone of movable curve classes, equivalently by [BDPP13] the dual of $\overline{\mathrm{Eff}}^1(X)$;

$\mathrm{CI}_1(X)$: the closure of the set of all curve classes of the form A^{n-1} for an ample divisor A;

With only a few exceptions, capital letters A, B, D, L will denote $\mathbb{R}$-Cartier divisor classes and greek letters α, β, γ will denote curve classes. For two curve classes α, β, we write $\alpha \succeq \beta$ (resp.

$\alpha \preceq \beta$) to denote that $\alpha - \beta$ (resp. $\beta - \alpha$) belongs to $\overline{\mathrm{Eff}}_1(X)$. We will do similarly for divisor classes, or two elements of a cone $\mathcal{C}$ if the cone is understood.

We will use the notation $\langle - \rangle$ for the positive product as in [BDPP13], [BFJ09] and [Bou02]. We make a few remarks on this construction for singular projective varieties. Suppose that X has dimension n. Then $N_{n-1}(X)$ denotes the vector space of $\mathbb{R}$-classes of Weil divisors up to numerical equivalence as in [Ful84, Chapter 19]. In this setting, the 1st and $(n-1)$st positive product should be interpreted respectively as maps $\overline{\mathrm{Eff}}^1(X) \to N_{n-1}(X)$ and $\overline{\mathrm{Eff}}^1(X)^{\times n-1} \to \mathrm{Mov}_1(X)$. We will also let $P_\sigma(L)$ denote the positive part in this sense – that is, pullback L to closer and closer Fujita approximations, take its positive part, and push the numerical class forward to X as a numerical Weil divisor class. With these conventions, we still have the crucial result of [BFJ09] and [LM09] that the derivative of the volume is controlled by intersecting against the positive part.

We define the movable cone of divisors $\mathrm{Mov}^1(X)$ to be the subset of $\overline{\mathrm{Eff}}^1(X)$ consisting of divisor classes L such that $N_\sigma(L) = 0$ and $P_\sigma(L) = L \cap [X]$. On any projective variety, by [Ful84, Example 19.3.3] capping with X defines an injective linear map $N^1(X) \to N_{n-1}(X)$. Thus if $D, L \in \mathrm{Mov}^1(X)$ have the same positive part in $N_{n-1}(X)$, then by the injectivity of the capping map we must have $D = L$.

To extend our results to arbitrary compact Kähler manifolds, we need to deal with transcendental objects which are not given by divisors or curves. Let X be a compact Kähler manifold of dimension n. By analogue with the projective situation, we need to deal with the following spaces and positive cones:

$H^{1,1}_{BC}(X, \mathbb{R})$: the real Bott-Chern cohomology group of bidegree $(1,1)$;

$H^{n-1,n-1}_{BC}(X, \mathbb{R})$: the real Bott-Chern cohomology group of bidegree $(n-1, n-1)$;

$\mathcal{N}(X)$: the cone of pseudo-effective $(n-1, n-1)$-classes;

$\mathcal{M}(X)$: the cone of movable $(n-1, n-1)$-classes;

$\overline{\mathcal{K}}(X)$: the cone of nef $(1,1)$-classes, i.e. the closure of the Kähler cone generated by Kähler classes;

$\mathcal{E}(X)$: the cone of pseudo-effective $(1,1)$-classes.

Recall that a $(1,1)$ (or $(n-1, n-1)$) class is a pseudo-effective class if it contains a d-closed positive current, and an $(n-1, n-1)$-class is a movable class if it is contained in the closure of the cone generated by the classes of the form $\mu_*(\widetilde{\omega}_1 \wedge ... \wedge \widetilde{\omega}_{n-1})$ where $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{\omega}_1, ..., \widetilde{\omega}_{n-1}$ are Kähler metrics on $\widetilde{X}$. For the basic theory of positive currents, we refer the reader to [Dem12].

If X is a smooth projective variety over $\mathbb{C}$, then we have the following relations (see e.g. [BDPP13])

$$\mathrm{Nef}^1(X) = \overline{\mathcal{K}}(X) \cap N^1(X),\ \overline{\mathrm{Eff}}^1(X) = \mathcal{E}(X) \cap N^1(X)$$

and

$$\overline{\mathrm{Eff}}_1(X) = \mathcal{N}(X) \cap N_1(X),\ \mathrm{Mov}_1(X) = \mathcal{M}(X) \cap N_1(X).$$

2.1 Khovanskii-Teissier inequalities

We collect several results which we will frequently use in our paper. In every case, the statement for arbitrary projective varieties follows from the familiar smooth versions via a pullback argument. Recall the well-known Khovanskii-Teissier inequalities for a pair of nef divisors over projective varieties (see e.g. [Tei79]).

- Let X be a projective variety and let A, B be two nef divisor classes on X. Then we have

$$A^{n-1} \cdot B \geqslant (A^n)^{n-1/n}(B^n)^{1/n}$$

We also need the characterization of the equality case in the above inequality as in [BFJ09, Theorem D] – see also [FX14b] for the analytic proof for transcendental classes in the Kähler setting. (We call this characterization Teissier's proportionality theorem as it was first proposed and studied by B. Teissier.)

- Let X be a projective variety and let A, B be two big and nef divisor classes on X. Then

$$A^{n-1} \cdot B = (A^n)^{n-1/n}(B^n)^{1/n}$$

  if and only if A and B are proportional.

We next prove a more general version of Teissier's proportionality theorem for n big and nef $(1,1)$-classes over compact Kähler manifolds (thus including projective varieties defined over $\mathbb{C}$) which follows easily from the result of [FX14b].

Theorem 2.1. *Let X be a compact Kähler manifold of dimension n, and let $B_1, ..., B_n$ be n big and nef $(1,1)$-classes over X. Then we have*

$$B_1 \cdot B_2 \cdots B_n \geqslant (B_1^n)^{1/n} \cdot (B_2^n)^{1/n} \cdots (B_n^n)^{1/n},$$

where the equality is obtained if and only if $B_1, ..., B_n$ are proportional.

We include a proof, since we are not aware of any reference in the literature. The proof reduces the global inequalities to the pointwise Brunn-Minkowski inequalities by solving Monge-Ampère equations [FX14b] (see also [Dem93] for related result), and then applies the result of [FX14b] – where the key technique and estimates go back to [FX14a] – for a pair of big and nef classes (see also [BFJ09, Theorem D] for divisor classes).

Recall that the ample locus $\mathrm{Amp}(D)$ of a big $(1,1)$-class D is the set of points $x \in X$ such that there is a strictly positive current $T_x \in D$ with analytic singularities which is smooth near x. When L is a big $\mathbb{R}$-divisor class on a smooth projective variety X, then the ample locus $\mathrm{Amp}(L)$ is equal to the complement of the augmented base locus $\mathbb{B}_+(L)$ (see [Bou04]).

Proof. Without loss of generality, we can assume all the $B_i^n = 1$. Then we need to prove

$$B_1 \cdot B_2 \cdots B_n \geqslant 1,$$

with the equality obtained if and only if $B_1, ..., B_n$ are equal.

To this end, we fix a smooth volume form Φ with $\mathrm{vol}(\Phi) = 1$. We choose a smooth $(1,1)$-form b_j in the class B_j. Then by [BEGZ10, Theorem C], for every class B_j we can solve the following singular Monge-Ampère equation

$$\langle (b_j + i\partial\bar{\partial}\psi_j)^n \rangle = \Phi,$$

where $\langle - \rangle$ denotes the non-pluripolar products of positive currents (see [BEGZ10, Definition 1.1 and Proposition 1.6]).

Denote $T_j = b_j + i\partial\bar{\partial}\psi_j$, then [BEGZ10, Theorem B] implies T_j is a positive current with minimal singularities in the class B_j. Moreover, T_j is a Kähler metric over the ample locus $\mathrm{Amp}(B_j)$ of the big class B_j by [BEGZ10, Theorem C].

Note that $\mathrm{Amp}(B_j)$ is a Zariski open set of X. Denote $\Omega = \mathrm{Amp}(B_1) \cap ... \cap \mathrm{Amp}(B_n)$, which is also a Zariski open set. By [BEGZ10, Definition 1.17], we then have

$$\begin{aligned} B_1 \cdot B_2 \cdots B_n &= \int_X \langle T_1 \wedge ... \wedge T_n \rangle \\ &= \int_\Omega T_1 \wedge ... \wedge T_n, \end{aligned}$$

where the second line follows because the non-pluripolar product $\langle T_1 \wedge ... \wedge T_n \rangle$ puts no mass on the subvariety $X \backslash \Omega$ and all the T_j are Kähler metrics over Ω.

For any point $x \in \Omega$, we have the following pointwise Brunn-Minkowski inequality

$$T_1 \wedge ... \wedge T_n \geqslant \left(\frac{T_1^n}{\Phi} \right)^{1/n} \cdots \left(\frac{T_n^n}{\Phi} \right)^{1/n} \Phi = \Phi$$

with equality if and only if the Kähler metrics T_j are proportional at x. Here the second equality follows because we have $T_j^n = \Phi$ on Ω. In particular, we get the Khovanskii-Teissier inequality

$$B_1 \cdot B_2 \cdots B_n \geqslant 1.$$

And we know the equality $B_1 \cdot B_2 \cdots B_n = 1$ holds if and only if the Kähler metrics T_j are pointwise proportional. At this step, we can not conclude that the Kähler metrics T_j are equal over Ω since we can not control the proportionality constants from the pointwise Brunn-Minkowski inequalities. However, for any pair of T_i and T_j, we have the following pointwise equality over Ω:

$$T_i^{n-1} \wedge T_j = \left(\frac{T_i^n}{\Phi} \right)^{n-1/n} \cdot \left(\frac{T_j^n}{\Phi} \right)^{1/n} \Phi,$$

since T_i and T_j are pointwise proportional over Ω. This implies the equality

$$B_i^{n-1} \cdot B_j = 1.$$

Then by the pointwise estimates of [FX14b], we know the currents T_i and T_j must be equal over X, which implies $B_i = B_j$.

In conclusion, we get that $B_1 \cdot B_2 \cdots B_n = 1$ if and only if the B_j are equal. $\square$

2.2 Complete intersection cone

Since the complete intersection cone plays an important role in the paper, we quickly outline its basic properties. Recall that $\mathrm{CI}_1(X)$ is the closure of the set of all curve classes of the form A^{n-1} for an ample divisor A. It naturally has the structure of a closed pointed cone.

Proposition 2.2. *Let X be a projective variety of dimension n. Suppose that $\alpha \in \mathrm{CI}_1(X)$ lies on the boundary of the cone. Then either*

1. *$\alpha = B^{n-1}$ for some big and nef divisor class B, or*
2. *α lies on the boundary of $\overline{\mathrm{Eff}}_1(X)$.*

Proof. We fix an ample divisor class K. Since $\alpha \in \mathrm{CI}_1(X)$ is a boundary point of the cone, we can write α as the limit of classes A_i^{n-1} for some sequence of ample divisor classes A_i.

First suppose that the values of $A_i \cdot K^{n-1}$ are bounded above as i varies. Then the classes of the divisor A_i vary in a compact set, so they have some nef accumulation point B. Clearly $\alpha = B^{n-1}$. Furthermore, if B is not big then α will lie on the boundary of $\overline{\mathrm{Eff}}_1(X)$ since in this case $B^{n-1} \cdot B = 0$. If B is big, then it is not ample, since the map $A \mapsto A^{n-1}$ from the ample cone of divisors to $N_1(X)$ is locally surjective. Thus in this case B is big and nef.

Now suppose that the values of $A_i \cdot K^{n-1}$ do not have any upper bound. Since the A_i^{n-1} limit to α, for i sufficiently large we have

$$2(\alpha \cdot K) > A_i^{n-1} \cdot K \geqslant \mathrm{vol}(A_i)^{n-1/n} \,\mathrm{vol}(K)^{1/n}$$

by the Khovanskii-Teissier inequality. In particular this shows that $\mathrm{vol}(A_i)$ admits an upper bound as i varies. Note that the classes $A_i/(K^{n-1} \cdot A_i)$ vary in a compact slice of the nef cone of divisors. Without loss of generality, we can assume they limit to a nef divisor class B. Then we have

$$\begin{aligned} B \cdot \alpha &= \lim_{i \to \infty} \frac{A_i}{K^{n-1} \cdot A_i} \cdot A_i^{n-1} \\ &= \lim_{i \to \infty} \frac{\mathrm{vol}(A_i)}{K^{n-1} \cdot A_i} \\ &= 0. \end{aligned}$$

The last equality holds because $\mathrm{vol}(A_i)$ is bounded above but $A_i \cdot K^{n-1}$ is not. So in this case α must be on the boundary of the pseudo-effective cone $\overline{\mathrm{Eff}}_1$. □

The complete intersection cone differs from most cones considered in birational geometry in that it is *not* convex. Since we are not aware of any such example in the literature, we give a toric example from [FS09] in Appendix B. The same example shows that the cone that is the closure of all products of $(n-1)$ ample divisors is also not convex.

Remark 2.3. It is still true that $\mathrm{CI}_1(X)$ is "locally convex". Let A, B be two ample divisor classes. If ϵ is sufficiently small, then

$$A^{n-1} + \epsilon B^{n-1} = A_\epsilon^{n-1}$$

for a unique ample divisor A_ϵ. The existence of A_ϵ follows from the Hard Lefschetz theorem. Consider the following smooth map

$$\Phi : N^1(X) \to N_1(X)$$

sending D to D^{n-1}. By the Hard Lefschetz theorem, the derivative $d\Phi$ is an isomorphism at the point A. Thus Φ is local diffeomorphism near A, yielding the existence of A_ϵ. The uniqueness follows from Teissier's proportionality theorem. (See [GT13] for a more in-depth discussion.)

Another natural question is:

Question 2.4. Suppose that X is a projective variety of dimension n and that $\{A_i\}_{i=1}^{n-1}$ are ample divisor classes on X. Then is $A_1 \cdot \ldots \cdot A_{n-1} \in \mathrm{CI}_1(X)$?

One can imagine that such a statement could be proved using an "averaging" method.

2.3 Fields of characteristic p

Almost all the results in the paper will hold for smooth varieties over an arbitrary algebraically closed field. The necessary technical generalizations are verified in the following references:

- [Laz04, Remark 1.6.5] checks that the Khovanskii-Teissier inequalities hold over an arbitrary algebraically closed field.
- The existence of Fujita approximations over an arbitrary algebraically closed field is proved in [Tak07].
- The basic properties of the σ-decomposition in positive characteristic are considered in [Mus13].
- The results of [Cut13] lay the foundations of the theory of positive products and volumes over an arbitrary field.
- [FL13] describes how the above results can be used to extend [BDPP13] and most of the results of [BFJ09] over an arbitrary algebraically closed field. In particular the description of the derivative of the volume function in [BFJ09, Theorem A] holds for smooth varieties in any characteristic.

2.4 Compact Kähler manifolds

The following results enable us to extend most of our results to arbitrary compact Kähler manifolds.

- The Khovanskii-Teissier inequalities for classes in the nef cone $\overline{\mathcal{K}}$ can be proved by the mixed Hodge-Riemann bilinear relations [DN06], or by solving complex Monge-Ampère equations [Dem93]; see also Theorem 2.1.
- Teissier's proportionality theorem for transcendental big and nef classes has recently been proved by [FX14b]; see also Theorem 2.1.

- The theory of positive intersection products for pseudo-effective $(1,1)$-classes has been developed by [Bou02, BDPP13, BEGZ10].
- The cone duality $\overline{\mathcal{K}}^{\vee} = \mathcal{N}$ follows from the numerical characterization of the Kähler cone of [DP04].

We remark that we need the cone duality $\overline{\mathcal{K}}^{\vee} = \mathcal{N}$ to extend the Zariski decompositions and Morse-type inequality for curves to positive currents of bidimension $(1,1)$.

Comparing with the projective situation, the main ingredient missing is Demailly's conjecture on the transcendental holomorphic Morse inequality, which is in turn implied by the expected identification of the derivative of the volume function on pseudo-effective $(1,1)$-classes as in [BFJ09]. Indeed, it is not hard to see these two expected results are equivalent (see e.g. [Xia14, Proposition 1.1] – which is essentially [BFJ09, Section 3.2]). And they would imply the duality of the cones $\mathcal{M}(X)$ and $\mathcal{E}(X)$. Thus, any of our results which relies on either the transcendental holomorphic Morse inequality, or the results of [BFJ09], is still conjectural in the Kähler setting. However, these conjectures are known if X is a compact hyperkähler manifold (see [BDPP13, Theorem 10.12]), so all of our results extend to compact hyperkähler manifolds.

3 Polar transforms

As explained in the introduction, Zariski decompositions capture the failure of the volume function to be strictly log concave. In this section and the next, we use some basic convex analysis to define a formal Zariski decomposition which makes sense for any non-negative homogeneous log concave function on a cone. The main tool is a Legendre-Fenchel type transform for such functions.

3.1 Duality transforms

Let V be a finite-dimensional $\mathbb{R}$-vector space of dimension n, and let V^* be its dual. We denote the pairing of $w^* \in V^*$ and $v \in V$ by $w^* \cdot v$. Let $\mathrm{Cvx}(V)$ denote the class of lower-semicontinuous convex functions on V. Then [AAM09, Theorem 1] shows that, up to composition with an additive linear function and a symmetric linear transformation, the Legendre-Fenchel transform is the unique order-reversing involution $\mathcal{L} : \mathrm{Cvx}(V) \to \mathrm{Cvx}(V^*)$. Motivated by this result, the authors define a duality transform to be an order-reversing involution of this type and characterize the duality transforms in many other contexts (see e.g. [AAM11], [AAM08]).

Below we study a duality transform for the set of non-negative homogeneous functions on a cone. This transform is the concave homogeneous version of the well-known polar transform; see [Roc70, Chapter 15] for the basic properties of this transform in a related context. This transform is also a special case of the generalized Legendre-Fenchel transform studied by [Mor67, Section 14], which is the usual Legendre-Fenchel transform with a "coupling function" – we would like to thank M. Jonsson for pointing this out to us. See also [Sin97, Section 0.6] and [Rub00, Chapter 1] for a brief introduction to this perspective. Finally, it is essentially the same as the transform $\mathcal{A}$ from [AAM11] when applied to homogeneous functions, and is closely related to other constructions of [AAM08]. [Rub00, Chapter 2] and [RD02] work in a different setting which nonetheless has some nice parallels with our situation.

Let $\mathcal{C} \subset V$ be a proper closed convex cone of full dimension and let $\mathcal{C}^* \subset V^*$ denote the dual cone of $\mathcal{C}$, that is,

$$\mathcal{C}^* = \{w^* \in V^* | \; w^* \cdot v \geqslant 0 \text{ for any } v \in \mathcal{C}\}.$$

We let $\mathrm{HConc}_s(\mathcal{C})$ denote the collection of functions $f : \mathcal{C} \to \mathbb{R}$ satisfying:

- f is upper-semicontinuous and homogeneous of weight $s > 1$;
- f is strictly positive in the interior of $\mathcal{C}$ (and hence non-negative on $\mathcal{C}$);
- f is s-concave: for any $v, x \in \mathcal{C}$ we have $f(v)^{1/s} + f(x)^{1/s} \leqslant f(v + x)^{1/s}$.

Note that since $f^{1/s}$ is homogeneous of degree 1, the definition of concavity for $f^{1/s}$ above coheres with the usual one. For any $f \in \mathrm{HConc}_s(\mathcal{C})$, the function $f^{1/s}$ can extend to a proper upper-semicontinuous concave function over V by letting $f^{1/s}(v) = -\infty$ whenever $v \notin \mathcal{C}$. Thus many tools developed for arbitrary concave functions on V also apply in our case.

Since an upper-semicontinuous function is continuous along decreasing sequences, the following continuity property of f follows immediately from the non-negativity and concavity of $f^{1/s}$.

Lemma 3.1. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ and $v \in \mathcal{C}$. For any element $x \in \mathcal{C}$ we have*

$$f(v) = \lim_{t \to 0^+} f(v + tx).$$

In particular, any $f \in \mathrm{HConc}_s(\mathcal{C})$ must vanish at the origin.

In this section we outline the basic properties of the polar transform $\mathcal{H}$ (following a suggestion of M. Jonsson). In contrast to abstract convex transforms, $\mathcal{H}$ retains all of the properties of the classical Lengendre-Fenchel transform. Despite the many mentions of this transform we have been unable to find a comprehensive reference, so we have included the proofs.

Recall that the polar transform $\mathcal{H}$ associates to a function $f \in \mathrm{HConc}_s(\mathcal{C})$ the function $\mathcal{H}f : \mathcal{C}^* \to \mathbb{R}$ defined as

$$\mathcal{H}f(w^*) := \inf_{v \in \mathcal{C}^\circ} \left(\frac{w^* \cdot v}{f^{1/s}(v)} \right)^{s/s-1}.$$

By Lemma 3.1 the definition is unchanged if we instead vary v over all elements of $\mathcal{C}$ where f is positive. The following proposition shows that $\mathcal{H}$ defines an order-reversing involution from $\mathrm{HConc}_s(\mathcal{C})$ to $\mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$.

Proposition 3.2. *Let $f, g \in \mathrm{HConc}_s(\mathcal{C})$. Then we have*

1. *$\mathcal{H}f \in \mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$.*
2. *If $f \leqslant g$ then $\mathcal{H}f \geqslant \mathcal{H}g$.*
3. *$\mathcal{H}^2 f = f$.*

The proof is closely related to results in the literature (see e.g. [Roc70, Theorem 15.1]).

Proof. We first show (1). It is clear that $\mathcal{H}f$ is non-negative, homogeneous of weight $s/s-1$, and has a concave $s/s-1$-root. Since $\mathcal{H}f$ is defined as a pointwise infimum of a family of continuous functions, $\mathcal{H}f$ is upper-semicontinuous. So it only remains to show that $\mathcal{H}f$ is positive in the interior of $\mathcal{C}^*$.

Let w^* be an interior point of $\mathcal{C}^*$, we need to verify $\mathcal{H}f(w^*) > 0$. To this end, take a fixed compact slice T of the cone $\mathcal{C}$, e.g. take T to be the intersection of $\mathcal{C}$ with some hyperplane of V. By homogeneity we can compute $\mathcal{H}f(w^*)$ by taking the infimum over $v \in T \cap \mathcal{C}^\circ$. Since w^* is an interior point, for any $v \in T$, $w^* \cdot v$ has a uniform strictly positive lower bound. On the other hand, by the upper semi-continuity of f, we have a uniform upper bound on $f(v)$ as $v \in T$ varies. These then imply $\mathcal{H}f(w^*) > 0$.

The second statement (2) is obvious.

For the third statement (3), we always have $\mathcal{H}^2 f \geqslant f$. Indeed, by Lemma 3.1 we can find a sequence $\{v_k\}$ of points in $\mathcal{C}^\circ$ such that $\lim_k f(v_k) = f(v)$. Then

$$\mathcal{H}^2 f(v) = \inf_{w^* \in \mathcal{C}^{*\circ}} \left(\frac{w^* \cdot v}{\mathcal{H}f(w^*)^{s-1/s}} \right)^s \geqslant \liminf_k \inf_{w^* \in \mathcal{C}^{*\circ}} \left(\frac{w^* \cdot v}{(w^* \cdot v_k)/f(v_k)^{1/s}} \right)^s = f(v).$$

So we need to show $\mathcal{H}^2 f \leqslant f$. Note that $f^{1/s}$ is the pointwise infimum of the set of all affine functions which are minorized by f (since $f^{1/s}$ can be extended to a proper upper-semicontinuous concave function on all of V by assigning formally $f^{1/s}(v) = -\infty$ outside of $\mathcal{C}$). Thus f is the pointwise infimum of the set of all functions L which are minorized by f and are the s-th powers of some affine function which is positive on $\mathcal{C}^\circ$. Such functions have the form $L : v \mapsto (w^* \cdot v + b)^s$ for some $w^* \in \mathcal{C}^*$ and $b \geqslant 0$. In fact it suffices to consider only those L which are s-th powers of linear functions – if the function $v \mapsto (w^* \cdot v + b)^s$ is minorized by f then by homogeneity the smaller function $v \mapsto (w^* \cdot v)^s$ is also minorized by f. For any $v \in \mathcal{C}$ we have

$$f(v) = \inf_L L(v) \qquad \text{and} \qquad \inf_L \mathcal{H}^2 L(v) \geqslant \mathcal{H}^2 f(v),$$

where the infimum is taken over all functions L of the form $L_{w^*} = (w^* \cdot v)^s$ for some $w^* \in \mathcal{C}^*$ such that $L_{w^*} \geqslant f$. Thus it suffices to prove the statement for L_{w^*}. Since

$$\mathcal{H}L_{w^*}(w^*)^{s-1/s} = \inf_{v \in \mathcal{C}^\circ} \frac{w^* \cdot v}{w^* \cdot v} = 1,$$

for any $v \in \mathcal{C}$ we get by taking limits as in Lemma 3.1

$$\mathcal{H}^2 L_{w^*}(v) \leqslant \left(\frac{w^* \cdot v}{\mathcal{H}L_{w^*}(w^*)^{s-1/s}} \right)^s = (w^* \cdot v)^s = L_{w^*}(v).$$

This finishes the proof of $\mathcal{H}^2 f = f$. □

It will be crucial to understand which points obtain the infimum in the definition of $\mathcal{H}f$.

Definition 3.3. Let $f \in \mathrm{HConc}_s(\mathcal{C})$. For any $w^* \in \mathcal{C}^*$, we define G_{w^*} to be the set of all $v \in \mathcal{C}$ which satisfy $f(v) > 0$ and which achieve the infimum in the definition of $\mathcal{H}f(w^*)$, so that

$$\mathcal{H}f(w^*) = \left(\frac{w^* \cdot v}{f(v)^{1/s}}\right)^{s/s-1}.$$

Remark 3.4. The set G_{w^*} is the analogue of supergradients of concave functions. In particular, in the following sections we will see that the differential of $\mathcal{H}f$ at w^* lies in G_{w^*} if $\mathcal{H}f$ is differentiable.

It is easy to see that $G_{w^*} \cup \{0\}$ is a convex subcone of $\mathcal{C}$. Note the symmetry in the definition: if $v \in G_{w^*}$ and $\mathcal{H}f(w^*) > 0$ then $w^* \in G_v$. Thus if $v \in \mathcal{C}$ and $w^* \in \mathcal{C}^*$ satisfy $f(v) > 0$ and $\mathcal{H}f(w^*) > 0$ then the conditions $v \in G_{w^*}$ and $w^* \in G_v$ are equivalent.

The analogue of the Young-Fenchel inequality in our situation is:

Proposition 3.5. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Then for any $v \in \mathcal{C}$ and $w^* \in \mathcal{C}^*$ we have*

$$\mathcal{H}f(w^*)^{s-1/s} f(v)^{1/s} \leqslant v \cdot w^*.$$

Furthermore, equality is obtained only if either $v \in G_{w^*}$ and $w^* \in G_v$, or at least one of $\mathcal{H}f(w^*)$ and $f(v)$ vanishes.

Proof. The statement is obvious if either $\mathcal{H}f(w^*) = 0$ or $f(v) = 0$. Otherwise by Lemma 3.1 there is a sequence of $v_k \in \mathcal{C}^\circ$ such that $\lim_{k\to\infty} f(v_k) = f(v)$ and for every k

$$\mathcal{H}f(w^*)^{s-1/s} \leqslant \frac{v_k \cdot w^*}{f(v_k)^{1/s}}.$$

We obtain the desired inequality by taking limits. The last statement follows from the definition and the symmetry in the definition of G noted above. □

Theorem 3.6. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$.*

1. *Fix $v \in \mathcal{C}$. Let $\{w_i^*\}$ be a sequence of elements of $\mathcal{C}^*$ with $\mathcal{H}f(w_i^*) = 1$ such that*

   $$f(v) = \lim_i (v \cdot w_i^*)^s > 0.$$

   Suppose that the sequence admits an accumulation point w^*. Then $f(v) = (v \cdot w^*)^s$ and $\mathcal{H}f(w^*) = 1$.

2. *For every $v \in \mathcal{C}^\circ$ we have that G_v is non-empty.*

3. *Fix $v \in \mathcal{C}^\circ$. Let $\{v_i\}$ be a sequence of elements of $\mathcal{C}^\circ$ whose limit is v and for each v_i choose $w_i^* \in G_{v_i}$ with $\mathcal{H}f(w_i^*) = 1$. Then the w_i^* admit an accumulation point w^*, and any accumulation point lies in G_v and satisfies $\mathcal{H}f(w^*) = 1$.*

Proof. (1) The limiting statement for $f(v)$ is clear. We have $\mathcal{H}f(w^*) \geqslant 1$ by upper semicontinuity, so that

$$f(v)^{1/s} = \lim_{i\to\infty} v \cdot w_i^* \geqslant \frac{v \cdot w^*}{\mathcal{H}f(w^*)^{s-1/s}} \geqslant f(v)^{1/s}.$$

Thus we have equality everywhere. If $\mathcal{H}f(w^*)^{s-1/s} > 1$ then we obtain a strict inequality in the middle, a contradiction.

(2) Let w_i^* be a sequence of points in $\mathcal{C}^{*\circ}$ with $\mathcal{H}f(w_i^*) = 1$ such that $f(v) = \lim_{i\to\infty}(w_i^* \cdot v)^s$. By (1) it suffices to see that the w_i^* vary in a compact set. But since v is an interior point, the set of points which have intersection with v less than $2f(v)^{1/s}$ is bounded.

(3) By (1) it suffices to show that the w_i^* vary in a compact set. For sufficiently large i we have that $2v_i - v \in \mathcal{C}$. By the log concavity of f on $\mathcal{C}$ we see that f must be continuous at v. Thus for any fixed $\epsilon > 0$, we have for sufficiently large i

$$w_i^* \cdot v \leqslant 2w_i^* \cdot v_i \leqslant 2(1+\epsilon)f(v)^{1/s}.$$

Since v lies in the interior of $\mathcal{C}$, this implies that the w_i^* must lie in a bounded set. □

We next identify the collection of points where f is controlled by $\mathcal{H}$.

Definition 3.7. Let $f \in \mathrm{HConc}_s(\mathcal{C})$. We define $\mathcal{C}_f$ to be the set of all $v \in \mathcal{C}$ such that $v \in G_{w^*}$ for some $w^* \in \mathcal{C}$ satisfying $\mathcal{H}f(w^*) > 0$.

Since $v \in G_{w^*}$ and $\mathcal{H}f(w^*) > 0$, Proposition 3.5 and the symmetry of G show that $w^* \in G_v$. Furthermore, we have $\mathcal{C}^\circ \subset \mathcal{C}_f$ by Theorem 3.6 and the symmetry of G.

3.2 Differentiability

Definition 3.8. We say that $f \in \mathrm{HConc}_s(\mathcal{C})$ is differentiable if it is $\mathcal{C}^1$ on $\mathcal{C}^\circ$. In this case we define the function

$$D : \mathcal{C}^\circ \to V^* \qquad \text{by} \qquad v \mapsto \frac{Df(v)}{s}.$$

The main properties of the derivative are:

Theorem 3.9. *Suppose that $f \in \mathrm{HConc}_s(\mathcal{C})$ is differentiable. Then*

1. *D defines an $(s-1)$-homogeneous function from $\mathcal{C}^\circ$ to $\mathcal{C}^*_{\mathcal{H}f}$.*
2. *D satisfies a Brunn-Minkowski inequality with respect to f: for any $v \in \mathcal{C}^\circ$ and $x \in \mathcal{C}$*

$$D(v) \cdot x \geqslant f(v)^{s-1/s} f(x)^{1/s}.$$

Moreover, we have $D(v) \cdot v = f(v) = \mathcal{H}f(D(v))$.

Proof. For (1), the homogeneity is clear. Note that for any $v \in \mathcal{C}^\circ$ and $x \in C$ we have $f(v+x) \geqslant f(v)$ by the non-negativity of f and the concavity of $f^{1/s}$. Thus D takes values in $\mathcal{C}^*$. The fact that it takes values in $\mathcal{C}^*_{\mathcal{H}f}$ is a consequence of (2) which shows that $D(v) \in G_v$.

For (2), we start with the inequality $f(v+\epsilon x)^{1/s} \geqslant f(v)^{1/s} + f(\epsilon x)^{1/s}$. Since we have equality when $\epsilon = 0$, by taking derivatives with respect to ϵ at 0, we obtain

$$\frac{Df(v)}{s} \cdot x \geqslant f(v)^{s-1/s} f(x)^{1/s}.$$

The equality $\mathcal{H}f(D(v)) = f(v)$ is a consequence of the Brunn-Minkowski inequality, and the equality $D(v) \cdot v = f(v)$ is a consequence of the homogeneity of f. □

We will need the following familiar criterion for the differentiability of f, which is an analogue of related results in convex analysis connecting the differentiability with the uniqueness of supergradient (see e.g. [Roc70, Theorem 25.1]).

Proposition 3.10. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Let $U \subset \mathcal{C}^\circ$ be an open set. Then $f|_U$ is differentiable if and only if for every $v \in U$ the set $G_v \cup \{0\}$ consists of a single ray. In this case $D(v)$ is defined by intersecting against the unique element $w^* \in G_v$ satisfying $\mathcal{H}f(w^*) = f(v)$.*

Proof. We first show the forward implication. Let $v \in \mathcal{C}^\circ$ and choose some $w^* \in G_v$ satisfying $\mathcal{H}f(w^*) = f(v)$. We claim that $w^* = D(v)$, which then shows that $G_v \cup \{0\}$ is the ray generated by $D(v)$. To this end, by the symmetry of G we first have $v \in G_{w^*}$. Thus for any $x \in \mathcal{C}^\circ$ and $t > 0$ we get

$$\frac{v+tx}{f(v+tx)^{1/s}} \cdot w^* \geqslant \frac{v \cdot w^*}{f(v)^{1/s}}$$

with equality at $t = 0$. Taking the derivative at $t = 0$ we have

$$x \cdot w^* \geqslant \frac{v \cdot w^*}{f(v)} \left(D(v) \cdot x\right).$$

Since by our choice of normalization $v \cdot w^* = f(v)$, we have

$$x \cdot (w^* - D(v)) \geqslant 0.$$

By the arbitrariness of x, we obtain that

$$w^* - D(v) \in \mathcal{C}^*.$$

Since v is an interior point and $(w^* - D(v)) \cdot v = 0$ by homogeneity, we must have $w^* = D(v)$.

We next show the the reverse implication. Suppose $v \in U$. Fix any $x \in V$, and for each sufficiently small t let w_t^* denote the unique element in G_{v+tx} satisfying $\mathcal{H}f(w_t^*) = 1$. By Theorem 3.6 the w_t^* admit an accumulation point $w^* \in G_v$ satisfying $\mathcal{H}f(w^*) = 1$, which indeed is a limit point. For any t we have

$$tx \cdot w_t^* \leqslant f(v+tx)^{1/s} - f(v)^{1/s} \leqslant tx \cdot w^*.$$

Thus we see that the derivative of $f^{1/s}$ at v in the direction of x exists and is given by intersecting against w^*. This shows the reverse implication and (after rescaling to derive f instead of $f^{1/s}$) the final statement as well. □

We next discuss the behaviour of the derivative along the boundary.

Definition 3.11. We say that $f \in \mathrm{HConc}_s(\mathcal{C})$ is $+$-differentiable if f is $\mathcal{C}^1$ on $\mathcal{C}^\circ$ and the derivative on $\mathcal{C}^\circ$ extends to a continuous function on all of $\mathcal{C}_f$.

It is easy to see that the $+$-differentiability implies continuity.

Lemma 3.12. *If $f \in \mathrm{HConc}_s(\mathcal{C})$ is $+$-differentiable then f is continuous on $\mathcal{C}_f$.*

Remark 3.13. For $+$-differentiable functions f, we define the function $D : \mathcal{C}_f \to V^*$ by extending continuously from $\mathcal{C}^\circ$. Many of the properties in Theorem 3.9 hold for D on all of $\mathcal{C}_f$. By taking limits and applying Lemma 3.1 we obtain the Brunn-Minkowski inequality. In particular, for any $x \in \mathcal{C}_f$ we still have

$$D(x) \cdot x = f(x) = \mathcal{H}f(D(x)).$$

Thus it is clear that $D(x) \in \mathcal{C}^*_{\mathcal{H}f}$ for any $x \in \mathcal{C}_f$.

Lemma 3.14. *Assume $f \in \mathrm{HConc}_s(\mathcal{C})$ is $+$-differentiable. For any $x \in \mathcal{C}_f$ and $y \in \mathcal{C}^\circ$, we have*

$$\left.\frac{d}{dt}\right|_{t=0^+} f(x+ty)^{1/s} = (D(x)\cdot y) f(x)^{1-s/s}.$$

Proof. Consider the concave function $f(x+ty)^{1/s}$ of t. By [Roc70, Theorem 24.1] we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0^+} f(x+ty)^{1/s} &= \lim_{\epsilon\downarrow 0} \left.\frac{d}{dt}\right|_{t=\epsilon^+} f(x+ty)^{1/s} \\ &= \lim_{\epsilon\downarrow 0}(D(x+\epsilon y)\cdot y) f(x+\epsilon y)^{1-s/s} \\ &= (D(x)\cdot y) f(x)^{1-s/s},\end{aligned}$$

where the second line follows from the differentiability of f, and the third line follows from the $+$-differentiability of f. □

We next analyze what we can deduce about f in a neighborhood of $v \in \mathcal{C}_f$ from the fact that $G_v \cup \{0\}$ is a unique ray.

Lemma 3.15. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Let $v \in \mathcal{C}_f$ and assume that $G_v \cup \{0\}$ consists of a single ray. Suppose $\{v_i\}$ is a sequence of elements of $\mathcal{C}_f$ converging to v. Let $w_i^* \in G_{v_i}$ be any point satisfying $\mathcal{H}f(w_i^*) = 1$. Then the w_i^* vary in a compact set. Any accumulation point w^* must be the unique point in G_v satisfying $\mathcal{H}f(w^*) = 1$.*

Proof. By Theorem 3.6 it suffices to prove that the w_i^* vary in a compact set. Otherwise, we must have that $w_i^* \cdot m$ is unbounded for some interior point $m \in \mathcal{C}^\circ$. By passing to a subsequence we may suppose that $w_i^* \cdot m \to \infty$. Consider the normalization

$$\hat{w}_i^* := \frac{w_i^*}{w_i^* \cdot m};$$

note that $\widehat{w}_i^*$ vary in a compact set. Take some convergent subsequence, which we still denote by $\widehat{w}_i^*$, and write $\widehat{w}_i^* \to \widehat{w}_0^*$. Since $\widehat{w}_0^* \cdot m = 1$ we see that $\widehat{w}_0^* \neq 0$.

We first prove $v \cdot \widehat{w}_0^* > 0$. Otherwise, $v \cdot \widehat{w}_0^* = 0$ implies

$$\frac{v \cdot (w^* + \widehat{w}_0^*)}{\mathcal{H}f(w^* + \widehat{w}_0^*)^{s-1/s}} \leqslant \frac{v \cdot w^*}{\mathcal{H}f(w^*)^{s-1/s}} = f(v)^{1/s}.$$

By our assumption on G_v, we get $w^* + \widehat{w}_0^*$ and w^* are proportional, which implies $\widehat{w}_0^*$ lies in the ray spanned by w^*. Since $\widehat{w}_0^* \neq 0$ and $v \cdot w^* > 0$, we get that $v \cdot \widehat{w}_0^* > 0$. So our assumption $v \cdot \widehat{w}_0^* = 0$ does not hold. On the other hand, $\mathcal{H}f(w_i^*) = 1$ implies

$$\mathcal{H}f(\widehat{w}_i^*)^{s-1/s} = \frac{1}{m \cdot w_i^*} \to 0.$$

By the upper-semicontinuity of f and the fact that $\lim v_i \cdot \widehat{w}_i^* = v \cdot \widehat{w}_0^* > 0$, we get

$$\begin{aligned} f(v)^{1/s} &\geqslant \limsup_{i \to \infty} f(v_i)^{1/s} \\ &= \limsup_{i \to \infty} \frac{v_i \cdot \widehat{w}_i^*}{\mathcal{H}f(\widehat{w}_i^*)^{s-1/s}} = \infty. \end{aligned}$$

This is a contradiction, thus the sequence w_i^* must vary in a compact set. □

Theorem 3.16. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Suppose that $U \subset \mathcal{C}_f$ is a relatively open set and $G_v \cup \{0\}$ consists of a single ray for any $v \in U$. If f is continuous on U then f is $+$-differentiable on U. In this case $D(v)$ is defined by intersecting against the unique element $w^* \in G_v$ satisfying $\mathcal{H}f(w^*) = f(v)$.*

Even if f is not continuous, we at least have a similar statement along the directions in which f is continuous (for example, any directional derivative toward the interior of the cone).

Proof. Theorem 3.10 shows that f is differentiable on $U \cap \mathcal{C}^\circ$ and is determined by intersections. By combining Lemma 3.15 with the continuity of f, we see that the derivative extends continuously to any point in U. □

Remark 3.17. Assume $f \in \mathrm{HConc}_s(\mathcal{C})$ is $+$-differentiable. In general, we can not conclude that $G_v \cup \{0\}$ contains a single ray if $x \in \mathcal{C}_f$ is not an interior point. An explicit example is in Section 5. Let X be a smooth projective variety of dimension n, let $\mathcal{C} = \mathrm{Nef}^1(X)$ be the cone of nef divisor classes and let $f = \mathrm{vol}$ be the volume function of divisors. Let B be a big and nef divisor class which is not ample. Then G_B contains the cone generated by all $B^{n-1} + \gamma$ with γ pseudo-effective and $B \cdot \gamma = 0$, which in general is more than a ray.

4 Formal Zariski decompositions

The Legendre-Fenchel transform relates the strict concavity of a function to the differentiability of its transform. The transform $\mathcal{H}$ will play the same role in our situation; however, one needs to interpret the strict concavity slightly differently. We will encapsulate this property using the notion of a Zariski decomposition.

Definition 4.1. Let $f \in \mathrm{HConc}_s(\mathcal{C})$ and let $U \subset \mathcal{C}$ be a non-empty subcone. We say that f admits a strong Zariski decomposition with respect to U if:

1. For every $v \in \mathcal{C}_f$ there are unique elements $p_v \in U$ and $n_v \in \mathcal{C}$ satisfying

$$v = p_v + n_v \qquad \text{and} \qquad f(v) = f(p_v).$$

   We call the expression $v = p_v + n_v$ the Zariski decomposition of v, and call p_v the positive part and n_v the negative part of v.

2. For any $v, w \in \mathcal{C}_f$ satisfying $v + w \in \mathcal{C}_f$ we have

$$f(v)^{1/s} + f(w)^{1/s} \leqslant f(v+w)^{1/s}$$

   with equality only if p_v and p_w are proportional.

Remark 4.2. Note that the vector n_v must satisfy $f(n_v) = 0$ by the non-negativity and log-concavity of f. In particular n_v lies on the boundary of $\mathcal{C}$. Furthermore, any $w^* \in G_v$ is also in G_{p_v} and must satisfy $w^* \cdot n_v = 0$.

Note also that the proportionality of p_v and p_w may not be enough to conclude that $f(v)^{1/s} + f(w)^{1/s} = f(v+w)^{1/s}$. This additional property turns out to rely on the strict log concavity of $\mathcal{H}f$.

The main principle of the section is that when f satisfies a differentiability property, $\mathcal{H}f$ admits some kind of Zariski decomposition. Usually the converse is false, due to the asymmetry of G when f or $\mathcal{H}f$ vanishes. However, the existence of a Zariski decomposition is usually strong enough to determine the differentiability of f along some subcone. We will give a version that takes into account the behavior of f along the boundary of $\mathcal{C}$.

Theorem 4.3. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Then we have the following results:*

- *If f is $+$-differentiable, then $\mathcal{H}f$ admits a strong Zariski decomposition with respect to the cone $D(\mathcal{C}_f) \cup \{0\}$.*
- *If $\mathcal{H}f$ admits a strong Zariski decomposition with respect to a cone U, then f is differentiable.*

Proof. First suppose f is $+$-differentiable; we must prove the function $\mathcal{H}f$ satisfies properties $(1), (2)$ in Definition 4.1.

We first show the existence of the Zariski decomposition in property (1). If $w^* \in \mathcal{C}^*_{\mathcal{H}f}$ then by definition there is some $v \in \mathcal{C}$ satisfying $f(v) > 0$ such that $w^* \in G_v$. In particular, by the symmetry of G we also have $v \in G_{w^*}$, thus $v \in \mathcal{C}_f$. Since $f(v) > 0$ we can define

$$p_{w^*} := \left(\frac{\mathcal{H}f(w^*)}{f(v)}\right)^{s-1/s} \cdot D(v), \qquad n_{w^*} = w^* - p_{w^*}.$$

Then $p_{w^*} \in D(\mathcal{C}_f)$ and

$$\begin{aligned}\mathcal{H}f(p_{w^*}) &= \mathcal{H}\left(\left(\frac{\mathcal{H}f(w^*)}{f(v)}\right)^{s-1/s} \cdot D(v)\right) \\ &= \frac{\mathcal{H}f(w^*)}{f(v)} \cdot \mathcal{H}f\left(D(v)\right) = \mathcal{H}f(w^*)\end{aligned}$$

where the final equality follows from Theorem 3.9 and Remark 3.13. We next show that $n_{w^*} \in \mathcal{C}^*$. Choose any $x \in \mathcal{C}^\circ$ and note that for any $t > 0$ we have the inequality

$$\frac{v + tx}{f(v+tx)^{1/s}} \cdot w^* \geqslant \frac{v}{f(v)^{1/s}} \cdot w^*$$

with equality when $t = 0$. By Lemma 3.14, taking derivatives at $t = 0$ we obtain

$$\frac{x \cdot w^*}{f(v)^{1/s}} - \frac{(v \cdot w^*)(D(v) \cdot x)}{f(v)^{(s+1)/s}} \geqslant 0,$$

or equivalently, identifying $v \cdot w^*/f(v)^{1/s} = \mathcal{H}f(w^*)^{s-1/s}$,

$$x \cdot \left(w^* - D(v) \cdot \frac{\mathcal{H}f(w^*)^{s-1/s}}{f(v)^{s-1/s}}\right) \geqslant 0.$$

Since this is true for any $x \in \mathcal{C}^\circ$, we see that $n_{w^*} \in \mathcal{C}^*$ as claimed.

We next show that p_{w^*} constructed above is the unique element of $D(\mathcal{C}_f)$ satisfying the two given properties. First, after some rescaling we can assume $\mathcal{H}f(w^*) = f(v)$, which then implies $w^* \cdot v = f(v)$. Suppose that $z \in \mathcal{C}_f$ and $D(z)$ is another vector satisfying $\mathcal{H}f(D(z)) = \mathcal{H}f(w^*)$ and $w^* - D(z) \in \mathcal{C}$. Note that by Remark 3.13 $f(z) = \mathcal{H}f(D(z)) = f(v)$. By Proposition 3.5 we have

$$\mathcal{H}f(D(z))^{s-1/s} f(v)^{1/s} \leqslant D(z) \cdot v \leqslant w^* \cdot v = f(v)$$

so we obtain equality everywhere. In particular, we have $D(z) \cdot v = f(v)$. By Theorem 3.9, for any $x \in \mathcal{C}$ we have

$$D(z) \cdot x \geqslant f(z)^{s-1/s} f(x)^{1/s}.$$

Set $x = v + \epsilon q$ where $\epsilon > 0$ and $q \in \mathcal{C}^\circ$. With this substitution, the two sides of the equation above are equal at $\epsilon = 0$, so taking an ϵ-derivative of the above equation and arguing as before, we see that $D(z) - D(v) \in \mathcal{C}^*$.

We claim that $D(z) = D(v)$. First we note that $D(v) \cdot z = f(z)$. Indeed, since $f(z) = f(v)$ and $D(v) \preceq D(z)$ we have

$$f(v)^{s-1/s} f(z)^{1/s} \leqslant D(v) \cdot z \leqslant D(z) \cdot z = f(z).$$

Thus we have equality everywhere, proving the equality $D(v) \cdot z = f(z)$. Then we can apply the same argument as before with the roles of v and z switched. This shows $D(v) \succeq D(z)$, so we must have $D(z) = D(v)$.

We next turn to (2). The inequality is clear, so we only need to characterize the equality. Suppose $w^*, y^* \in \mathcal{C}^*_{\mathcal{H}f}$ satisfy

$$\mathcal{H}f(w^*)^{s-1/s} + \mathcal{H}f(y^*)^{s-1/s} = \mathcal{H}f(w^* + y^*)^{s-1/s}$$

and $w^* + y^* \in \mathcal{C}^*_{\mathcal{H}f}$. We need to show they have proportional positive parts. By assumption $G_{w^*+y^*}$ is non-empty, so we may choose some $v \in G_{w^*+y^*}$. Then also $v \in G_{w^*}$ and $v \in G_{y^*}$. Note that by homogeneity v is also in G_{aw^*} and G_{by^*} for any positive real numbers a and b. Thus by rescaling w^* and y^*, we may suppose that both have intersection $f(v)$ against v, so that $\mathcal{H}f(w^*) = \mathcal{H}f(y^*) = f(v)$. Then we need to verify the positive parts of w^* and y^* are equal. But they both coincide with $D(v)$ by the argument in the proof of (1).

Conversely, suppose that $\mathcal{H}f$ admits a strong Zariski decomposition with respect to the cone U. We claim that f is differentiable. By Proposition 3.10 it suffices to show that $G_v \cup \{0\}$ is a single ray for any $v \in \mathcal{C}^\circ$.

For any two elements w^*, y^* in G_v we have

$$\mathcal{H}f(w^*)^{1/s} + \mathcal{H}f(y^*)^{1/s} = \frac{w^* \cdot v}{f(v)^{1/s}} + \frac{y^* \cdot v}{f(v)^{1/s}} \geqslant \mathcal{H}f(w^* + y^*)^{1/s}.$$

Since w^*, y^* and their sum are all in $\mathcal{C}^*_{\mathcal{H}f}$, we conclude by the strong Zariski decomposition condition that w^* and y^* have proportional positive parts. After rescaling so that $\mathcal{H}f(w^*) = f(v) = \mathcal{H}f(y^*)$ we have $p_{w^*} = p_{y^*}$. Thus it suffices to prove $w^* = p_{w^*}$. Note that $\mathcal{H}f(w^*) = \mathcal{H}f(p_{w^*})$ as p_{w^*} is the positive part. If $w^* \neq p_{w^*}$, then $v \cdot w^* > v \cdot p_{w^*}$ since v is an interior point. This implies

$$f(v) = \inf_{y^* \in \mathcal{C}^{*\circ}} \left(\frac{v \cdot y^*}{\mathcal{H}f(y^*)^{s-1/s}} \right)^s < \left(\frac{v \cdot w^*}{\mathcal{H}f(w^*)^{s-1/s}} \right)^s,$$

contradicting with $w^* \in G_v$. Thus $w^* = p_{w^*}$ and $G_v \cup \{0\}$ must be a single ray. □

Remark 4.4. It is worth emphasizing that if f is $+$-differentiable and $w^* \in \mathcal{C}^*_{\mathcal{H}f}$, we can construct a positive part for w^* by choosing *any* $v \in G_{w^*}$ with $f(v) > 0$ and taking an appropriate rescaling of $D(v)$.

Remark 4.5. It would also be interesting to study some kind of weak Zariski decomposition. For example, one can define a weak Zariski decomposition as a decomposition $v = p_v + n_v$ only demanding $f(v) = f(p_v)$ and the strict log concavity of f over the set of positive parts. Appropriately interpreted, the existence of a weak decomposition for $\mathcal{H}f$ should correspond to the differentiability of f.

Under some additional conditions, we can get the continuity of the Zariski decompositions.

Theorem 4.6. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable. Then the function taking an element $w^* \in \mathcal{C}^{*\circ}$ to its positive part p_{w^*} is continuous.*

If furthermore $G_v \cup \{0\}$ is a unique ray for every $v \in \mathcal{C}_f$ and $\mathcal{H}f$ is continuous on all of $\mathcal{C}^*_{\mathcal{H}f}$, then the Zariski decomposition is continuous on all of $\mathcal{C}^*_{\mathcal{H}f}$.

Proof. Fix any $w^* \in \mathcal{C}^{*\circ}$ and suppose that w_i^* is a sequence whose limit is w^*. For each choose some $v_i \in G_{w_i^*}$ with $f(v_i) = 1$. By Theorem 3.6, the v_i admit an accumulation point $v \in G_{w^*}$ with $f(v) = 1$. By the symmetry of G, each v_i and also v lies in $\mathcal{C}_f$. The $D(v_i)$ limit to $D(v)$ by the continuity of D. Recall that by the argument in the proof of Theorem 4.3 we have $p_{w_i^*} = \mathcal{H}f(w_i^*)^{s-1/s}D(v_i)$ and similarly for w^*. Since $\mathcal{H}f$ is continuous at interior points, we see that the positive parts vary continuously as well.

The last statement follows by a similar argument using Lemma 3.15. □

Example 4.7. Suppose that q is a bilinear form on V and $f(v) = q(v, v)$. Let $\mathcal{P}$ denote one-half of the positive cone of vectors satisfying $f(v) \geqslant 0$. It is easy to see that f is 2-concave and non-trivial on $\mathcal{P}$ if and only if q has signature $(1, \dim V - 1)$. Identifying V with V^* under q, we have $\mathcal{P} = \mathcal{P}^*$ and $\mathcal{H}f = f$ by the usual Hodge inequality argument.

Now suppose $\mathcal{C} \subset \mathcal{P}$. Then $\mathcal{C}^*$ contains $\mathcal{C}$. As discussed above, by the Hodge inequality $\mathcal{H}f|_{\mathcal{C}} = f$. Note that f is everywhere differentiable and $D(v) = v$ for classes in $\mathcal{C}$. Thus on $\mathcal{C}$ the polar transform $\mathcal{H}f$ agrees with f, but outside of $\mathcal{C}$ the function $\mathcal{H}f$ is controlled by a Zariski decomposition involving a projection to $\mathcal{C}$.

This is of course just the familiar picture for curves on a surface identifying f with the self-intersection on the nef cone and $\mathcal{H}f$ with the volume on the pseudo-effective cone. More precisely, for big curve classes the decomposition constructed in this way is the numerical version of Zariski's original construction. Along the boundary of $\mathcal{C}^*$, the function $\mathcal{H}f$ vanishes identically so that Theorem 4.3 does not apply. The linear algebra arguments of [Zar62], [Bau09] give a way of explicitly constructing the vector computing the minimal intersection as above.

Example 4.8. Fix a spanning set of unit vectors $\mathcal{Q}$ in $\mathbb{R}^n$. Recall that the polytopes whose unit facet normals are a subset of $\mathcal{Q}$ naturally define a cone $\mathcal{C}$ in a finite dimensional vector space V which parametrizes the constant terms of the bounding hyperplanes. One can also consider the cone $\mathcal{C}_\Sigma$ which is the closure of those polytopes whose normal fan is Σ. The volume function vol defines a weight-n homogeneous function on $\mathcal{C}$ and (via restriction) vol_Σ on $\mathcal{C}_\Sigma$, and it is interesting to ask for the behavior of the polar transforms. (Note that this is somewhat different from the link between polar sets and polar functions, which is described for example in [AAM11].)

The dual space V^* consists of the Minkowski weights on $\mathcal{Q}$. We will focus on the subcone $\mathcal{M}$ of strictly positive Minkowski weights, which is contained in the dual of both cones. By Minkowski's theorem, a strictly positive Minkowski weight determines naturally a polytope in $\mathcal{C}$, so we can identify $\mathcal{M}$ with the interior of $\mathcal{C}$. As explained in Section 8, the Brunn-Minkowski inequality shows that $\mathcal{H}\,\mathrm{vol}\,|_{\mathcal{M}}$ coincides with the volume function on $\mathcal{M}$. However, calculating $\mathcal{H}\,\mathrm{vol}_\Sigma\,|_{\mathcal{M}}$ is more subtle, and is an interesting special case of an isoperimetric inequality as in the introduction.

It would be very interesting to extend this duality to all convex sets, perhaps by working on an infinite dimensional space.

4.1 Teissier proportionality

In this section, we give some conditions which are equivalent to the strict log concavity. The prototype is the volume function of divisors over the cone of big and movable divisor classes.

Definition 4.9. Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable and let $\mathcal{C}_T$ be a non-empty subcone of $\mathcal{C}_f$. We say that f satisfies Teissier proportionality with respect to $\mathcal{C}_T$ if for any $v, x \in \mathcal{C}_T$ satisfying

$$D(v) \cdot x = f(v)^{s-1/s} f(x)^{1/s}$$

we have that v and x are proportional.

Note that we do not assume that $\mathcal{C}_T$ is convex – indeed, in examples it is important to avoid this condition. However, since f is defined on the convex hull of $\mathcal{C}_T$, we can (somewhat abusively) discuss the strict log concavity of $f|_{\mathcal{C}_T}$:

Definition 4.10. Let $\mathcal{C}' \subset \mathcal{C}$ be a (possibly non-convex) subcone. We say that f is strictly s-concave on $\mathcal{C}'$ if

$$f(v)^{1/s} + f(x)^{1/s} < f(v + x)^{1/s}$$

holds whenever $v, x \in \mathcal{C}'$ are not proportional. Note that this definition makes sense even when $\mathcal{C}'$ is not itself convex.

Theorem 4.11. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable. For any non-empty subcone $\mathcal{C}_T$ of $\mathcal{C}_f$, consider the following conditions:*

1. *The restriction $f|_{\mathcal{C}_T}$ is strictly s-concave (in the sense defined above).*
2. *f satisfies Teissier proportionality with respect to $\mathcal{C}_T$.*
3. *The restriction of D to $\mathcal{C}_T$ is injective.*

Then we have (1) $\implies$ (2) $\implies$ (3). If $\mathcal{C}_T$ is convex, then we have (2) $\implies$ (1). If $\mathcal{C}_T$ is an open subcone, then we have (3) $\implies$ (1).

Proof. We first prove (1) $\implies$ (2). Let $v, x \in \mathcal{C}_T$ satisfy $D(v) \cdot x = f(v)^{s-1/s} f(x)^{1/s}$ and $f(v) = f(x)$. Assume for a contradiction that $v \neq x$. Since $f|_{\mathcal{C}_T}$ is strictly s-concave, for any two $v, x \in \mathcal{C}_T$ which are not proportional we have

$$f(x)^{1/s} < f(v)^{1/s} + \frac{D(v) \cdot (x - v)}{f(v)^{s-1/s}}.$$

Since we have assumed $D(v) \cdot x = f(v)^{s-1/s} f(x)^{1/s}$ and $f(v) = f(x)$, we must have

$$f(x)^{1/s} = f(v)^{1/s} + \frac{D(v) \cdot (x - v)}{f(v)^{s-1/s}}$$

since $D(v) \cdot v = f(v)$. This is a contradiction, so we must have $v = x$. This then implies f satisfies Teissier proportionality.

We next show (2) $\implies$ (3). Let $v_1, v_2 \in \mathcal{C}_T$ with $D(v_1) = D(v_2)$. Then we have

$$\begin{aligned} f(v_1) = D(v_1) \cdot v_1 &= D(v_2) \cdot v_1 \\ &\geqslant f(v_2)^{s-1/s} f(v_1)^{1/s}, \end{aligned}$$

which implies $f(v_1) \geqslant f(v_2)$. By symmetry, we get $f(v_1) = f(v_2)$. So we must have

$$D(v_1) \cdot v_2 = f(v_1)^{s-1/s} f(v_2)^{1/s}.$$

By the Teissier proportionality we see that v_1, v_2 are proportional, and since $f(v_1) = f(v_2)$ they must be equal.

We next show that if $\mathcal{C}_T$ is convex then (2) $\implies$ (1). Fix y in the interior of $\mathcal{C}_T$ and fix $\epsilon > 0$. Then

$$f(v + x + \epsilon y)^{1/s} - f(v)^{1/s} = \int_0^1 (D(v + t(x + \epsilon y)) \cdot x) f(v + t(x + \epsilon y))^{1-s/s} dt.$$

The integrand is bounded by a positive constant independent of ϵ as we let ϵ go to 0 due to the +-differentiability of f (which also implies the continuity of f). Using Lemma 3.1, the dominanted convergence theorem shows that

$$f(v + x)^{1/s} - f(v)^{1/s} = \int_0^1 (D(v + tx) \cdot x) f(v + tx)^{1-s/s} dt.$$

This immediately shows the strict log concavity.

Finally, we show that if $\mathcal{C}_T$ is open then (3) $\implies$ (1). By [Roc70, Corollary 26.3.1], it is clear that for any convex open set $U \subset \mathcal{C}_T$ the injectivity of D over U is equivalent to the strict log concavity of $f|_U$. Using the global log concavity of f, we obtain the conclusion. More precisely, assume $x, y \in \mathcal{C}_T$ are not proportional, then by the strict log concavity of f near x and the global log concavity on $\mathcal{C}$, for $t > 0$ sufficiently small we have

$$\begin{aligned} f^{1/s}(x + y) &\geqslant f^{1/s}(x + ty) + (1 - t) f^{1/s}(y) \\ &> (f^{1/s}(x) + f^{1/s}(x + 2ty))/2 + (1 - t) f^{1/s}(y) \\ &\geqslant f^{1/s}(x) + f^{1/s}(y). \end{aligned}$$

□

Another useful observation is:

Proposition 4.12. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be differentiable and suppose that f is strictly s-concave on an open subcone $\mathcal{C}_T \subset \mathcal{C}^\circ$. Then $\mathcal{H}f$ is differentiable on $D(\mathcal{C}_T)$ and the derivative is determined by the prescription*

$$D(D(v)) = v.$$

Proof. We first show that $D(\mathcal{C}_T) \subset \mathcal{C}^{*\circ}$. Suppose that there were some $v \in \mathcal{C}_T$ such that $D(v)$ lay on the boundary of $\mathcal{C}^*$. Choose $x \in \mathcal{C}$ satisfying $x \cdot D(v) = 0$. By openness we have $v + tx \in \mathcal{C}_T$ for sufficiently small t. Since $D(v) \in G_{v+tx}$, we must have that $D(v)$ and $D(v + tx)$ are proportional by Proposition 3.10. This is a contradiction by Theorem 4.11.

Now suppose $w^* = D(v) \in D(\mathcal{C}_T)$. By the strict log concavity of f on $\mathcal{C}_T$ (and the global log concavity), we must have that $G_{w^*} \cup \{0\}$ consists only of the ray spanned by v. Applying Proposition 3.10, we obtain the statement. □

Combining all the results above, we obtain a very clean property of D under the strongest possible assumptions.

Theorem 4.13. *Assume $f \in \mathrm{HConc}_s(\mathcal{C})$ and its polar transform $\mathcal{H}f \in \mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$ are $+$-differentiable. Let $U = D(\mathcal{C}^*_{\mathcal{H}f}) \cup \{0\}$ and $U^* = D(\mathcal{C}_f) \cup \{0\}$. Then we have:*

- *f and $\mathcal{H}f$ admit a strong Zariski decomposition with respect to the cone U and the cone U^* respectively;*
- *For any $v \in \mathcal{C}_f$ we have $D(v) = D(p_v)$ (and similarly for $w \in \mathcal{C}^*_{\mathcal{H}f}$);*
- *D defines a bijection $D : U^\circ \to U^{*\circ}$ with inverse also given by D. In particular, f and $\mathcal{H}f$ satisfy Teissier proportionality with respect to the open cone U° and $U^{*\circ}$ respectively.*

Proof. Note that $U^* \subset \mathcal{C}^*_{\mathcal{H}f}$ (and $U \subset \mathcal{C}_f$) since for any $v \in \mathcal{C}_f$ we have $D(v) \in G_v$ and $f(v) > 0$.

The first statement is immediate from Theorem 4.3.

We next show the second statement. By the definition of positive parts, we have $G_v \subset G_{p_v}$. Since both $v, p_v \in \mathcal{C}_f$, we know by the argument of Theorem 4.3 that $D(v)$ and $D(p_v)$ are both proportional to the (unique) positive part of any $w^* \in G_v$ with positive $\mathcal{H}f$.

Finally we show the third statement. We start by proving the Teissier proportionality on U°. By part (2) of the Zariski decomposition condition f is strictly s-concave on U°, and Teissier proportionality follows by Theorem 4.11. Furthermore, the argument of Proposition 4.12 then shows that $D(U^\circ) \subset \mathcal{C}^{*\circ}$ and $D(D(U^\circ)) = U^\circ$.

We must show that $D(U^\circ) \subset U^{*\circ}$. Suppose that $v \in U^\circ$ had that $D(v)$ was on the boundary of U^*. Since $D(v) \in \mathcal{C}^{*\circ}$, there must be some sequence $w_i^* \in C^{*\circ} - U^*$ whose limit is $D(v)$. We note that each $D(w_i^*)$ lies on the boundary of $\mathcal{C}$, thus must lie on the boundary of U. Indeed, by the second statement we have $D(w_i^*) = D(w_i^* + tn_{w_i^*})$ for any $t > 0$, which would violate the uniqueness of $G_{D(w_i^*)}$ as in Proposition 3.10 if it were an interior point. Using the continuity of D we see that $v = D(D(v))$ lies on the boundary of U, a contradiction.

In all, we have shown that $D : U^\circ \to U^{*\circ}$ is an isomorphism onto its image with inverse D. By symmetry we also have $D(U^{*\circ}) \subset U^\circ$, and we conclude after taking D the reverse inclusion $U^{*\circ} \subset D(U^\circ)$. ☐

4.2 Morse-type inequality

The polar transform $\mathcal{H}$ also gives a natural way of translating cone positivity conditions from $\mathcal{C}$ to $\mathcal{C}^*$.

Definition 4.14. Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable. We say that f satisfies a Morse-type inequality if for any $v \in \mathcal{C}_f$ and $x \in \mathcal{C}$ satisfying the inequality

$$f(v) - sD(v) \cdot x > 0$$

we have that $v - x \in \mathcal{C}^\circ$.

Note that the prototype of the Morse-type inequality is the well known algebraic Morse inequality for nef divisors.

In order to translate the positivity in $\mathcal{C}$ to $\mathcal{C}^*$, we need the following “reverse” Khovanskii-Teissier inequality.

Proposition 4.15. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable and satisfy a Morse-type inequality. Then we have*

$$s(y^* \cdot v)(D(v) \cdot x) \geqslant f(v)(y^* \cdot x),$$

for any $y^* \in \mathcal{C}^*$, $v \in \mathcal{C}_f$ and $x \in \mathcal{C}$.

Proof. The inequality holds when $y^* = 0$, so we need to deal with the case when $y^* \neq 0$. Since both sides are homogeneous in all the arguments, we may rescale to assume that $y^* \cdot v = y^* \cdot x$. Then we need to show that $sD(v) \cdot x \geqslant f(v)$. If not, then

$$f(v) - sD(v) \cdot x > 0,$$

so that $v - x \in \mathcal{C}^\circ$ by the Morse-type inequality. But then we conclude that $y^* \cdot v > y^* \cdot x$, a contradiction. ☐

Theorem 4.16. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable and satisfy a Morse-type inequality. Then for any $v \in \mathcal{C}_f$ and $y^* \in \mathcal{C}^*$ satisfying*

$$\mathcal{H}f(D(v)) - sv \cdot y^* > 0,$$

we have $D(v) - y^* \in \mathcal{C}^{*\circ}$. In particular, we have $D(v) - y^* \in \mathcal{C}^*_{\mathcal{H}f}$ and

$$\begin{aligned} \mathcal{H}f(D(v) - y^*)^{s-1/s} &\geqslant (\mathcal{H}f(D(v)) - sv \cdot y^*)\mathcal{H}f(D(v))^{-1/s} \\ &= (f(v) - sv \cdot y^*)f(v)^{-1/s}. \end{aligned}$$

As a consequence, we get

$$\mathcal{H}f(D(v) - y^*) \geqslant f(v) - \frac{s^2}{s-1} v \cdot y^*.$$

Proof. Note that $\mathcal{H}f(D(v)) = f(v)$. First we claim that the inequality $f(v) - sv \cdot y^* > 0$ implies $D(v) - y^* \in \mathcal{C}^{*\circ}$. To this end, fix some sufficiently small $y'^* \in \mathcal{C}^{*\circ}$ such that $y^* + y'^*$ still satisfies $f(v) - sv \cdot (y^* + y'^*) > 0$.

Then by the "reverse" Khovanskii-Teissier inequality, for some $\delta > 0$ and any $x \in \mathcal{C}$ we have

$$D(v) \cdot x \geqslant \left(\frac{f(v)}{s(y^* + y'^*) \cdot v} \right) (y^* + y'^*) \cdot x \geqslant (1 + \delta)(y^* + y'^*) \cdot x.$$

This implies $D(v) - y^* \in \mathcal{C}^{*\circ}$.

By the definition of $\mathcal{H}f$ we have

$$\begin{aligned} \mathcal{H}f(D(v) - y^*) &= \inf_{x \in \mathcal{C}^\circ} \left(\frac{(D(v) - y^*) \cdot x}{f(x)^{1/s}} \right)^{s/s-1} \\ &\geqslant \left(\frac{f(v) - sy^* \cdot v}{f(v)} \right)^{s/s-1} \inf_{x \in \mathcal{C}^\circ} \left(\frac{D(v) \cdot x}{f(x)^{1/s}} \right)^{s/s-1} \\ &= \mathcal{H}f(D(v)) \left(\frac{f(v) - sy^* \cdot v}{f(v)} \right)^{s/s-1}, \end{aligned}$$

where the second line follows from "reverse" Khovanskii-Teissier inequality. To obtain the desired inequality, we only need to use the equality $\mathcal{H}f(D(v)) = f(v)$ again.

To show the last inequality, we only need to note that the function $(1-x)^\alpha$ is convex for $x \in [0,1)$ if $\alpha \geqslant 1$. This implies $(1-x)^\alpha \geqslant 1 - \alpha x$. Applying this inequality in our situation, we get

$$\begin{aligned}\mathcal{H}f(D(v) - y^*) &\geqslant \left(1 - \frac{sv \cdot y^*}{f(v)}\right)^{s/s-1} f(v) \\ &\geqslant f(v) - \frac{s^2}{s-1} v \cdot y^*.\end{aligned}$$

□

4.3 Boundary conditions

Under certain conditions we can control the behaviour of $\mathcal{H}f$ near the boundary, and thus obtain the continuity.

Definition 4.17. Let $f \in \mathrm{HConc}_s(\mathcal{C})$ and let $\alpha \in (0,1)$. We say that f satisfies the sublinear boundary condition of order α if for any non-zero v on the boundary of $\mathcal{C}$ and for any x in the interior of $\mathcal{C}$, there exists a constant $C := C(v,x) > 0$ such that $f(v + \epsilon x)^{1/s} \geqslant C\epsilon^\alpha$.

Note that the condition is always satisfied at v if $f(v) > 0$. Furthermore, the condition is satisfied for any v, x with $\alpha = 1$ by homogeneity and log-concavity, so the crucial question is whether we can decrease α slightly.

Using this sublinear condition, we get the vanishing of $\mathcal{H}f$ along the boundary.

Proposition 4.18. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ satisfy the sublinear boundary condition of order α. Then $\mathcal{H}f$ vanishes along the boundary. As a consequence, $\mathcal{H}f$ extends to a continuous function over V^* by setting $\mathcal{H}f = 0$ outside $\mathcal{C}^*$.*

Proof. Let w^* be a boundary point of $\mathcal{C}^*$. Then there exists some non-zero $v \in \mathcal{C}$ such that $w^* \cdot v = 0$. Fix $x \in \mathcal{C}^\circ$. By the definition of $\mathcal{H}f$ we get

$$\mathcal{H}f(w^*)^{s-1/s} \leqslant \frac{w^* \cdot (v + \epsilon x)}{f^{1/s}(v + \epsilon x)} \leqslant \frac{\epsilon w^* \cdot x}{C\epsilon^\alpha}.$$

Letting ϵ tend to zero, we see $\mathcal{H}f(w^*) = 0$.

To show the continuity, by Lemma 3.1 we only need to verify

$$\lim_{\epsilon \to 0} \mathcal{H}f(w^* + \epsilon y^*) = 0$$

for some $y^* \in \mathcal{C}^{*\circ}$ (as any other limiting sequence is dominated by such a sequence). This follows easily from

$$\begin{aligned}\mathcal{H}f(w^* + \epsilon y^*)^{s-1/s} &\leqslant \frac{(w^* + \epsilon y^*) \cdot (v + \epsilon x)}{f^{1/s}(v + \epsilon x)} \\ &\leqslant \frac{\epsilon(y^* \cdot v + w^* \cdot x + \epsilon y^* \cdot x)}{C\epsilon^\alpha}.\end{aligned}$$

□

Remark 4.19. If f satisfies the sublinear condition, then $\mathcal{C}^*_{\mathcal{H}f} = \mathcal{C}^{*\circ}$. This makes the statements of the previous results very clean. In the following sections, the function $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ both have this nice property.

5 Positivity for curves

We now study the basic properties of $\widehat{\mathrm{vol}}$ and of the Zariski decompositions for curves. Some aspects of the theory will follow immediately from the formal theory of Section 4; others will require a direct geometric argument.

We first outline how to apply the results of Section 4. Recall that $\widehat{\mathrm{vol}}$ is the polar transform of the volume function for divisors restricted to the nef cone. More precisely, we are now in the situation:

$$\mathcal{C} = \mathrm{Nef}^1(X), \quad f = \mathrm{vol}, \quad \mathcal{C}^* = \overline{\mathrm{Eff}}_1(X), \quad \mathcal{H}f = \widehat{\mathrm{vol}}.$$

Thus, to understand the properties of $\widehat{\mathrm{vol}}$ we need to recall the basic features of the volume function on the nef cone of divisors. It is an elementary fact that the volume function on the nef cone of divisors is differentiable everywhere (with $D(A) = A^{n-1}$). In the notation of Section 3 the cone $\mathrm{Nef}^1(X)_{\mathrm{vol}}$ coincides with the big and nef cone. The Khovanskii-Teissier inequality (with Teissier proportionality) holds on the big and nef cone as recalled in Section 2. Finally, the volume for nef divisors satisfies the sublinear boundary condition of order $n - 1/n$: this follows from an elementary intersection calculation using the fact that $N \cdot A^{n-1} \neq 0$ for any non-zero nef divisor N and ample divisor A.

Remark 5.1. Due to the outline above, the proofs in this section depend only upon elementary facts about intersection theory, the Khovanskii-Teissier inequality and Teissier's proportionality theorem. As discussed in the preliminaries, the arguments in this section thus extend immediately to smooth varieties over an arbitrary algebraically closed field and to the Kähler setting.

5.1 Basic properties

The following theorems collect the various analytic consequences for $\widehat{\mathrm{vol}}$.

Theorem 5.2. *Let X be a projective variety of dimension n. Then:*

1. *$\widehat{\mathrm{vol}}$ is continuous and homogeneous of weight $n/n-1$ on $\overline{\mathrm{Eff}}_1(X)$ and is positive precisely for the big classes.*

2. *For any big and nef divisor class A, we have $\widehat{\mathrm{vol}}(A^{n-1}) = \mathrm{vol}(A)$.*

3. *For any big curve class α, there is a big and nef divisor class B such that*

$$\widehat{\mathrm{vol}}(\alpha) = \left(\frac{B \cdot \alpha}{\mathrm{vol}(B)^{1/n}}\right)^{n/n-1}.$$

We say that the class B computes $\widehat{\mathrm{vol}}(\alpha)$.

The first two were already proved in [Xia15, Theorem 3.1].

Proof. (1) follows immediately from Propositions 3.2 and 4.18. Since $D(A) = A^{n-1}$, (2) follows from the computation

$$\widehat{\mathrm{vol}}(A^{n-1}) = D(A) \cdot A = A^n.$$

The existence in (3) follows from Theorem 3.6. $\square$

We also note the following easy basic linearity property, which follows immediately from the Khovanskii-Teissier inequalities.

Theorem 5.3. *Let X be a projective variety of dimension n and let α be a big curve class. If A computes $\widehat{\mathrm{vol}}(\alpha)$, it also computes $\widehat{\mathrm{vol}}(c_1\alpha + c_2A^{n-1})$ for any positive constants c_1 and c_2.*

After constructing Zariski decompositions below, we will see that in fact we can choose a possibly negative c_2 so long as $c_1\alpha + c_2A^{n-1}$ is a big class.

5.2 Zariski decompositions for curves

The following theorem is the basic result establishing the existence of Zariski decompositions for curve classes.

Theorem 5.4. *Let X be a projective variety of dimension n. Any big curve class α admits a unique Zariski decomposition: there is a unique pair consisting of a big and nef divisor class B_α and a pseudo-effective curve class γ satisfying $B_\alpha \cdot \gamma = 0$ and*

$$\alpha = B_\alpha^{n-1} + \gamma.$$

In fact $\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(B_\alpha^{n-1}) = \mathrm{vol}(B_\alpha)$. In particular B_α computes $\widehat{\mathrm{vol}}(\alpha)$, and any big and nef divisor computing $\widehat{\mathrm{vol}}(\alpha)$ is proportional to B_α.

Proof. The existence of the Zariski decomposition and the uniqueness of the positive part B_α^{n-1} follow from Theorem 4.3. The uniqueness of B_α follows from Teissier proportionality for big and nef divisor classes. It is clear that B_α computes $\widehat{\mathrm{vol}}(\alpha)$ by Theorem 4.3. The last claim follows from Teissier proportionality and the fact that $\alpha \geq B_\alpha^{n-1}$. $\square$

As discussed before, conceptually the Zariski decomposition $\alpha = B_\alpha^{n-1} + \gamma$ captures the failure of log concavity of $\widehat{\mathrm{vol}}$: the term B_α^{n-1} captures all the of the positivity encoded by $\widehat{\mathrm{vol}}$ and is positive in a very strong sense, while the negative part γ lies on the boundary of the pseudo-effective cone.

Example 5.5. Let X be the projective bundle over $\mathbb{P}^1$ defined by $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)$. There are two natural divisor classes on X: the class f of the fibers of the projective bundle and the class ξ of the sheaf $\mathcal{O}_{X/\mathbb{P}^1}(1)$. Using for example [Ful11, Theorem 1.1] and [FL13, Proposition 7.1], one sees that f and ξ generate the algebraic cohomology classes with the relations $f^2 = 0$, $\xi^2 f = -\xi^3 = 1$ and

$$\overline{\mathrm{Eff}}^1(X) = \mathrm{Mov}^1(X) = \langle f, \xi \rangle \qquad \mathrm{Nef}^1(X) = \langle f, \xi + f \rangle$$

and

$$\overline{\mathrm{Eff}}_1(X) = \langle \xi f, \xi^2 \rangle \qquad \mathrm{Nef}_1(X) = \langle \xi f, \xi^2 + \xi f \rangle$$
$$\mathrm{CI}_1(X) = \langle \xi f, \xi^2 + 2\xi f \rangle.$$

Using this explicit computation of the nef cone of the divisors, we have

$$\widehat{\mathrm{vol}}(x\xi f + y\xi^2) = \inf_{a,b \geqslant 0} \frac{ay + bx}{(3ab^2 + 2b^3)^{1/3}}$$

This is essentially a one-variable minimization problem due to the homogeneity in a, b. It is straightforward to compute directly that for non-negative values of x, y:

$$\begin{aligned} \widehat{\mathrm{vol}}(x\xi f + y\xi^2) &= \left(\frac{3}{2}x - y\right) y^{1/2} && \text{if } x \geqslant 2y; \\ &= \frac{x^{3/2}}{2^{1/2}} && \text{if } x < 2y. \end{aligned}$$

Note that when $x < 2y$, the class $x\xi f + y\xi^2$ no longer lies in the complete intersection cone – to obtain $\widehat{\mathrm{vol}}$, Theorem 5.4 indicates that we must project α onto the complete intersection cone in the y-direction. This exactly coheres with the calculation above.

The Zariski decomposition for curves is continuous.

Theorem 5.6. *Let X be a projective variety of dimension n. The function sending a big class α to its positive part B_α^{n-1} or to the corresponding divisor B_α is continuous.*

Proof. The first statement follows from Theorem 4.6. The second then follows from the continuity of the inverse map to the $n-1$-power map. □

It is interesting to study whether the Zariski projection taking α to its positive part is $\mathcal{C}^1$. This is true on the ample cone – the map Φ sending an ample divisor class A to A^{n-1} is a $\mathcal{C}^1$ diffeomorphism by the argument in Remark 2.3.

Remark 5.7. The continuity of the Zariski decomposition does not extend to the entire pseudo-effective cone, even for surfaces. For example, suppose that a surface S admits a nef class N which is a limit of (rescalings of) irreducible curve classes which each have negative self-intersection. (A well-known example of such a surface is $\mathbb{P}^2$ blown up at 9 general points.) For any $c \in [0, 1]$ one can find a sequence of big divisors $\{L_i\}$ whose limit is N but whose positive parts have limit cN.

An important feature of the σ-decomposition for divisors is its concavity: given two big divisors L_1, L_2 we have

$$P_\sigma(L_1 + L_2) \geq P_\sigma(L_1) + P_\sigma(L_2).$$

However, the analogous property fails for curves:

Example 5.8. Let X be a smooth projective variety such that $\mathrm{CI}_1(X)$ is not convex. (An explicit example is given in Appendix B.) Then there are complete intersection classes $\alpha = B_\alpha^{n-1}$ and $\beta = B_\beta^{n-1}$ such that $\alpha+\beta$ is not a complete intersection class. Let $B_{\alpha+\beta}^{n-1}$ denote the positive part of the Zariski decomposition for $\alpha+\beta$. Then

$$B_{\alpha+\beta}^{n-1} \preceq B_\alpha^{n-1} + B_\beta^{n-1}.$$

Furthermore, we can not have equality since the sum is not a complete intersection class. Thus

$$B_{\alpha+\beta}^{n-1} \precneq B_\alpha^{n-1} + B_\beta^{n-1}.$$

However, one can still ask:

Question 5.9. Fix $\alpha \in \overline{\mathrm{Eff}}_1(X)$. Is there a fixed class $\xi \in \mathrm{CI}_1(X)$ such that for any $\epsilon > 0$ there is a $\delta > 0$ satisfying

$$B_{\alpha+\delta\beta}^{n-1} \preceq B_{\alpha+\epsilon\xi}^{n-1}$$

for every $\beta \in N_1(X)$ of bounded norm?

This question is crucial for making sense of the Zariski decomposition of a curve class on the boundary of $\overline{\mathrm{Eff}}_1(X)$ via taking a limit.

5.3 Strict log concavity

The following theorem is an immediate consequence of Theorem 4.3, which gives the strict log concavity of $\widehat{\mathrm{vol}}$.

Theorem 5.10. *Let X be a projective variety of dimension n. For any two pseudo-effective curve classes α, β we have*

$$\widehat{\mathrm{vol}}(\alpha+\beta)^{\frac{n-1}{n}} \geqslant \widehat{\mathrm{vol}}(\alpha)^{\frac{n-1}{n}} + \widehat{\mathrm{vol}}(\beta)^{\frac{n-1}{n}}.$$

Furthermore, if α and β are big, then we obtain an equality if and only if the positive parts of α and β are proportional.

5.4 Differentiability

In [BFJ09] the derivative of the volume function was calculated using the positive product: given a big divisor class L and any divisor class E, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(L+tE) = n\langle L^{n-1}\rangle \cdot E.$$

In this section we prove an analogous statement for curve classes. For curves, the big and nef divisor class B occurring in the Zariski decomposition plays the role of the positive product, and the homogeneity constant $n/n-1$ plays the role of n.

Theorem 5.11. *Let X be a projective variety of dimension n, and let α be a big curve class with Zariski decomposition $\alpha = B^{n-1} + \gamma$. Let β be any curve class. Then $\widehat{\mathrm{vol}}(\alpha + t\beta)$ is differentiable at 0 and*

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{\mathrm{vol}}(\alpha + t\beta) = \frac{n}{n-1} B \cdot \beta.$$

In particular, the function $\widehat{\mathrm{vol}}$ is $\mathcal{C}^1$ on the big cone of curves. If C is an irreducible curve on X, then we can instead write

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{\mathrm{vol}}(\alpha + tC) = \frac{n}{n-1} \mathrm{vol}(B|_C).$$

Proof. This follows immediately from Proposition 3.10 since $G_\alpha \cup \{0\}$ consists of a single ray by the last statement of Theorem 5.4. □

Example 5.12. We return to the setting of Example 5.5: let X be the projective bundle over $\mathbb{P}^1$ defined by $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)$. Using our earlier notation we have

$$\overline{\mathrm{Eff}}_1(X) = \langle \xi f, \xi^2 \rangle$$

and

$$\begin{aligned} \widehat{\mathrm{vol}}(x\xi f + y\xi^2) &= \left(\frac{3}{2}x - y\right) y^{1/2} && \text{if } x \geqslant 2y; \\ &= \frac{x^{3/2}}{2^{1/2}} && \text{if } x < 2y. \end{aligned}$$

We focus on the complete intersection region where $x \geqslant 2y$. Then we have

$$x\xi f + y\xi^2 = \left(\frac{x-2y}{2y^{1/2}} f + y^{1/2}(\xi + f)\right)^2.$$

The divisor in the parentheses on the right hand side is exactly the B appearing in the Zariski decomposition expression for $x\xi f + y\xi^2$. Thus, we can calculate the directional derivative of $\widehat{\mathrm{vol}}$ along a curve class β by intersecting against this divisor.

For a very concrete example, set $\alpha = 3\xi f + \xi^2$, and consider the behavior of $\widehat{\mathrm{vol}}$ for

$$\alpha_t := 3\xi f + \xi^2 - t(2\xi f + \xi^2).$$

Note that α_t is pseudo-effective precisely for $t \leqslant 1$. In this range, the explicit expression for the volume above yields

$$\begin{aligned} \widehat{\mathrm{vol}}(\alpha_t) &= \left(\frac{7}{2} - 2t\right)(1-t)^{1/2}, \\ \frac{d}{dt}\widehat{\mathrm{vol}}(\alpha_t) &= -3(1-t)^{1/2} - \frac{3}{4}(1-t)^{-1/2}. \end{aligned}$$

Note that this calculation agrees with the prediction of Theorem 5.11, which states that if B_t is the divisor defining the positive part of α_t then

$$\begin{aligned}\frac{d}{dt}\widehat{\mathrm{vol}}(\alpha_t) &= \frac{3}{2}B_t \cdot (2\xi f + \xi^2) \\ &= \frac{-3}{2}\left(\frac{(3-2t)-2(1-t)}{2(1-t)^{1/2}} + 2(1-t)^{1/2}\right).\end{aligned}$$

In particular, the derivative decreases to $-\infty$ as t approaches 1 (and the coefficients of the divisor B also increase without bound). This is a surprising contrast to the situation for divisors. Note also that $\widehat{\mathrm{vol}}$ is not convex on this line segment, while vol is convex in any pseudo-effective direction in the nef cone of divisors by the Morse inequality.

5.5 Negative parts

We next analyze the structure of the negative part of the Zariski decomposition. First we have:

Lemma 5.13. *Let X be a projective variety. Suppose α is a big curve class and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. If $\gamma \neq 0$ then $\gamma \notin \mathrm{Mov}_1(X)$.*

Proof. Since B is big and $B \cdot \gamma = 0$, γ cannot be movable if it is non-zero. □

For the Zariski decomposition under $\widehat{\mathrm{vol}}$, we can not guarantee the negative part is a curve class of effective curve. As in [FL13], it is more reasonable to ask if the negative part is the pushforward of a pseudo-effective class from a proper subvariety. Note that this property is automatic when the negative part is represented by an effective class, and for surfaces it is actually equivalent to asking that the negative part be effective. In general this subtle property of pseudo-effective classes is crucial for inductive arguments on dimension.

Proposition 5.14. *Let X be a projective variety of dimension n. Let α be a big curve class and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. There is a proper subscheme $i : V \subsetneq X$ and a pseudo-effective class $\gamma' \in N_1(V)$ such that $i_*\gamma' = \gamma$.*

Proof. We may choose an effective nef $\mathbb{R}$-Cartier divisor D whose class is B. By resolving the base locus of a sufficiently high multiple of D we obtain a blow-up $\phi : Y \to X$, a birational morphism $\psi : Y \to Z$ and an effective ample divisor A on Z such that after replacing D by some numerically equivalent divisor we have $\phi^*D \geqslant \psi^*A$. Write E for the difference of these two divisors and set V_Y to be the union of $\mathrm{Supp}(E)$ with the ψ-exceptional locus.

There is a pseudo-effective curve class γ_Y on Y which pushes forward to γ and thus satisfies $\phi^*D \cdot \gamma_Y = 0$. There is an infinite sequence of effective 1-cycles C_i such that $\lim_{i\to\infty}[C_i] = \gamma_Y$. Each effective cycle C_i can be decomposed as a sum $C_i = T_i + T_i'$ where T_i' consists of the components contained in V_Y and T_i consists of the rest.

Note that

$$\lim_{i\to\infty} A \cdot \psi_* T_i \leqslant \lim_{i\to\infty} \phi^*D \cdot T_i = 0.$$

This shows that $\lim_{i\to\infty}[T_i]$ converges to a pseudo-effective curve class $\beta \in N_1(Y)$ satisfying $\psi_*\beta = 0$.

Clearly $\lim_{i\to\infty}[T_i']$ is the pushforward of a pseudo-effective curve class from V_Y. [DJV13, Theorem 4.1] (which holds in the singular case by the same argument) shows that β is also the pushforward of a pseudo-effective curve class on V_Y. Thus γ_Y is the pushforward of a pseudo-effective curve class on V_Y. Pushing forward to X, we see that γ is the pushforward of a pseudo-effective curve class on $V := \phi(V_Y)$. Note that V is a proper subset of X since ϕ is birational. □

Remark 5.15. In contrast, for the Zariski decomposition of curves in the sense of Boucksom (see [Xia15, Theorem 3.3 and Lemma 3.5]) the negative part can always be represented by an effective curve which is very rigidly embedded in X. This has a similar feel as the σ-decomposition of [Nak04] for curve classes.

5.6 Birational behavior

We next use the Zariski decomposition to analyze the behavior of positivity of curves under birational maps $\phi : Y \to X$. Note that (in contrast to divisors) the birational pullback can only decrease the positivity for curve classes: we have

$$\widehat{\mathrm{vol}}(\alpha) \geqslant \widehat{\mathrm{vol}}(\phi^*\alpha).$$

In fact pulling back does not preserve pseudo-effectiveness, and even for a movable class we can have a strict inequality of $\widehat{\mathrm{vol}}$ (for example, a big movable class can pull back to a movable class on the pseudo-effective boundary). Again guided by [FL13], the right approach is to consider all ϕ_*-preimages of α at once.

Proposition 5.16. *Let $\phi : Y \to X$ be a birational morphism of projective varieties of dimension n. Let α be a big curve class on X with Zariski decomposition $B^{n-1} + \gamma$. Let $\mathcal{A}$ be the set of all pseudo-effective curve classes α' on Y satisfying $\phi_*\alpha' = \alpha$. Then*

$$\sup_{\alpha'\in\mathcal{A}} \widehat{\mathrm{vol}}(\alpha') = \widehat{\mathrm{vol}}(\alpha).$$

This supremum is achieved by an element $\alpha_Y \in \mathcal{A}$.

Proof. Suppose $\alpha' \in \mathcal{A}$. Since $\phi_*\alpha' = \alpha$, it is clear from the projection formula that $\widehat{\mathrm{vol}}(\alpha') \leqslant \widehat{\mathrm{vol}}(\alpha)$. Conversely, set γ_Y to be any pseudo-effective curve class on Y pushing forward to γ. Define $\alpha_Y = \phi^*B^{n-1} + \gamma_Y$. Since $\phi^*B \cdot \gamma_Y = 0$, by Theorem 5.4 this expression is the Zariski decomposition for α_Y. In particular $\widehat{\mathrm{vol}}(\alpha_Y) = \widehat{\mathrm{vol}}(\alpha)$. □

This proposition indicates the existence of some "distinguished" preimages of α with maximum $\widehat{\mathrm{vol}}$. In fact, these distinguished preimages also have a very nice structure.

Proposition 5.17. *Let $\phi : Y \to X$ be a birational morphism of projective varieties of dimension n. Let α be a big curve class on X with Zariski decomposition $B^{n-1} + \gamma$. Set $\mathcal{A}'$ to be the set of all pseudo-effective curve class α' on Y satisfying $\phi_*\alpha' = \alpha$ and $\widehat{\mathrm{vol}}(\alpha') = \widehat{\mathrm{vol}}(\alpha)$. Then*

1. *Every $\alpha' \in \mathcal{A}'$ has a Zariski decomposition of the form*

$$\alpha' = \phi^* B^{n-1} + \gamma'.$$

Thus $\mathcal{A}' = \{\phi^* B^{n-1} + \gamma' \mid \gamma' \in \overline{\mathrm{Eff}}_1(Y), \phi_* \gamma' = \gamma\}$ *is determined by the set of pseudo-effective preimages of* γ.

2. *These Zariski decompositions are stable under adding ϕ-exceptional curves: if ξ is a pseudo-effective curve class satisfying $\phi_* \xi = 0$, then for any $\alpha' \in \mathcal{A}'$ we have*

$$\alpha' + \xi = \phi^* B^{n-1} + (\gamma' + \xi)$$

is the Zariski decomposition for $\alpha' + \xi$.

Proof. To see (1), note that

$$\frac{\phi^* B}{\mathrm{vol}(B)^{1/n}} \cdot \alpha' = \frac{B}{\mathrm{vol}(B)^{1/n}} \cdot \alpha = \widehat{\mathrm{vol}}(\alpha).$$

Thus if $\widehat{\mathrm{vol}}(\alpha') = \widehat{\mathrm{vol}}(\alpha)$ then $\widehat{\mathrm{vol}}(\alpha')$ is computed by $\phi^* B$. By Theorem 5.4 we obtain the statement.

(2) follows immediately from (1), since

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(\alpha') \leqslant \widehat{\mathrm{vol}}(\alpha' + \xi) \leqslant \widehat{\mathrm{vol}}(\alpha)$$

by Proposition 5.16. □

While there is not necessarily a *uniquely* distinguished ϕ_*-preimage of α, there *is* a uniquely distinguished complete intersection class on Y whose ϕ-pushforward lies beneath α – namely, the positive part of any sufficiently large class pushing forward to α. This is the analogue in our setting of the "movable transform" of [FL13].

5.7 Morse-type inequality for curves

In this section we prove a Morse-type inequality for curves under the volume function $\widehat{\mathrm{vol}}$. First let us recall the algebraic Morse inequality for nef divisor classes over smooth projective varieties. If A, B are nef divisor classes on a smooth projective variety X of dimension n, then by [Laz04, Example 2.2.33] (see also [Dem85], [Siu93], [Tra95])

$$\mathrm{vol}(A - B) \geqslant A^n - nA^{n-1} \cdot B.$$

In particular, if $A^n - nA^{n-1} \cdot B > 0$, then $A - B$ is big. This gives us a very useful bigness criterion for the difference of two nef divisors.

By analogy with the divisor case, we can ask:

- Let X be a projective variety of dimension n, and let $\alpha, \gamma \in \overline{\mathrm{Eff}}_1(X)$ be two nef curve classes. Is there a criterion for the bigness of $\alpha - \gamma \in \overline{\mathrm{Eff}}_1(X)$ using only intersection numbers defined by α, γ?

Inspired by [Xia13], we give such a criterion using the $\widehat{\mathrm{vol}}$ function. In Section 6, we answer the above question by giving a slightly different criterion which needs the refined structure of the movable cone of curves; see Theorem 6.18. The following results follow from Theorem 4.16.

Theorem 5.18. *Let X be a projective variety of dimension n. Let α be a big curve class and let β be a movable curve class. Write $\alpha = B^{n-1} + \gamma$ for the Zariski decomposition of α. Then*

$$\begin{aligned}\widehat{\mathrm{vol}}(\alpha-\beta)^{n-1/n} &\geqslant (\widehat{\mathrm{vol}}(\alpha) - nB\cdot\beta)\cdot\widehat{\mathrm{vol}}(\alpha)^{-1/n}\\ &= (B^n - nB\cdot\beta)\cdot(B^n)^{-1/n}.\end{aligned}$$

In particular, we have

$$\widehat{\mathrm{vol}}(\alpha-\beta) \geqslant B^n - \frac{n^2}{n-1}B\cdot\beta.$$

Proof. The theorem follows immediately from Theorem 4.16 and the fact that $\alpha \succeq B^{n-1}$. □

Corollary 5.19. *Let X be a projective variety of dimension n. Let α be a big curve class and let β be a movable curve class. Write $\alpha = B^{n-1} + \gamma$ for the Zariski decomposition of α. If*

$$\widehat{\mathrm{vol}}(\alpha) - nB\cdot\beta > 0$$

then $\alpha - \beta$ is big.

Remark 5.20. Superficially, the above theorem appears to differ from the classical algebraic Morse inequality for nef divisors, since α can be any big curve class. However, using the Zariski decomposition one sees that the statement for α is essentially equivalent to the statement for the positive part of α, so that Theorem 5.18 is really a claim about nef curve classes.

Example 5.21. The constant n is optimal in Corollary 5.19. Indeed, for any $\epsilon > 0$ there exists a projective variety X such that

$$\widehat{\mathrm{vol}}(\alpha) - (n-\epsilon)B_\alpha\cdot\gamma > 0,$$

for some $\alpha \in \overline{\mathrm{Eff}}_1(X)$ and $\gamma \in \mathrm{Mov}_1(X)$ but $\alpha - \gamma$ is not a big curve class.

To find such a variety, let E be an elliptic curve with complex multiplication and set $X = E^{\times n}$. The pseudo-effective cone of divisors $\overline{\mathrm{Eff}}^1(X)$ is identified with the cone of constant positive $(1,1)$-forms, while the pseudo-effective cone of curves $\overline{\mathrm{Eff}}_1(X)$ is identified with the cone of constant positive $(n-1,n-1)$-forms. Furthermore, every strictly positive $(n-1,n-1)$-form is a $(n-1)$-self-product of a strictly positive $(1,1)$-form.

Set $B_\alpha = i\sum_{j=1}^n dz^j \wedge d\bar{z}^j$ and $B_\gamma = i\sum_{j=1}^n \lambda_j dz^j \wedge d\bar{z}^j$ with $\lambda_j > 0$. Let $\alpha = B_\alpha^{n-1}$ and $\gamma = B_\gamma^{n-1}$. Then $\widehat{\mathrm{vol}}(\alpha) - (n-\epsilon)B_\alpha\cdot\gamma > 0$ is equivalent to

$$\sum_{j=1}^n \lambda_1 ... \widehat{\lambda_j} ... \lambda_n < \frac{n}{n-\epsilon},$$

and $\alpha - \gamma$ being big is equivalent to

$$\lambda_1 ... \widehat{\lambda_j} ... \lambda_n < 1$$

for every j. Now it is easy to see we can always choose $\lambda_1, ..., \lambda_n$ such that the first inequality holds but the second does not hold.

Remark 5.22. Using the cone duality $\overline{\mathcal{K}}^{\vee} = \mathcal{N}$ and Theorem 12.1 in Appendix A, it is easy to extend the above Morse-type inequality for curves to positive currents of bidimension $(1,1)$ over compact Kähler manifolds.

One wonders if Theorem 5.18 can be improved:

Question 5.23. Let X be a projective variety of dimension n. Let α be a big curve class and let β be a movable curve class. Write $\alpha = B^{n-1} + \gamma$ for the Zariski decomposition of α. Is

$$\widehat{\mathrm{vol}}(\alpha - \beta) \geqslant \mathrm{vol}(\alpha) - nB \cdot \beta?$$

Remark 5.24. By Theorem 5.18, if $\widehat{\mathrm{vol}}(\alpha) - nB \cdot \beta > 0$ then $\widehat{\mathrm{vol}}$ is $\mathcal{C}^1$ at the point $\alpha - s\beta$ for every $s \in [0,1]$. The derivative formula of $\widehat{\mathrm{vol}}$ implies

$$\widehat{\mathrm{vol}}(\alpha - \beta) - \widehat{\mathrm{vol}}(\alpha) = \int_0^1 -\frac{n}{n-1} B_{\alpha - s\beta} \cdot \beta \, ds,$$

where $B_{\alpha - s\beta}$ is the big and nef divisor class defining the Zariski decomposition of $\alpha - s\beta$. To give an affirmative answer to Question 5.23, we conjecture the following:

$$B_{\alpha - s\beta} \cdot \beta \leqslant (n-1) B_{\alpha} \cdot \beta \text{ for every } s \in [0,1].$$

Without loss of generality, we can assume $B_\alpha \cdot \beta > 0$. Then by continuity of the decomposition, this inequality holds for s in a neighbourhood of 0. At this moment, we do not know how to see this neighbourhood covers $[0,1]$.

6 Positive products and movable curves

In this section, we study the movable cone of curves and its relationship to the positive product of divisors. A key tool in this study is the following function of [Xia15, Definition 2.2]:

Definition 6.1 (see [Xia15] Definition 2.2)**.** Let X be a projective variety of dimension n. For any curve class $\alpha \in \mathrm{Mov}_1(X)$ define

$$\mathfrak{M}(\alpha) = \inf_{L \text{ big } \mathbb{R}\text{-divisor}} \left(\frac{L \cdot \alpha}{\mathrm{vol}(L)^{1/n}} \right)^{n/n-1}.$$

We say that a big class L computes $\mathfrak{M}(\alpha)$ if this infimum is achieved by L. When α is a curve class that is not movable, we set $\mathfrak{M}(\alpha) = 0$.

In other words, $\mathfrak{M}$ is the function on $\mathrm{Mov}_1(X)$ defined as the polar transform of the volume function on $\overline{\mathrm{Eff}}^1(X)$, so we are in the situation:

$$\mathcal{C} = \overline{\mathrm{Eff}}^1(X), \quad f = \mathrm{vol}, \quad \mathcal{C}^* = \mathrm{Mov}_1(X), \quad \mathcal{H}f = \mathfrak{M}.$$

Note that $\mathcal{C}^* = \mathrm{Mov}_1(X)$ follows from the main result of [BDPP13].

While the definition is a close analogue of $\widehat{\mathrm{vol}}$, the function $\mathfrak{M}$ exhibits somewhat different behavior. We will show that $\mathfrak{M}$ measures the volume of the "$(n-1)$st root" of α, in a sense described below. In order to establish some deeper properties of the function $\mathfrak{M}$, we need to better understand the volume function for divisors.

We first extend several well known results on big and nef divisors to big and movable divisors.

6.1 The volume function on big and movable divisors

The key will be an extension of Teissier's proportionality theorem for big and nef divisors (see Section 2) to big and movable divisors.

Lemma 6.2. *Let X be a projective variety of dimension n. Let L_1 and L_2 be big movable divisor classes. Set s to be the largest real number such that $L_1 - sL_2$ is pseudo-effective. Then*

$$s^n \leqslant \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}$$

with equality if and only if L_1 and L_2 are proportional.

Proof. We first prove the case when X is smooth. Certainly we have $\mathrm{vol}(L_1) \geqslant \mathrm{vol}(sL_2) = s^n \mathrm{vol}(L_2)$. If they are equal, then since sL_2 is movable and $L_1 - sL_2$ is pseudo-effective we get a Zariski decomposition of

$$L_1 = sL_2 + (L_1 - sL_2)$$

in the sense of [FL13]. By [FL13, Proposition 5.3], this decomposition coincides with the numerical version of the σ-decomposition of [Nak04] so that $P_\sigma(L_1) = sL_2$. Since L_1 is movable, we obtain equality $L_1 = sL_2$.

For arbitrary X, let $\phi : X' \to X$ be a resolution. The inequality follows by pulling back L_1 and L_2 and replacing them by their positive parts. Indeed using the numerical analogue of [Nak04, III.1.14 Proposition] we see that $\phi^* L_1 - sP_\sigma(\phi^* L_2)$ is pseudo-effective if and only if $P_\sigma(\phi^* L_1) - sP_\sigma(\phi^* L_2)$ is pseudo-effective, so that s can only go up under this operation. To characterize the equality, recall that if L_1 and L_2 are movable and $P_\sigma(\phi^* L_1) = sP_\sigma(\phi^* L_2)$, then $L_1 = sL_2$ by the injectivity of the capping map. □

Proposition 6.3. *Let X be a projective variety of dimension n. Let L_1, L_2 be big and movable divisor classes. Then*

$$\langle L_1^{n-1} \rangle \cdot L_2 \geqslant \mathrm{vol}(L_1)^{n-1/n} \mathrm{vol}(L_2)^{1/n}$$

with equality if and only if L_1 and L_2 are proportional.

Proof. We first suppose X is smooth. Set s_L to be the largest real number such that $L_1 - s_L L_2$ is pseudo-effective, and fix an ample divisor H on X.

For any $\epsilon > 0$, by taking sufficiently good Fujita approximations we may find a birational map $\phi_\epsilon : Y_\epsilon \to X$ and ample divisor classes $A_{1,\epsilon}$ and $A_{2,\epsilon}$ such that

- $\phi_\epsilon^* L_i - A_{i,\epsilon}$ is pseudo-effective for $i = 1, 2$;
- $\mathrm{vol}(A_{i,\epsilon}) > \mathrm{vol}(L_i) - \epsilon$ for $i = 1, 2$;
- $\phi_{\epsilon *} A_{i,\epsilon}$ is in an ϵ-ball around L_i for $i = 1, 2$.

Furthermore:

- By applying the argument of [FL13, Theorem 6.22], we may ensure

$$\phi_\epsilon^*(\langle L_1^{n-1} \rangle - \epsilon H^{n-1}) \preceq A_{1,\epsilon}^{n-1} \preceq \phi_\epsilon^*(\langle L_1^{n-1} \rangle + \epsilon H^{n-1}).$$

- Set s_ϵ to be the largest real number such that $A_{1,\epsilon} - s_\epsilon A_{2,\epsilon}$ is pseudo-effective. Then we may ensure that $s_\epsilon < s_L + \epsilon$.

By the Khovanskii-Teissier inequality for nef divisor classes, we have

$$(A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{n/n-1} \geqslant \operatorname{vol}(A_{1,\epsilon}) \operatorname{vol}(A_{2,\epsilon})^{1/n-1}.$$

Note that $\langle L^{n-1} \rangle \cdot L_2$ is approximated by $A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon}$ by the projection formula. Taking a limit as ϵ goes to 0, we see that

$$\langle L_1^{n-1} \rangle \cdot L_2 \geqslant \operatorname{vol}(L_1)^{n-1/n} \operatorname{vol}(L_2)^{1/n}. \qquad (\star)$$

On the other hand, the Diskant inequality for big and nef divisors in [BFJ09, Theorem F] implies that

$$\begin{aligned}
(A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{n/n-1} - &\operatorname{vol}(A_{1,\epsilon}) \operatorname{vol}(A_{2,\epsilon})^{1/n-1} \\
&\geqslant \left((A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{1/n-1} - s_\epsilon \operatorname{vol}(A_{2,\epsilon})^{1/n-1} \right)^n \\
&\geqslant \left((A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{1/n-1} - (s_L + \epsilon) \operatorname{vol}(A_{2,\epsilon})^{1/n-1} \right)^n .
\end{aligned}$$

Taking a limit as ϵ goes to 0 again, we see that

$$\begin{aligned}
(\langle L_1^{n-1} \rangle \cdot L_2)^{n/n-1} - &\operatorname{vol}(L_1) \operatorname{vol}(L_2)^{1/n-1} \\
&\geqslant \left((\langle L_1^{n-1} \rangle \cdot L_2)^{1/n-1} - s_L \operatorname{vol}(L_2)^{1/n-1} \right)^n .
\end{aligned}$$

Thus we extend the Diskant inequality to big and movable divisor classes. Lemma 6.2, equation $(\star)$ and the above Diskant inequality together show that

$$\langle L_1^{n-1} \rangle \cdot L_2 = \operatorname{vol}(L_1)^{n-1/n} \operatorname{vol}(L_2)^{1/n}$$

if and only if L_1 and L_2 are proportional.

Now suppose X is singular. The inequality can be computed by passing to a resolution $\phi : X' \to X$ and replacing L_1 and L_2 by their positive parts, since the left hand side can only decrease under this operation. To characterize the equality, recall that if L_1 and L_2 are movable and $P_\sigma(\phi^* L_1) = s P_\sigma(\phi^* L_2)$, then $L_1 = s L_2$ by the injectivity of the capping map. $\square$

Remark 6.4. As a byproduct of the proof above, we get the Diskant inequality for big and movable divisor classes.

Remark 6.5. In the analytic setting, applying Proposition 6.3 and the same method in the proof of Theorem 2.1, it is not hard to generalize Theorem 2.1 to big and movable divisor classes:

- Let $L_1, ..., L_n$ be n big divisor classes over a smooth complex projective variety X, then we have

$$\langle L_1 \cdot ... \cdot L_n \rangle \geqslant \operatorname{vol}(L_1)^{1/n} \cdot ... \cdot \operatorname{vol}(L_n)^{1/n}$$

where the equality is obtained if and only if $P_\sigma(L_1), ..., P_\sigma(L_n)$ are proportional.

We only need to characterize the equality situation. To see this, we need the fact that the above positive intersection $\langle L_1 \cdot ... \cdot L_n \rangle$ depends only on the positive parts $P_\sigma(L_i)$, which follows from the analytic construction of positive product [Bou02, Proposition 3.2.10]. Then by the method in the proof of Theorem 2.1 where we apply [BEGZ10] or [DDG+14, Theorem D], we reduce it to the case of a pair of divisor classes, e.g. we get

$$\langle P_\sigma(L_1)^{n-1} \cdot P_\sigma(L_2) \rangle = \mathrm{vol}(L_1)^{n-1/n} \mathrm{vol}(L_2)^{1/n}.$$

By the definition of positive product we always have

$$\langle P_\sigma(L_1)^{n-1} \cdot P_\sigma(L_2) \rangle \geqslant \langle P_\sigma(L_1)^{n-1} \rangle \cdot P_\sigma(L_2) \geqslant \mathrm{vol}(L_1)^{n-1/n} \mathrm{vol}(L_2)^{1/n},$$

this then implies the equality

$$\langle P_\sigma(L_1)^{n-1} \rangle \cdot P_\sigma(L_2) = \mathrm{vol}(L_1)^{n-1/n} \mathrm{vol}(L_2)^{1/n}.$$

By Proposition 6.3, we immediately obtain the desired result.

Corollary 6.6. *Let X be a smooth projective variety of dimension n. Let $\alpha \in \mathrm{Mov}_1(X)$ be a big movable curve class. All big divisor classes L satisfying $\alpha = \langle L^{n-1} \rangle$ have the same positive part $P_\sigma(L)$.*

Proof. Suppose L_1 and L_2 have the same positive product. We have $\mathrm{vol}(L_1) = \langle L_2^{n-1} \rangle \cdot L_1$ so that $\mathrm{vol}(L_1) \geqslant \mathrm{vol}(L_2)$. By symmetry we obtain the reverse inequality, hence equality everywhere, and we conclude by Proposition 6.3 and the σ-decomposition for smooth varieties. □

As a consequence of Proposition 6.3, we show the strict log concavity of the volume function vol on the cone of big and movable divisors.

Proposition 6.7. *Let X be a projective variety of dimension n. Then the volume function* vol *is strictly n-concave on the cone of big and movable divisor classes.*

Proof. Since the big and movable cone is convex, this follows from Proposition 6.3 and Theorem 4.11. □

6.2 The function $\mathfrak{M}$

We now return to the study of the function $\mathfrak{M}$. As preparation for using the polar transform theory of Section 4, we note the following features of the volume function of divisors on smooth varieties. By [BFJ09] the volume function on the pseudo-effective cone of divisors is differentiable on the big cone (with $D(L) = \langle L^{n-1} \rangle$). In the notation of Section 3 the cone $\overline{\mathrm{Eff}}^1(X)_{\mathrm{vol}}$ coincides with the big cone, so that vol is +-differentiable. The volume function is n-concave, and is strictly n-concave on the big and movable cone by Proposition 6.7. Furthermore, it admits a strong Zariski decomposition with respect to the movable cone of divisors using the σ-decomposition of [Nak04] and Proposition 6.7.

Remark 6.8. Note that if X is not smooth (or at least $\mathbb{Q}$-factorial), then it is unclear whether vol admits a Zariski decomposition structure with respect to the cone of movable divisors. For this reason, we will focus on smooth varieties in this section. See Remark 6.22 for more details.

Note that the sublinearity condition does not hold for the volume function. Thus our first task is to understand the behaviour of $\mathfrak{M}$ on the boundary of the movable cone of curves.

Lemma 6.9. *Let X be a smooth projective variety of dimension n and let α be a movable curve class. Then $\mathfrak{M}(\alpha) = 0$ if and only if α has vanishing intersection a non-zero movable divisor class L.*

Proof. We first show that if there exists some nonzero movable divisor class M such that $\alpha \cdot M = 0$ then $\mathfrak{M}(\alpha) = 0$. Fix an ample divisor class A. Note that $M + \epsilon A$ is big and movable for any $\epsilon > 0$. Thus there exists some modification $\mu_\epsilon : Y_\epsilon \to X$ and an ample divisor class A_ϵ on Y_ϵ such that $M + \frac{\epsilon}{2}A = \mu_{\epsilon *}A_\epsilon$. So we can write

$$M + \epsilon A = \mu_{\epsilon *}\left(A_\epsilon + \frac{\epsilon}{2}\mu_\epsilon^* A\right),$$

which implies

$$\begin{aligned}
\mathrm{vol}(M + \epsilon A) &= \mathrm{vol}\left(\mu_{\epsilon *}\left(A_\epsilon + \frac{\epsilon}{2}\mu_\epsilon^* A\right)\right)\\
&\geqslant \mathrm{vol}\left(A_\epsilon + \frac{\epsilon}{2}\mu_\epsilon^* A\right)\\
&\geqslant n\left(\frac{\epsilon}{2}\mu_\epsilon^* A\right)^{n-1} \cdot A_\epsilon\\
&\geqslant c\epsilon^{n-1} A^{n-1} \cdot M.
\end{aligned}$$

Consider the following intersection number

$$\rho_\epsilon = \alpha \cdot \frac{M + \epsilon A}{\mathrm{vol}(M + \epsilon A)^{1/n}}.$$

The above estimate shows that ρ_ϵ tends to zero as ϵ tends to zero, and so $\mathfrak{M}(\alpha) = 0$.

Conversely, suppose that $\mathfrak{M}(\alpha) = 0$. From the definition of $\mathfrak{M}(\alpha)$, we can take a sequence of big divisor classes L_k with $\mathrm{vol}(L_k) = 1$ such that

$$\lim_{k \to \infty} (\alpha \cdot L_k)^{\frac{n}{n-1}} = \mathfrak{M}(\alpha).$$

Moreover, let $P_\sigma(L_k)$ be the positive part of L_k. Then we have $\mathrm{vol}(P_\sigma(L_k)) = 1$ and

$$\alpha \cdot P_\sigma(L_k) \leqslant \alpha \cdot L_k$$

since α is movable. Thus we can assume the sequence of big divisor classes L_k is movable in the beginning.

Fix an ample divisor A of volume 1, and consider the classes $L_k/(A^{n-1} \cdot L_k)$. These lie in a compact slice of the movable cone, so they must have a non-zero movable accumulation point L, which without loss of generality we may assume is a limit.

Choose a modification $\mu_\epsilon : Y_\epsilon \to X$ and an ample divisor class $A_{\epsilon,k}$ on Y such that

$$A_{\epsilon,k} \leqslant \mu_\epsilon^* L_k, \quad \mathrm{vol}(A_{\epsilon,k}) > \mathrm{vol}(L_k) - \epsilon$$

Then

$$L_k \cdot A^{n-1} \geqslant A_{\epsilon,k} \cdot \mu_\epsilon^* A^{n-1} \geqslant \mathrm{vol}(A_{\epsilon,k})^{1/n}$$

by the Khovanskii-Teissier inequality. Taking a limit over all ϵ, we find $L_k \cdot A^{n-1} \geqslant \mathrm{vol}(L_k)^{1/n}$. Thus

$$L \cdot \alpha = \lim_{k\to\infty} \frac{L_k \cdot \alpha}{L_k \cdot A^{n-1}} \leqslant \mathfrak{M}(\alpha)^{n-1/n} = 0.$$

$\square$

Example 6.10. Note that a movable curve class α with positive $\mathfrak{M}$ need not lie in the interior of the movable cone of curves. A simple example is when X is the blow-up of $\mathbb{P}^2$ at one point, H denotes the pullback of the hyperplane class. For surfaces the functions $\mathfrak{M}$ and vol coincide, so $\mathfrak{M}(H) = 1$ even though H is on the boundary of $\mathrm{Mov}_1(X) = \mathrm{Nef}^1(X)$.

It is also possible for a big movable curve class α to have $\mathfrak{M}(\alpha) = 0$. This occurs for the projective bundle $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1))$ analyzed in Example 5.5. Keeping the notation there, we see that the big and movable curve class $\xi^2 + \xi f$ has vanishing intersection against the movable divisor ξ so that $\mathfrak{M}(\xi^2 + \xi f) = 0$ by Lemma 6.9.

Remark 6.11. Another perspective on Lemma 6.9 is provided by the numerical dimension of [Nak04] and [Bou04]. We recall from [Leh13a] the fact that on a smooth variety the following conditions are equivalent for a class $L \in \overline{\mathrm{Eff}}^1(X)$. (They both correspond to the non-vanishing of the numerical dimension.)

- Fix an ample divisor class A. For any big class D, there is a positive constant C such that $Ct^{n-1} < \mathrm{vol}(L + tA)$ for all $t > 0$.
- $P_\sigma(L) \neq 0$.

In particular, this implies that vol satisfies the sublinear boundary condition of order $n - 1/n$ when restricted to the movable cone, and can be used in the previous proof. A variant of this statement in characteristic p is proved by [CHMS14].

In many ways it is most natural to define $\mathfrak{M}$ using the movable cone of divisors instead of the pseudo-effective cone of divisors. Conceptually, this coheres with the fact that the polar transform can be calculated using the positive part of a Zariski decomposition. Indeed, the argument above (passing to the positive part) shows that when X is smooth, for any $\alpha \in \mathrm{Mov}_1(X)$ we have

$$\mathfrak{M}(\alpha) = \inf_{D \text{ big and movable}} \left(\frac{D \cdot \alpha}{\mathrm{vol}(D)^{1/n}} \right)^{n/n-1}.$$

All in all, for X smooth it is better to consider the following polar transform:

$$\mathcal{C} = \mathrm{Mov}^1(X), \quad f = \mathrm{vol}, \quad \mathcal{C}^* = \mathrm{Mov}^1(X)^*, \quad \mathcal{H}f = \mathfrak{M}'.$$

In particular, since vol satisfies a sublinear condition on $\mathrm{Mov}^1(X)$, the function $\mathfrak{M}'$ is strictly positive exactly in $\mathrm{Mov}^1(X)^{*\circ}$ and extends to a continuous function over $N_1(X)$.

Since this polar function admits a Zariski decomposition onto $\mathrm{Mov}_1(X)$, we continue to focus on the subcone $\mathrm{Mov}_1(X) \subset \mathrm{Mov}^1(X)^*$ where there is interesting behavior and apply $\mathfrak{M}'|_{\mathrm{Mov}_1(X)} = \mathfrak{M}$. Note however an important consequence of this perspective: Lemma 6.9 shows that the subcone of $\mathrm{Mov}_1(X)$ where $\mathfrak{M}$ is positive lies in the interior of $\mathrm{Mov}^1(X)^*$. Thus this region agrees with $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ and $\mathfrak{M}$ extends to a differentiable function on an open set containing this cone by applying Theorem 4.3. In particular $\mathfrak{M}$ is continuous on $\mathrm{Mov}_1(X)$.

We next prove a refined structure of the movable cone of curves. Recall that by [BDPP13] the movable cone of curves $\mathrm{Mov}_1(X)$ is generated by the $(n-1)$-self positive products of big divisors. In other words, any curve class in the interior of $\mathrm{Mov}_1(X)$ is a *convex combination* of such positive products. We show that $\mathrm{Mov}_1(X)$ actually coincides with the closure of such products (which naturally form a cone).

Theorem 6.12. *Let X be a smooth projective variety of dimension n. Then any movable curve class α with $\mathfrak{M}(\alpha) > 0$ has the form*

$$\alpha = \langle L_\alpha^{n-1} \rangle$$

for a unique big and movable divisor class L_α. We then have $\mathfrak{M}(\alpha) = \mathrm{vol}(L_\alpha)$ and any big and movable divisor computing $\mathfrak{M}(\alpha)$ is proportional to L_α.

Proof. Applying Theorem 4.3 to $\mathfrak{M}'$, we get

$$\alpha = D(L_\alpha) + n_\alpha$$

where L_α is a big movable class computing $\mathfrak{M}(\alpha)$ and $n_\alpha \in \mathrm{Mov}^1(X)^*$. As D is the differential of $\mathrm{vol}^{1/n}$ on big and movable divisor classes, we have $D(L_\alpha) = \langle L_\alpha^{n-1} \rangle$. Note that $\mathfrak{M}(\alpha) = \langle L_\alpha^{n-1} \rangle \cdot L_\alpha = \mathrm{vol}(L_\alpha)$.

To finish the proof, we observe that $n_\alpha \in \mathrm{Mov}_1(X)$. This follows since α is movable: by the definition of L_α, for any pseudo-effective divisor class E and $t \geqslant 0$ we have

$$\frac{\alpha \cdot L_\alpha}{\mathrm{vol}(L_\alpha)^{1/n}} \leqslant \frac{\alpha \cdot P_\sigma(L_\alpha + tE)}{\mathrm{vol}(L_\alpha + tE)^{1/n}} \leqslant \frac{\alpha \cdot (L_\alpha + tE)}{\mathrm{vol}(L_\alpha + tE)^{1/n}}$$

with equality at $t = 0$. This then implies

$$n_\alpha \cdot E \geqslant 0.$$

Thus $n_\alpha \in \mathrm{Mov}_1(X)$. Intersecting against L_α, we have $n_\alpha \cdot L_\alpha = 0$. This shows $n_\alpha = 0$ because L_α is an interior point of $\overline{\mathrm{Eff}}^1(X)$ and $\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X)$.

So we have

$$\alpha = D(L_\alpha) = \langle L_\alpha^{n-1} \rangle.$$

Finally, the uniqueness follows from Corollary 6.6. □

We note in passing that we immediately obtain:

Corollary 6.13. *Let X be a projective variety of dimension n. Then the rays spanned by classes of irreducible curves which deform to cover X are dense in* $\mathrm{Mov}_1(X)$.

Proof. It suffices to prove this on a resolution of X. By Theorem 6.12 it suffices to show that any class of the form $\langle L^{n-1}\rangle$ for a big divisor L is a limit of rescalings of classes of irreducible curves which deform to cover X. Indeed, we may even assume that L is a $\mathbb{Q}$-Cartier divisor. Then the positive product is a limit of the pushforward of complete intersections of ample divisors on birational models, whence the result. □

We can also describe the boundary of $\mathrm{Mov}_1(X)$, in combination with Lemma 6.9.

Corollary 6.14. *Let X be a smooth projective variety of dimension n. Let α be a movable class with $\mathfrak{M}(\alpha) > 0$ and let L_α be the unique big movable divisor whose positive product is α. Then α is on the boundary of $\mathrm{Mov}_1(X)$ if and only if L_α is on the boundary of $\mathrm{Mov}^1(X)$.*

Proof. Note that α is on the boundary of $\mathrm{Mov}_1(X)$ if and only if it has vanishing intersection against a class D lying on an extremal ray of $\overline{\mathrm{Eff}}^1(X)$. Lemma 6.9 shows that in this case D is not movable, so by [Nak04, Chapter III.1] D is (after rescaling) the class of an integral divisor on X which we continue to call D. By [BFJ09, Proposition 4.8 and Theorem 4.9], the equation $\langle L_\alpha^{n-1}\rangle \cdot D = 0$ holds if and only if $D \in \mathbb{B}_+(L_\alpha)$. Altogether, we see that α is on the boundary of $\mathrm{Mov}_1(X)$ if and only if L_α is on the boundary of $\mathrm{Mov}^1(X)$. □

Arguing just as in Section 5, we obtain most of the other analytic properties of $\mathfrak{M}$.

Theorem 6.15. *Let X be a smooth projective variety of dimension n. For any movable curve class α with $\mathfrak{M}(\alpha) > 0$, let L_α denote the unique big and movable divisor class satisfying $\langle L_\alpha^{n-1}\rangle = \alpha$. As we vary α in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$, L_α depends continuously on α.*

Theorem 6.16. *Let X be a smooth projective variety of dimension n. For a curve class $\alpha = \langle L_\alpha^{n-1}\rangle$ in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ and for an arbitrary curve class $\beta \in N_1(X)$ we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathfrak{M}(\alpha + t\beta) = \frac{n}{n-1} P_\sigma(L_\alpha) \cdot \beta.$$

Theorem 6.17. *Let X be a smooth projective variety of dimension n. Let α_1, α_2 be two big and movable curve classes in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Then*

$$\mathfrak{M}(\alpha_1 + \alpha_2)^{n-1/n} \geqslant \mathfrak{M}(\alpha_1)^{n-1/n} + \mathfrak{M}(\alpha_2)^{n-1/n}$$

with equality if and only if α_1 and α_2 are proportional.

Another application of the results in this section is the promised Morse-type bigness criterion for movable curve classes, which is slightly different from Theorem 5.18.

Theorem 6.18. *Let X be a smooth projective variety of dimension n. Let α, β be two curve classes lying in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Write $\alpha = \langle L_\alpha^{n-1}\rangle$ and $\beta = \langle L_\beta^{n-1}\rangle$ for the unique big and movable divisor classes L_α, L_β given by Theorem 6.12. Then we have*

$$\begin{aligned}\mathfrak{M}(\alpha - \beta)^{n-1/n} &\geqslant (\mathfrak{M}(\alpha) - nL_\alpha \cdot \beta) \cdot \mathfrak{M}(\alpha)^{-1/n} \\ &= (\mathrm{vol}(L_\alpha) - nL_\alpha \cdot \beta) \cdot \mathrm{vol}(L_\alpha)^{-1/n}.\end{aligned}$$

In particular, we have

$$\mathfrak{M}(\alpha - \beta) \geqslant \mathrm{vol}(L_\alpha) - \frac{n^2}{n-1} L_\alpha \cdot \beta$$

and the curve class $\alpha - \beta$ is big whenever $\mathfrak{M}(\alpha) - nL_\alpha \cdot \beta > 0$.

Proof. By Theorem 4.16 the above inequality follows if we have a Morse-type bigness criterion for the difference of two movable divisor classes. So we need to prove $L - M$ is big whenever

$$\langle L^n \rangle - n\langle L^{n-1} \rangle \cdot M > 0.$$

This is proved (in the Kähler setting) in [Xia14, Theorem 1.1]. $\square$

Remark 6.19. We remark that we can not extend this Morse-type criterion from big and movable divisors to arbitrary pseudo-effective divisor classes. A very simple construction provides the counter examples, e.g. the blow up of $\mathbb{P}^2$ (see [Tra95, Example 3.8]).

Combining Theorem 6.12 and Theorem 6.15, we obtain:

Corollary 6.20. *Let X be a smooth projective variety of dimension n. Then*

$$\Phi : \mathrm{Mov}^1(X)_{\mathrm{vol}} \to \mathrm{Mov}_1(X)_{\mathfrak{M}}, \qquad L \mapsto \langle L^{n-1} \rangle$$

is a homeomorphism.

Remark 6.21. Corollary 6.20 gives a systematic way of translating between "chamber decompositions" on $\mathrm{Mov}_1(X)$ and $\mathrm{Mov}^1(X)$. This relationship could be exploited to elucidate the geometry underlying chamber decompositions.

One potential application is in the study of stability conditions. For example, [Neu10] studies a decomposition of $\mathrm{Mov}_1(X)$ into chambers defining different Harder-Narasimhan filtrations of the tangent bundle of X with respect to movable curves. Let α be a movable curve class. Denote by $\mathrm{HNF}(\alpha, TX)$ the Harder-Narasimhan filtration of the tangent bundle with respect to the class α. Then we have the following "destabilizing chambers":

$$\Delta_\alpha := \{\beta \in \mathrm{Mov}_1(X) |\, \mathrm{HNF}(\beta, TX) = \mathrm{HNF}(\alpha, TX)\}.$$

By [Neu10, Theorem 3.3.4, Proposition 3.3.5], the destabilizing chambers are pairwise disjoint and provide a decomposition of the movable cone $\mathrm{Mov}_1(X)$. Moreover, the decomposition is locally finite in $\mathrm{Mov}_1(X)^\circ$ and the destabilizing chambers are convex cones whose closures are locally polyhedral in $\mathrm{Mov}_1(X)^\circ$. In particular, if $\mathrm{Mov}_1(X)$ is polyhedral, then the chamber structure is finite.

For Fano threefolds, [Neu10] shows that the destabilizing subsheaves are all relative tangent sheaves of some Mori fibration on X. See also [Keb13] for potential applications of this analysis. It would be interesting to study whether the induced filtrations on TX are related to the geometry of the movable divisors L in the Φ-inverse of the corresponding chamber of $\mathrm{Mov}_1(X)$.

Remark 6.22. Modified versions of many of the results in this section hold for singular varieties as well (see Remark 6.8). For example, by similar arguments we can see that any element in the interior of $\mathrm{Mov}_1(X)$ is the positive product of some big divisor class regardless of singularities. Conversely, whenever $\mathfrak{M}$ is $+$-differentiable we obtain a Zariski decomposition structure for vol by Theorem 4.3.

Remark 6.23. All the results above extend to smooth varieties over algebraically closed fields. However, for compact Kähler manifolds some results rely on Demailly's conjecture on the transcendental holomorphic Morse-type inequality, or equivalently, on the extension of the results of [BFJ09] to the Kähler setting. Since the results of [BFJ09] are used in an essential way in the proof of Theorems 6.12 and 6.2 (via the proof of [FL13, Proposition 5.3]), the only statement in this section which extends unconditionally to the Kähler setting is Lemma 6.9.

7 Comparing the complete intersection cone and the movable cone

Consider the functions $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ on the movable cone of curves $\mathrm{Mov}_1(X)$. By their definitions we always have $\widehat{\mathrm{vol}} \geqslant \mathfrak{M}$ on the movable cone, and [Xia15, Remark 3.1] asks whether one can characterize when equality holds. In this section we show:

Theorem 7.1. *Let X be a smooth projective variety of dimension n and let α be a big and movable class. Then $\widehat{\mathrm{vol}}(\alpha) > \mathfrak{M}(\alpha)$ if and only if $\alpha \notin \mathrm{CI}_1(X)$.*

Thus $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ can be used to distinguish whether a big movable curve class lies in $\mathrm{CI}_1(X)$ or not.

Proof. If $\alpha = B^{n-1}$ is a complete intersection class, then $\widehat{\mathrm{vol}}(\alpha) = \mathrm{vol}(B) = \mathfrak{M}(\alpha)$. By continuity the equality holds true for any big curve class in $\mathrm{CI}_1(X)$.

Conversely, suppose that α is not in the complete intersection cone. The claim is clearly true if $\mathfrak{M}(\alpha) = 0$, so by Theorem 6.12 it suffices to consider the case when there is a big and movable divisor class L such that $\alpha = \langle L^{n-1}\rangle$. Note that L can not be big and nef since $\alpha \notin \mathrm{CI}_1(X)$.

We prove $\widehat{\mathrm{vol}}(\alpha) > \mathfrak{M}(\alpha)$ by contradiction. First, by the definition of $\widehat{\mathrm{vol}}$ we always have

$$\widehat{\mathrm{vol}}(\langle L^{n-1}\rangle) \geqslant \mathfrak{M}(\langle L^{n-1}\rangle) = \mathrm{vol}(L).$$

Suppose $\widehat{\mathrm{vol}}(\langle L^{n-1}\rangle) = \mathrm{vol}(L)$. For convenience, we assume $\mathrm{vol}(L) = 1$. By Theorem 5.2.(3), there exists a big and nef divisor class B with $\mathrm{vol}(B) = 1$ computing $\widehat{\mathrm{vol}}(\langle L^{n-1}\rangle)$. For the divisor class B we get

$$\langle L^{n-1}\rangle \cdot B = 1 = \mathrm{vol}(L)^{n-1/n}\,\mathrm{vol}(B)^{1/n}.$$

By Proposition 6.3, this implies L and B are proportional which contradicts the non-nefness of L. Thus we must have $\widehat{\mathrm{vol}}(\langle L^{n-1}\rangle) > \mathrm{vol}(L) = \mathfrak{M}(\langle L^{n-1}\rangle)$. □

Remark 7.2. Alternatively, suppose that $\alpha = \langle L^{n-1}\rangle$ for a big movable divisor L that is not nef. Note that there is a pseudo-effective curve class β satisfying $L \cdot \beta < 0$. As we subtract a small amount of β from α, Theorems 5.11 and 6.16 show that $\widehat{\mathrm{vol}}$ decreases but $\mathfrak{M}$ increases. Since we always have an inequality $\widehat{\mathrm{vol}} \geqslant \mathfrak{M}$ for movable classes, we can not have an equality at α.

We also obtain:

Proposition 7.3. *Let X be a smooth projective variety of dimension n and let α be a big and movable curve class. Assume that $\widehat{\mathrm{vol}}(\phi^*\alpha) = \widehat{\mathrm{vol}}(\alpha)$ for any birational morphism ϕ. Then $\alpha \in \mathrm{CI}_1(X)$.*

Proof. We first consider the case when $\mathfrak{M}(\alpha) > 0$. Let L be a big movable divisor class satisfying $\langle L^{n-1}\rangle = \alpha$. Choose a sequence of birational maps $\phi_\epsilon : Y_\epsilon \to X$ and ample divisor classes A_ϵ on Y_ϵ defining an ϵ-Fujita approximation for L. Then $\mathrm{vol}(L) \geqslant \mathrm{vol}(A_\epsilon) > \mathrm{vol}(L) - \epsilon$ and the classes $\phi_{\epsilon *}A_\epsilon$ limit to L. Note that $A_\epsilon \cdot \phi_\epsilon^*\alpha = \phi_{\epsilon *}A_\epsilon \cdot \alpha$. This implies that for any $\epsilon > 0$ we have

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(\phi_\epsilon^*\alpha) \leqslant \frac{(\alpha \cdot \phi_{\epsilon *}A_\epsilon)^{n/n-1}}{\mathrm{vol}(L)^{1/n-1}}.$$

As ϵ shrinks the right hand side approaches $\mathrm{vol}(L) = \mathfrak{M}(\alpha)$, and we conclude by Theorem 7.1.

Next we consider the case when $\mathfrak{M}(\alpha) > 0$. Choose a class ξ in the interior of $\mathrm{Mov}_1(X)$ and consider the classes $\alpha + \delta\xi$ for $\delta > 0$. The argument above shows that for any $\epsilon > 0$, there is a birational model $\phi_\epsilon : Y_\epsilon \to X$ such that

$$\widehat{\mathrm{vol}}(\phi_\epsilon^*(\alpha + \delta\xi)) < \mathfrak{M}(\alpha + \delta\xi) + \epsilon.$$

But we also have $\widehat{\mathrm{vol}}(\phi_\epsilon^*\alpha) \leqslant \widehat{\mathrm{vol}}(\phi_\epsilon^*(\alpha + \delta\xi))$ since the pullback of the nef curve class $\delta\xi$ is pseudo-effective. Taking limits as $\epsilon \to 0$, $\delta \to 0$, we see that we can make the volume of the pullback of α arbitrarily small, a contradiction to the assumption and the bigness of α. □

Let L be a big divisor class and let $\alpha = \langle L^{n-1}\rangle$ be the corresponding big movable curve class. From the Zariski decomposition $\alpha = B^{n-1} + \gamma$, we get a "canonical" map π from a big divisor class to a big and nef divisor class, that is, $\pi(L) := B$. Note that the map π is continuous and satisfies $\pi^2 = \pi$. It is natural to ask whether we can compare L and B. However, if $P_\sigma(L)$ is not nef then L and B can not be compared:

- if $L \succeq B$ then we have $\mathrm{vol}(L) \geqslant \mathrm{vol}(B)$ which contradicts with Theorem 7.1;
- if $L \preceq B$ then we have $\langle L^{n-1}\rangle \preceq B^{n-1}$ which contradicts with $\gamma \neq 0$.

If we modify the map π a little bit, we can always get a "canonical" nef divisor class lying below the big divisor class.

Theorem 7.4. *Let X be a smooth projective variety of dimension n, and let α be a big movable curve class. Let L be a big divisor class such that $\alpha = \langle L^{n-1}\rangle$, and let $\alpha = B^{n-1} + \gamma$ be the Zariski decomposition of α. Define the map $\widehat{\pi}$ from the cone of big divisor classes to the cone of big and nef divisor classes as*

$$\widehat{\pi}(L) := \left(1 - \left(1 - \frac{\mathfrak{M}(\alpha)}{\widehat{\mathrm{vol}}(\alpha)}\right)^{1/n}\right) B.$$

Then $\widehat{\pi}$ is a surjective continuous map satisfying $L \succeq \widehat{\pi}(L)$ and $\widehat{\pi}^2 = \widehat{\pi}$.

Proof. It is clear if L is nef then we have $\widehat{\pi}(L) = L$, and this implies $\widehat{\pi}$ is surjective and $\widehat{\pi}^2 = \widehat{\pi}$. By Theorem 5.6 and Theorem 6.15, we get the continuity of $\widehat{\pi}$. So we only need to verify $L \succeq \widehat{\pi}(L)$. And this follows from the Diskant inequality for big and movable divisor classes.

Let s be the largest real number such that $L \succeq sB$. By the properties of σ-decompositions, s is also the largest real number such that $P_\sigma(L) \succeq sB$. First, observe that $s \leqslant 1$ since

$$\mathrm{vol}(L) = \mathfrak{M}(\alpha) \leqslant \widehat{\mathrm{vol}}(\alpha) = \mathrm{vol}(B).$$

Applying the Diskant inequality to $P_\sigma(L)$ and B, we have

$$\begin{aligned}(\langle P_\sigma(L)^{n-1}\rangle \cdot B)^{n/n-1} &- \mathrm{vol}(L)\,\mathrm{vol}(B)^{1/n-1} \\ &\geqslant ((\langle P_\sigma(L)^{n-1}\rangle \cdot B)^{1/n-1} - s\,\mathrm{vol}(B)^{1/n-1})^n.\end{aligned}$$

Note that $\widehat{\mathrm{vol}}(\alpha) = (\langle P_\sigma(L)^{n-1}\rangle \cdot \frac{B}{\mathrm{vol}(B)^{1/n}})^{n/n-1}$ and $\mathfrak{M}(\alpha) = \mathrm{vol}(L)$. The above inequality implies

$$s \geqslant 1 - \left(1 - \frac{\mathfrak{M}(\alpha)}{\widehat{\mathrm{vol}}(\alpha)}\right)^{1/n},$$

which yields the desired relation $L \succeq \widehat{\pi}(L)$. □

Example 7.5. Let X be a Mori Dream Space. Recall that a small $\mathbb{Q}$-factorial modification (henceforth SQM) $\phi : X \dashrightarrow X'$ is a birational contraction (i.e. does not extract any divisors) defined in codimension 1 such that X' is projective $\mathbb{Q}$-factorial. [HK00] shows that for any SQM the strict transform defines an isomorphism $\phi_* : N^1(X) \to N^1(X')$ which preserves the pseudo-effective and movable cones of divisors. (More generally, any birational contraction induces an injective pullback $\phi^* : N^1(X') \to N^1(X)$ and dually a surjection $\phi_* : N_1(X) \to N_1(X')$.) The SQM structure induces a chamber decomposition of the pseudo-effective and movable cones of divisors.

One would like to see a "dual picture" in $N_1(X)$ of this chamber decomposition. However, it does not seem interesting to simply dualize the divisor decomposition: the resulting cones are no longer pseudo-effective and are described as intersections instead of unions. Motivated by the Zariski decomposition for curves, we define a chamber structure on the movable cone of curves as a union of the complete intersection cones on SQMs.

Note that for each SQM we obtain by duality an isomorphism $\phi_* : N_1(X) \to N_1(X')$ which preserves the movable cone of curves. We claim that the strict transforms of the various complete intersection cones define a chamber structure on $\mathrm{Mov}_1(X)$. More precisely, given any birational contraction $\phi : X \dashrightarrow X'$ with X' normal projective, define

$$\mathrm{CI}^\circ_\phi := \bigcup_{A \text{ ample on } X'} \langle \phi^* A^{n-1}\rangle.$$

Then

- $\mathrm{Mov}_1(X)$ is the union over all SQMs $\phi : X \dashrightarrow X'$ of $\overline{\mathrm{CI}^\circ_\phi} = \phi_*^{-1}\,\mathrm{CI}_1(X')$, and the interiors of the $\overline{\mathrm{CI}^\circ_\phi}$ are disjoint.

- The set of classes in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ is the disjoint union over all birational contractions $\phi : X \dashrightarrow X'$ of the CI°_ϕ.

To see this, first recall that for a pseudo-effective divisor L the σ-decomposition of L and the volume are preserved by ϕ_*. We know that each $\alpha \in \mathrm{Mov}_1(X)_{\mathfrak{M}}$ has the form $\langle L^{n-1}\rangle$ for a unique big and movable divisor L. If $\phi : X \dashrightarrow X'$ denotes the birational canonical model obtained by running the L-MMP, and A denotes the corresponding ample divisor on X', then $\phi_*\alpha = A^{n-1}$ and $\alpha = \langle \phi^* A^{n-1}\rangle$. The various claims now can be deduced from the properties of divisors and the MMP for Mori Dream Spaces as in [HK00, 1.11 Proposition].

Since the volume of divisors behaves compatibly with strict transforms of pseudo-effective divisors, the description of ϕ_* above shows that $\mathfrak{M}$ also behaves compatibly with strict transforms of movable curves under an SQM. However, the volume function can change: we may well have $\widehat{\mathrm{vol}}(\phi_*\alpha) \neq \widehat{\mathrm{vol}}(\alpha)$. The reason is that the pseudo-effective cone of curves is also changing as we vary ϕ. In particular, the set

$$C_{\alpha,\phi} := \{\phi_*\alpha - \gamma | \gamma \in \overline{\mathrm{Eff}}_1(X')\}$$

will look different as we vary ϕ. Since $\widehat{\mathrm{vol}}$ is the same as the maximum value of $\mathfrak{M}(\beta)$ for $\beta \in C_{\alpha,\phi}$, the volume and Zariski decomposition for a given model will depend on the exact shape of $C_{\alpha,\phi}$.

Remark 7.6. Theorem 7.1 and Theorem 7.4 also hold for smooth varieties over any algebraically closed field. However, since they rely on the results of Section 6 we do not know if they hold in the Kähler setting.

8 Toric varieties

In this section X will denote a simplicial projective toric variety of dimension n. In terms of notation, X will be defined by a fan Σ in a lattice N with dual lattice M. We let $\{v_i\}$ denote the primitive generators of the rays of Σ and $\{D_i\}$ denote the corresponding classes of T-divisors. Our goal is to interpret the properties of the functions $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ in terms of toric geometry.

8.1 Positive product on toric varieties

Suppose that L is a big movable divisor class on the toric variety X. Then L naturally defines a (non-lattice) polytope Q_L: if we choose an expression $L = \sum a_i D_i$, then

$$Q_L = \{u \in M_{\mathbb{R}} | \langle u, v_i\rangle + a_i \geqslant 0\}$$

and changing the choice of representative corresponds to a translation of Q_L. Conversely, suppose that Q is a full-dimensional polytope such that the unit normals to the facets of Q form a subset of the rays of Σ. Then Q uniquely determines a big movable divisor class L_Q on X. The divisors in the interior of the movable cone correspond to those polytopes whose facet normals coincide with the rays of Σ.

Given polytopes $Q_1, \ldots, Q_n$, let $V(Q_1, \ldots, Q_n)$ denote the mixed volume of the polytopes. [BFJ09] explains that the positive product of big movable divisors $L_1, \ldots, L_n$ can be interpreted via the mixed volume of the corresponding polytopes:

$$\langle L_1 \cdot \ldots \cdot L_n \rangle = n! V(Q_1, \ldots, Q_n).$$

8.2 The function $\mathfrak{M}$

In this section we use a theorem of Minkowski to describe the function $\mathfrak{M}$. We thank J. Huh for a conversation working out this picture.

Recall that a class $\alpha \in \mathrm{Mov}_1(X)$ defines a non-negative Minkowski weight on the rays of the fan Σ – that is, an assignment of a positive real number t_i to each vector v_i such that $\sum t_i v_i = 0$. From now on we will identify α with its Minkowski weight. We will need to identify which movable curve classes are positive along a set of rays which span $\mathbb{R}^n$.

Lemma 8.1. *Suppose $\alpha \in \mathrm{Mov}_1(X)$ satisfies $\mathfrak{M}(\alpha) > 0$. Then α is positive along a spanning set of rays of Σ.*

We will soon see that the converse is also true in Theorem 8.2.

Proof. Suppose that there is a hyperplane V which contains every ray of Σ along which α is positive. Since X is projective, Σ has rays on both sides of V. Let D be the effective toric divisor consisting of the sum over all the primitive generators of rays of Σ not contained in V. It is clear that the polytope defined by D has non-zero projection onto the subspace spanned by $V^{\perp}$, and in particular, that the polytope defined by D is non-zero. Thus $P_{\sigma}(D) \neq 0$ and so α has vanishing intersection against a non-zero movable divisor. Lemma 6.9 shows that $\mathfrak{M}(\alpha) = 0$. $\square$

Minkowski's theorem asserts the following. Suppose that $u_1, \ldots, u_s$ are unit vectors which span $\mathbb{R}^n$ and that $r_1, \ldots, r_s$ are positive real numbers. Then there exists a polytope P with unit normals $u_1, \ldots, u_s$ and with corresponding facet volumes $r_1, \ldots, r_s$ if and only if the u_i satisfy the balanced condition

$$r_1 u_1 + \ldots + r_s u_s = 0.$$

Moreover, the resulting polytope is unique up to translation. (See [Kla04] for a proof which is compatible with the results below.) If a vector u is a unit normal to a facet of P, we will use the notation $\mathrm{vol}(P^u)$ to denote the volume of the facet corresponding to u.

If α is positive on a spanning set of rays, then it canonically defines a polytope (up to translation) via Minkowski's theorem by choosing the vectors u_i to be the unit vectors in the directions v_i and assigning to each the constant

$$r_i = \frac{t_i |v_i|}{(n-1)!}.$$

Note that this is the natural choice of volume for the corresponding facet, as it accounts for:

- the discrepancy in length between u_i and v_i, and

- the factor $\frac{1}{(n-1)!}$ relating the volume of an $(n-1)$-simplex to the determinant of its edge vectors.

We denote the corresponding polytope in $M_{\mathbb{R}}$ defined by the theorem of Minkowski by P_α.

Theorem 8.2. *Suppose α is a movable curve class which is positive on a spanning set of rays and let P_α be the corresponding polytope. Then*

$$\mathfrak{M}(\alpha) = n! \operatorname{vol}(P_\alpha).$$

Furthermore, the big movable divisor L_α corresponding to the polytope P_α satisfies $\langle L_\alpha^{n-1} \rangle = \alpha$.

Proof. Let $L \in \operatorname{Mov}^1(X)$ be a big movable divisor class and denote the corresponding polytope by Q_L. We claim that the intersection number can be interpreted as a mixed volume:

$$L \cdot \alpha = n! V(P_\alpha^{n-1}, Q_L).$$

To see this, define for a compact convex set K the function $h_K(u) = \sup_{v \in K}\{v \cdot u\}$. Using [Kla04, Equation (5)]

$$\begin{aligned} V(P_\alpha^{n-1}, Q_L) &= \frac{1}{n} \sum_{u \text{ a facet of } P_\alpha + Q_L} h_{Q_L}(u) \operatorname{vol}(P_\alpha^u) \\ &= \frac{1}{n} \sum_{\text{rays } v_i} \left(\frac{a_i}{|v_i|} \right) \left(\frac{t_i |v_i|}{(n-1)!} \right) \\ &= \frac{1}{n!} \sum_{\text{rays } v_i} a_i t_i = \frac{1}{n!} L \cdot \alpha. \end{aligned}$$

Note that we actually have equality in the second line because L is big and movable. Recall that by the Brunn-Minkowski inequality

$$V(P_\alpha^{n-1}, Q_L) \geqslant \operatorname{vol}(P_\alpha)^{n-1/n} \operatorname{vol}(Q_L)^{1/n}$$

with equality only when P_α and Q_L are homothetic. Thus

$$\begin{aligned} \mathfrak{M}(\alpha) &= \inf_{L \text{ big movable class}} \left(\frac{L \cdot \alpha}{\operatorname{vol}(L)^{1/n}} \right)^{n/n-1} \\ &= \inf_{L \text{ big movable class}} \left(\frac{n! V(P_\alpha^{n-1}, Q_L)}{n!^{1/n} \operatorname{vol}(Q_L)^{1/n}} \right)^{n/n-1} \\ &\geqslant n! \operatorname{vol}(P_\alpha). \end{aligned}$$

Furthermore, the equality is achieved for divisors L whose polytope is homothetic to P_α, showing the computation of $\mathfrak{M}(\alpha)$. Furthermore, since the divisor L_α defined by the polytope computes $\mathfrak{M}(\alpha)$ we see that $\langle L_\alpha^{n-1} \rangle$ is proportional to α. By computing $\mathfrak{M}$ we deduce the equality:

$$\mathfrak{M}(\langle L_\alpha^{n-1} \rangle) = \operatorname{vol}(L) = n! \operatorname{vol}(P_\alpha) = \mathfrak{M}(\alpha).$$

□

8.3 Zariski decompositions

The work of the previous section shows:

Corollary 8.3. *Let α be a curve class in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Then $\alpha \in \mathrm{CI}_1(X)$ if and only if the normal fan to the corresponding polytope P_α is refined by Σ. In this case we have*

$$\widehat{\mathrm{vol}}(\alpha) = n! \,\mathrm{vol}(P_\alpha).$$

Proof. By the uniqueness in Theorem 6.12, $\alpha \in \mathrm{CI}_1(X)$ if and only if the corresponding divisor L_α as in Theorem 8.2 is big and nef. ☐

The nef cone of divisors and pseudo-effective cone of curves on X can be computed algorithmically. Thus, for any face F of the nef cone, by considering the $(n-1)$-product and adding on any curve classes in the dual face, one can easily divide $\overline{\mathrm{Eff}}_1(X)$ into regions where the positive product is determined by a class on F. In practice this is a good way to compute the Zariski decomposition (and hence the volume) of curve classes on X.

In the other direction, suppose we start with a big curve class α. On a toric variety, every big and nef divisor is semi-ample (that is, the pullback of an ample divisor on a toric birational model). Thus, the Zariski decomposition is characterized by the existence of a birational toric morphism $\pi : X \to X'$ such that:

- the class $\pi_*\alpha \in N_1(X')$ coincides with A^{n-1} for some ample divisor A, and
- $\alpha - (\pi^*A)^{n-1}$ is pseudo-effective.

Thus one can compute the Zariski decomposition and volume for α by the following procedure.

1. For each toric birational morphism $\pi : X \to X'$, check whether $\pi_*\alpha$ is in the complete intersection cone. If so, there is a unique big and nef divisor $A_{X'}$ such that $A_{X'}^{n-1} = \pi_*\alpha$.
2. Check if $\alpha - (\pi^*A_{X'})^{n-1}$ is pseudo-effective.

The first step involves solving polynomial equations to deduce the equality of coefficients of numerical classes, but otherwise this procedure is completely algorithmic. Thus this procedure can be viewed as a solution to our isoperimetric problem. (Note that there may be no natural pullback from $\overline{\mathrm{Eff}}_1(X')$ to $\overline{\mathrm{Eff}}_1(X)$, and in particular, the calculation of $(\pi^*A_{X'})^{n-1}$ is not linear in $A_{X'}^{n-1}$.)

Example 8.4. Let X be the toric variety defined by a fan in $N = \mathbb{Z}^3$ on the rays

$$\begin{array}{lll} v_1 = (1,0,0) & v_2 = (0,1,0) & v_3 = (1,1,1) \\ v_4 = (-1,0,0) & v_5 = (0,-1,0) & v_6 = (0,0,-1) \end{array}$$

with maximal cones

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_2, v_6\rangle, \langle v_1, v_3, v_5\rangle, \langle v_1, v_5, v_6\rangle,$$
$$\langle v_2, v_3, v_4\rangle, \langle v_2, v_4, v_6\rangle, \langle v_3, v_4, v_5\rangle, \langle v_4, v_5, v_6\rangle.$$

The Picard rank of X is 3. Letting D_i and C_{ij} be the divisors and curves corresponding to v_i and $\overline{v_i v_j}$ respectively, we have intersection product

	D_1	D_2	D_3
C_{12}	-1	-1	1
C_{13}	0	1	0
C_{23}	1	0	0

Standard toric computations show that:

$$\overline{\mathrm{Eff}}^1(X) = \langle D_1, D_2, D_3 \rangle \qquad \mathrm{Nef}^1(X) = \langle D_1 + D_3, D_2 + D_3, D_3 \rangle$$
$$\mathrm{Mov}^1(X) = \langle D_1 + D_2, D_1 + D_3, D_2 + D_3, D_3 \rangle$$

and

$$\overline{\mathrm{Eff}}_1(X) = \langle C_{12}, C_{13}, C_{23} \rangle \qquad \mathrm{Nef}_1(X) = \langle C_{12} + C_{13} + C_{23}, C_{13}, C_{23} \rangle.$$

X admits a unique flip and has only one birational contraction corresponding to the face of $\mathrm{Nef}^1(X)$ generated by $D_1 + D_3$ and $D_2 + D_3$. Set $B_{a,b} = aD_1 + bD_2 + (a+b)D_3$. The complete intersection cone is given by taking the convex hull of the boundary classes

$$B_{a,b}^2 = T_{a,b} = 2abC_{12} + (a^2 + 2ab)C_{13} + (b^2 + 2ab)C_{23}$$

and the face of $\mathrm{Nef}_1(X)$ spanned by C_{13}, C_{23}.

For any big class α not in $\mathrm{CI}_1(X)$, the positive part can be computed on the unique toric birational contraction $\pi : X \to X'$ given by contracting C_{12}. In practice, the procedure above amounts to solving $\alpha - tC_{12} = T_{a,b}$ for some a, b, t. If $\alpha = xC_{12} + yC_{13} + zC_{23}$, this yields the quadratic equation $4(y - x + t)(z - x + t) = (x - t)^2$. Solving this for t tells us $\gamma = tC_{12}$, and the volume can then easily be computed.

9 Hyperkähler manifolds

Throughout this section X will denote a hyperkähler variety of dimension n (with $n = 2m$). We will continue to work in the projective setting. However, as explained in Section 2.4, Demailly's conjecture on transcendental Morse inequality is known for hyperkähler manifolds. Thus all of the material in the previous sections will hold in the Kähler setting for hyperkähler varieties with no qualifications, and all the results in this section can extended accordingly.

Let σ be a symplectic holomorphic form on X. For a real divisor class $D \in N^1(X)$ the Beauville-Bogomolov quadratic form is defined as

$$q(D) = D^2 \cdot \{(\sigma \wedge \bar{\sigma})\}^{n/2-1},$$

where we normalize the symplectic form σ such that

$$q(D)^{n/2} = D^n.$$

As proved in [Bou04, Section 4], the bilinear form q is compatible with the volume function and σ-decomposition for divisors in the following way:

1. The cone of movable divisors is q-dual to the pseudo-effective cone.

2. If D is a movable divisor then $\mathrm{vol}(D) = q(D,D)^{n/2} = D^n$.

3. For a pseudo-effective divisor D write $D = P_\sigma(D) + N_\sigma(D)$ for its σ-decomposition. Then $q(P_\sigma(D), N_\sigma(D)) = 0$, and if $N_\sigma(D) \neq 0$ then $q(N_\sigma(D), N_\sigma(D)) < 0$.

The bilinear form q induces an isomorphism $\psi : N^1(X) \to N_1(X)$ by sending a divisor class D to the curve class defining the linear function $q(D, -)$. We obtain an induced bilinear form q on $N_1(X)$ via the isomorphism ψ, so that for curve classes α, β

$$q(\alpha, \beta) = q(\psi^{-1}\alpha, \psi^{-1}\beta) = \psi^{-1}\alpha \cdot \beta.$$

In particular, two cones $\mathcal{C}, \mathcal{C}'$ in $N^1(X)$ are q-dual if and only if $\psi(\mathcal{C})$ is dual to $\mathcal{C}'$ under the intersection pairing (and similarly for cones of curves). In this section we verify that the bilinear form q on $N_1(X)$ is compatible with the volume and Zariski decomposition for curve classes in the same way as for divisors.

Remark 9.1. Since the signature of the Beauville-Bogomolov form is $(1, \dim N^1(X) - 1)$, one can use the Hodge inequality to analyze the Zariski decomposition as in Example 4.7. We will instead give a direct geometric argument to emphasize the ties with the divisor theory.

We first need the following proposition.

Proposition 9.2. *Let D be a big movable divisor class on X. Then $\mathfrak{M}(\psi(D)) = \mathrm{vol}(D)^{1/n-1}$ and*

$$\psi(D) = \frac{\langle D^{n-1} \rangle}{\mathrm{vol}(D)^{n-2/n}}.$$

In particular, the complete intersection cone coincides with the ψ-image of the nef cone of divisors and if A is a big and nef divisor then $\widehat{\mathrm{vol}}(\psi(A)) = \mathrm{vol}(A)^{1/n-1}$.

Proof. First note that $\psi(D)$ is contained in $\mathrm{Mov}_1(X)$. Indeed, since the movable cone of divisors is q-dual to the pseudo-effective cone of divisors by [Bou04, Proposition 4.4], the ψ-image of the movable cone of divisors is dual to the pseudo-effective cone of divisors.

For any big movable divisor L, the basic equality for bilinear forms shows that

$$L \cdot \psi(D) = q(L, D) = \frac{1}{2}(\mathrm{vol}(L + D)^{2/n} - \mathrm{vol}(L)^{2/n} - \mathrm{vol}(D)^{2/n}).$$

Proposition 6.7 shows that $\mathrm{vol}(L + D)^{1/n} \geqslant \mathrm{vol}(L)^{1/n} + \mathrm{vol}(D)^{1/n}$ with equality if and only if L and D are proportional. Squaring and rearranging, we see that

$$\frac{L \cdot \psi(D)}{\mathrm{vol}(L)^{1/n}} \geqslant \mathrm{vol}(D)^{1/n}$$

with equality if and only if L is proportional to D. Thus $\mathfrak{M}(\psi(D)) = \mathrm{vol}(D)^{1/n-1}$ and this quantity is computed by the movable and big divisor D. This implies that

$$\psi(D) = \frac{\langle D^{n-1} \rangle}{\mathrm{vol}(D)^{n-2/n}}$$

by Theorem 6.12. The final statements follow immediately. □

Theorem 9.3. *Let q denote the Beauville-Bogomolov form on $N_1(X)$. Then:*

1. *The complete intersection cone of curves is q-dual to the pseudo-effective cone of curves.*
2. *If α is a complete intersection curve class then $\widehat{\mathrm{vol}}(\alpha) = q(\alpha, \alpha)^{n/2(n-1)}$.*
3. *For a big class α write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. Then $q(B^{n-1}, \gamma) = 0$ and if γ is non-zero then $q(\gamma, \gamma) < 0$.*

Proof. For (1), since the complete intersection cone coincides with $\psi(\mathrm{Nef}^1(X))$ it is q-dual to the dual cone of $\mathrm{Nef}^1(X)$. For (2), by Proposition 9.2 we have

$$\begin{aligned} q(\psi(A), \psi(A)) &= q(A, A) = \mathrm{vol}(A)^{2/n} \\ &= \widehat{\mathrm{vol}}(\psi(A))^{2(n-1)/n}. \end{aligned}$$

For (3), we have

$$q(B^{n-1}, \gamma) = \psi^{-1}(B^{n-1}) \cdot \gamma = \mathrm{vol}(B)^{n-2/n} B \cdot \gamma = 0.$$

For the final statement $q(\gamma, \gamma) < 0$, note that

$$q(\alpha, \alpha) = q(B^{n-1}, B^{n-1}) + q(\gamma, \gamma)$$

so it suffices to show that $q(\alpha, \alpha) < q(B^{n-1}, B^{n-1})$. Set $D = \psi^{-1}\alpha$. The desired inequality is clear if $q(D, D) \leqslant 0$, so by [Huy99, Corollary 3.10 and Erratum Proposition 1] it suffices to restrict our attention to the case when D is big. (Note that the case when $-D$ is big can not occur, since $q(D, A) = A \cdot \alpha > 0$ for an ample divisor class A.) Let $D = P_\sigma(D) + N_\sigma(D)$ be the σ-decomposition of D. By [Bou04, Proposition 4.2] we have $q(N_\sigma(D), B) \geqslant 0$. Thus

$$\begin{aligned} \mathrm{vol}(B)^{2(n-1)/n} &= q(B^{n-1}, B^{n-1}) = q(\alpha, B^{n-1}) \\ &= \mathrm{vol}(B)^{n-2/n} q(D, B) \geqslant \mathrm{vol}(B)^{n-2/n} q(P_\sigma(D), B). \end{aligned}$$

Arguing just as in the proof of Proposition 9.2, we see that

$$q(P_\sigma(D), B) \geqslant \mathrm{vol}(P_\sigma(D))^{1/n} \mathrm{vol}(B)^{1/n}$$

with equality if and only if $P_\sigma(D)$ and B are proportional. Combining the two previous equations we obtain

$$\mathrm{vol}(B)^{n-1/n} \geqslant \mathrm{vol}(P_\sigma(D))^{1/n}.$$

and equality is only possible if B and $P_\sigma(D)$ are proportional. Then we calculate:

$$\begin{aligned} q(\alpha, \alpha) &= q(D, D) \\ &\leqslant q(P_\sigma(D), P_\sigma(D)) \text{ by [Bou04, Theorem 4.5]} \\ &= \mathrm{vol}(P_\sigma(D))^{2/n} \\ &\leqslant \mathrm{vol}(B)^{2(n-1)/n} = q(B, B). \end{aligned}$$

If $P_\sigma(D)$ and B are not proportional, we obtain a strict inequality at the last step. If $P_\sigma(D)$ and B are proportional, then $N_\sigma(D) > 0$ (since otherwise $D = B$ and α is a complete intersection class). Then by [Bou04, Theorem 4.5] we have a strict inequality $q(P_\sigma(D), P_\sigma(D)) > q(D, D)$ on the second line. In either case we conclude $q(\alpha, \alpha) < q(B, B)$ as desired.

□

Remark 9.4. Suppose that α is a nef curve class that is not in the complete intersection cone. Then $q(\alpha,\alpha) = \mathfrak{M}(\alpha)^{2(n-1)/n}$ and $q(B^{n-1},B^{n-1}) = \widehat{\mathrm{vol}}(\alpha)^{2(n-1)/n}$. Theorem 7.1 shows that $q(\alpha,\alpha) < q(B^{n-1},B^{n-1})$, giving another proof of the final statement of Theorem 9.3.(3) in this special case.

10 Comparison with mobility

In this section we compare the volume function with the mobility function. Recall from the introduction that we are trying to show:

Theorem 10.1. *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then:*

1. $\widehat{\mathrm{vol}}(\alpha) \leqslant \mathrm{mob}(\alpha) \leqslant n!\widehat{\mathrm{vol}}(\alpha)$.
2. *Assume Conjecture 1.12, then* $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$.

The upper bound improves the related result [Xia15, Theorem 3.2].

Proof. (1) We first prove the upper bound. By continuity and homogeneity it suffices to prove the upper bound for a class α in the natural sublattice of integral classes $N_1(X)_{\mathbb{Z}}$. Suppose that $p : U \to W$ is a family of curves representing $m\alpha$ of maximal mobility count for a positive integer m. Suppose that a general member of p decomposes into irreducible components $\{C_i\}$; arguing as in [Leh13b, Corollary 4.10], we must have $\mathrm{mc}(p) = \sum_i \mathrm{mc}(U_i)$, where U_i represents the closure of the family of deformations of C_i. We also let β_i denote the numerical class of C_i.

Suppose that $\mathrm{mc}(U_i) > 1$. Then we may apply Proposition 11.1 with all $k_i = 1$ and $r = \mathrm{mc}(U_i) - 1$ to deduce that

$$\widehat{\mathrm{vol}}(\beta_i) \geqslant \mathrm{mc}(U_i) - 1.$$

If $\mathrm{mc}(U_i) \leqslant 1$ then Proposition 11.1 does not apply but at least we still know that $\widehat{\mathrm{vol}}(\beta_i) \geqslant 0 \geqslant \mathrm{mc}(U_i) - 1$. Fix an ample Cartier divisor A, and note that the number of components C_i is at most $mA \cdot \alpha$. All told, we have

$$\begin{aligned} \widehat{\mathrm{vol}}(m\alpha) &\geqslant \sum_i \widehat{\mathrm{vol}}(\beta_i) \\ &\geqslant \sum_i (\mathrm{mc}(U_i) - 1) \\ &\geqslant \mathrm{mc}(m\alpha) - mA \cdot \alpha. \end{aligned}$$

Thus,

$$\begin{aligned} \widehat{\mathrm{vol}}(\alpha) &= \limsup_{m\to\infty} \frac{\widehat{\mathrm{vol}}(m\alpha)}{m^{n/n-1}} \\ &\geqslant \limsup_{m\to\infty} \frac{\mathrm{mc}(m\alpha) - mA\cdot\alpha}{m^{n/n-1}} = \frac{\mathrm{mob}(\alpha)}{n!}. \end{aligned}$$

The lower bound relies on the Zariski decomposition of curves in Theorem 5.4. By [Leh13b, Theorem 6.11 and Example 6.2] we have

$$B^n \leqslant \mathrm{mob}(B^{n-1})$$

for any nef divisor B. With Theorem 5.2, this implies

$$\widehat{\mathrm{vol}}(B^{n-1}) \leqslant \mathrm{mob}(B^{n-1}).$$

In general, for a big curve class α we have

$$\begin{aligned}
\mathrm{mob}(\alpha) &\geqslant \sup_{B \text{ nef},\ \alpha \succeq B^{n-1}} \mathrm{mob}(B^{n-1}) \\
&\geqslant \sup_{B \text{ nef},\ \alpha \succeq B^{n-1}} B^n \\
&= \widehat{\mathrm{vol}}(\alpha).
\end{aligned}$$

where the last equality follows from Theorem 5.4. This finishes the proof of the first statement.

(2) To prove the second half of Theorem 10.1, we need a result of [FL13]:

Lemma 10.2 (see [FL13] Corollary 6.16)**.** *Let X be a smooth projective variety of dimension n and let α be a big curve class. Then there is a big movable curve class β satisfying $\beta \preceq \alpha$ such that*

$$\mathrm{mob}(\alpha) = \mathrm{mob}(\beta) = \mathrm{mob}(\phi^* \beta)$$

for any birational map $\phi : Y \to X$ from a smooth variety Y.

We now prove the statement via a sequence of claims.

Claim: Assume Conjecture 1.12. If β is a movable curve class with $\mathfrak{M}(\beta) > 0$, then for any $\epsilon > 0$ there is a birational map $\phi_\epsilon : Y_\epsilon \to X$ such that

$$\mathfrak{M}(\beta) - \epsilon \leqslant \mathrm{mob}(\phi_\epsilon^* \beta) \leqslant \mathfrak{M}(\beta) + \epsilon.$$

□

By Theorem 6.12, we may suppose that there is a big divisor L such that $\beta = \langle L^{n-1} \rangle$. Without loss of generality we may assume that L is effective. Fix an ample effective divisor G as in [FL13, Proposition 6.24]; the proposition shows that for any sufficiently small ϵ there is a birational morphism $\phi_\epsilon : Y_\epsilon \to X$ and a big and nef divisor A_ϵ on Y_ϵ satisfying

$$A_\epsilon \leqslant P_\sigma(\phi_\epsilon^* L) \leqslant A_\epsilon + \epsilon \phi_\epsilon^* G.$$

Note that $\mathrm{vol}(A_\epsilon) \leqslant \mathrm{vol}(L) \leqslant \mathrm{vol}(A_\epsilon + \epsilon \phi_\epsilon^* G)$. Furthermore, we have

$$\mathrm{vol}(A_\epsilon + \epsilon \phi_\epsilon^* G) \leqslant \mathrm{vol}(\phi_{\epsilon *} A_\epsilon + \epsilon G) \leqslant \mathrm{vol}(L + \epsilon G).$$

Applying [FL13, Lemma 6.21] and the invariance of the positive product under passing to positive parts, we have

$$A_\epsilon^{n-1} \preceq \phi_\epsilon^* \beta \preceq (A_\epsilon + \epsilon \phi_\epsilon^* G)^{n-1}.$$

Applying Conjecture 1.12 (which is only stated for ample divisors but applies to big and nef divisors by continuity of mob), we find

$$\mathrm{vol}(A_\epsilon) = \mathrm{mob}(A_\epsilon^{n-1}) \leqslant \mathrm{mob}(\phi_\epsilon^*\beta) \leqslant \mathrm{mob}((A_\epsilon + \epsilon\phi_\epsilon^*(G))^{n-1}) = \mathrm{vol}(A_\epsilon + \epsilon\phi_\epsilon^*G).$$

As ϵ shrinks the two outer terms approach $\mathrm{vol}(L) = \mathfrak{M}(\beta)$.

Claim: Assume Conjecture 1.12. If a big movable curve class β satisfies $\mathrm{mob}(\beta) = \mathrm{mob}(\phi^*\beta)$ for every birational ϕ then we must have $\beta \in \mathrm{CI}_1(X)$. $\square$

When $\mathfrak{M}(\beta) > 0$, by the previous claim we see from taking a limit that $\mathrm{mob}(\beta) = \mathfrak{M}(\beta)$. By Theorem 10.1.(1) and Theorem 7.1 we get

$$\widehat{\mathrm{vol}}(\beta) \leqslant \mathfrak{M}(\beta) \leqslant \widehat{\mathrm{vol}}(\beta)$$

and Theorem 7.1 implies the result. When $\mathfrak{M}(\beta) = 0$, fix a class ξ in the interior of the movable cone and consider $\beta + \delta\xi$ for $\delta > 0$. By the previous claim, for any $\epsilon > 0$ we can find a sufficiently small δ and a birational map $\phi_\epsilon : Y_\epsilon \to X$ such that $\mathrm{mob}(\phi_\epsilon^*(\beta + \delta\xi)) < \epsilon$. We also have $\mathrm{mob}(\phi_\epsilon^*\beta) \leqslant \mathrm{mob}(\phi_\epsilon^*(\beta + \delta\xi))$ since the pullback of the nef curve class $\delta\xi$ is pseudo-effective. By the assumption on the birational invariance of $\mathrm{mob}(\beta)$, we can take a limit to obtain $\mathrm{mob}(\beta) = 0$, a contradiction to the bigness of β.

To finish the proof, recall that Lemma 10.2 implies that the mobility of α must coincide with the mobility of a movable class β lying below α and satisfying $\mathrm{mob}(\pi^*\beta) = \mathrm{mob}(\beta)$ for any birational map π. Thus we have shown

$$\mathrm{mob}(\alpha) = \sup_{B \text{ nef}, \, \alpha \succeq B^{n-1}} \mathrm{mob}(B^{n-1}).$$

By Conjecture 1.12 again, we obtain

$$\mathrm{mob}(\alpha) = \sup_{B \text{ nef}, \, \alpha \succeq B^{n-1}} B^n.$$

But the right hand side agrees with $\widehat{\mathrm{vol}}(\alpha)$ by Theorem 5.4. This proves the equality $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$ under the Conjecture 1.12. $\square$

Theorem 10.1 yields two interesting consequences:

- The theorem indicates (loosely speaking) that if the mobility count of complete intersection classes is optimized by complete intersection curves, then the mobility count of *any* curve class is optimized by complete intersection curves lying below the class. This result is very surprising: it indicates that the "positivity" of a curve class is coming from ample divisors in a strong sense.
- The theorem suggests that the Zariski decomposition constructed in [FL13] for curves is not optimal: instead of defining a positive part in the movable cone, if Conjecture 1.12 is true we should instead define a positive part in the complete intersection cone. It would be interesting to see an analogous improvement for higher dimension cycles.

Remark 10.3. We expect Theorem 10.1 to also hold over any algebraically closed field, but we have not thoroughly checked the results on asymptotic multiplier ideals used in the proof of [FL13, Proposition 6.24].

10.1 Weighted mobility

The weighted mobility of a class α is defined similarly to the mobility, but it gives a higher "weight" to singular points. This better reflects the intersection theory on the blow-up of the points and indicates the close connection between the weighted mobility and Seshadri constants. We first define the weighted mobility count of a class $\alpha \in N_1(X)_{\mathbb{Z}}$ (see [Leh13b, Definition 8.7]):

$$\mathrm{wmc}(\alpha) = \sup_{\mu} \max \left\{ b \in \mathbb{Z}_{\geqslant 0} \;\middle|\; \begin{array}{c} \text{there is an effective cycle of class } \mu\alpha \\ \text{through any } b \text{ points of } X \text{ with} \\ \text{multiplicity at least } \mu \text{ at each point} \end{array} \right\}.$$

The supremum is shown to exist in [Leh13b] – it is then clear that the supremum is achieved by some positive integer μ. We define the weighted mobility to be

$$\mathrm{wmob}(\alpha) = \limsup_{m \to \infty} \frac{\mathrm{wmc}(m\alpha)}{m^{\frac{n}{n-k}}}.$$

Note that we no longer need the correction factor of $n!$. [Leh13b] shows that the weighted mobility is continuous and homogeneous on $\overline{\mathrm{Eff}}_1(X)$ and is 0 precisely along the boundary.

Theorem 10.4. *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then $\widehat{\mathrm{vol}}(\alpha) = \mathrm{wmob}(\alpha)$.*

The key advantage is that the analogue of Conjecture 1.12 is known for the weighted mobility: [Leh13b, Example 8.22] shows that for any big and nef divisor B we have $\mathrm{wmob}(B^{n-1}) = B^n$.

Proof. We first prove the inequality $\geqslant$. The argument is essentially identical to the upper bound in Theorem 10.1.(1): by continuity and homogeneity it suffices to prove it for classes in $N_1(X)_{\mathbb{Z}}$. Choose a positive integer μ and a family of class $\mu m\alpha$ achieving $\mathrm{wmc}(m\alpha)$. By splitting up into components and applying Proposition 11.1 with equal weight μ at every point we see that for any component U_i with class β_i we have

$$\widehat{\mathrm{vol}}(\beta_i) \geqslant \mu^{n/n-1}(\mathrm{wmc}(U_i) - 1)$$

Arguing as in Theorem 10.1.(1), we see that for any fixed ample Cartier divisor A we have

$$\widehat{\mathrm{vol}}(m\mu\alpha) \geqslant \mu^{n/n-1}(\mathrm{wmc}(m\alpha) - mA \cdot \alpha).$$

Rescaling by μ and taking a limit proves the statement.

We next prove the inequality $\leqslant$. Again, the argument is identical to the lower bound in Theorem 10.1.(1). It is clear that the weighted mobility can only increase upon adding an effective class. Using continuity and homogeneity, the same is true for any pseudo-effective class. Thus we have

$$\begin{aligned}\mathrm{wmob}(\alpha) &\geqslant \sup_{B \text{ nef}, \, \alpha \succeq B^{n-1}} \mathrm{wmob}(B^{n-1}) \\ &= \sup_{B \text{ nef}, \, \alpha \succeq B^{n-1}} B^n \\ &= \widehat{\mathrm{vol}}(\alpha).\end{aligned}$$

where the second equality follows from [Leh13b, Example 8.22]. □

11 Applications to birational geometry

We end with a discussion of several connections between positivity of curves and other constructions in birational geometry. There is a large body of literature relating the positivity of a divisor at a point to its intersections against curves through that point. One can profitably reinterpret these relationships in terms of the volume of curve classes. A key result conceptually is:

Proposition 11.1. *Let X be a smooth projective variety of dimension n. Choose positive integers $\{k_i\}_{i=1}^r$. Suppose that $\alpha \in \mathrm{Mov}_1(X)$ is represented by a family of irreducible curves such that for any collection of general points $x_1, x_2, \ldots, x_r, y$ of X, there is a curve in our family which contains y and contains each x_i with multiplicity $\geqslant k_i$. Then*

$$\widehat{\mathrm{vol}}(\alpha)^{\frac{n-1}{n}} \geqslant \frac{\sum_i k_i}{r^{1/n}}.$$

This is just a rephrasing of well-known results in birational geometry; see for example [Kol96, V.2.9 Proposition].

Proof. By continuity and rescaling invariance, it suffices to show that if L is a big and nef Cartier divisor class then

$$\left(\sum_{i=1}^r k_i\right) \frac{\mathrm{vol}(L)^{1/n}}{r^{1/n}} \leqslant L \cdot C.$$

A standard argument (see for example [Leh13b, Example 8.22]) shows that for any $\epsilon > 0$ and any general points $\{x_i\}_{i=1}^r$ of X there is a positive integer m and a Cartier divisor M numerically equivalent to mL and such that $\mathrm{mult}_{x_i} M \geqslant mr^{-1/n}\,\mathrm{vol}(L)^{1/n} - \epsilon$ for every i. By the assumption on the family of curves we may find an irreducible curve C with multiplicity $\geqslant k_i$ at each x_i that is not contained M. Then

$$m(L \cdot C) \geqslant \sum_{i=1}^r k_i \,\mathrm{mult}_{x_i} M \geqslant \left(\sum_{i=1}^r k_i\right)\left(\frac{m\,\mathrm{vol}(L)^{1/n}}{r^{1/n}} - \epsilon\right).$$

Divide by m and let ϵ go to 0 to conclude. □

Example 11.2. The most important special case is when α is the class of a family of irreducible curves such that for any two general points of X there is a curve in our family containing them. Proposition 11.1 then shows that $\widehat{\mathrm{vol}}(\alpha) \geqslant 1$.

11.1 Seshadri constants

Let X be a smooth projective variety of dimension n and let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Recall that for points $\{x_i\}_{i=1}^r$ on X the Seshadri constant of A along the $\{x_i\}$ is

$$\varepsilon(x_1, \ldots, x_r, A) := \inf_{C \ni x_i} \frac{A \cdot C}{\sum_i \operatorname{mult}_{x_i} C}.$$

where the infimum is taken over all reduced irreducible curves C containing at least one of the points x_i. An easy intersection calculation on the blow-up of X at the r points shows that

$$\varepsilon(x_1, \ldots, x_r, A) \leqslant \frac{\operatorname{vol}(A)^{1/n}}{r^{1/n}}.$$

When the r points are very general, r is large, and A is sufficiently ample, one "expects" the two sides of the inequality to be close. This heuristic can fail badly, but it is interesting to analyze how close it is to being true. In particular, the Seshadri constant should only be very small compared to the volume in the presence of a "Seshadri-exceptional fibration" (see [EKL95], [HK03]). This motivates the following definition:

Definition 11.3. Let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Set $\varepsilon_r(A)$ to be the Seshadri constant of A along r points $\mathbf{x} := \{x_i\}$ of X. We define the Seshadri ratio of A to be

$$sr_{\mathbf{x}}(A) := \frac{r^{1/n}\varepsilon(x_1, \ldots, x_r, A)}{\operatorname{vol}(A)^{1/n}}.$$

Note that the Seshadri ratio is at most 1, and that low values should only arise in special geometric situations. The principle established by [EKL95], [HK03] is that if the Seshadri ratio for A is small, then the curves which approximate the bound in the Seshadri constant can not "move too much."

In this section we revisit these known results on Seshadri constants from the perspective of the volume of curves. In particular we demonstrate how the Zariski decomposition can be used to bound the classes of curves C which give small values in the Seshadri computations above.

Proposition 11.4. *Let X be a smooth projective variety of dimension n and let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Fix $\delta > 0$ and fix r points $x_1, \ldots, x_r$. Suppose that C is a curve containing at least one of the x_i and such that*

$$\varepsilon(x_1, \ldots, x_r, A)(1 + \delta) > \frac{A \cdot C}{\sum_i \operatorname{mult}_{x_i} C}.$$

Letting α denote the numerical class of C, we have

$$sr_{\mathbf{x}}(A)(1 + \delta) \geqslant r^{1/n} \frac{\widehat{\operatorname{vol}}(\alpha)^{n-1/n}}{\sum_i \operatorname{mult}_{x_i} C}$$

In fact, this estimate is rather crude; with better control on the relationship between A and α, one can do much better.

Proof. One simply multiplies both sides of the first inequality by $r^{1/n}/\operatorname{vol}(A)^{1/n}$ to deduce that

$$sr_{\mathbf{x}}(A)(1+\delta) \geqslant r^{1/n}\frac{A\cdot C}{\operatorname{vol}(A)^{1/n}\sum_i \operatorname{mult}_{x_i} C}$$

and then uses the obvious inequality $(A\cdot C)/\operatorname{vol}(A)^{1/n} \geqslant \widehat{\operatorname{vol}}(C)^{n-1/n}$. □

We can then bound the Seshadri ratio of A in terms of the Zariski decomposition of the curve.

Proposition 11.5. *Let X be a smooth projective variety of dimension n and let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Fix $\delta > 0$ and fix r distinct points $x_i \in X$. Suppose that C is a curve containing at least one of the x_i such that the class α of C is big and*

$$\varepsilon(x_1,\ldots,x_r,A)(1+\delta) > \frac{A\cdot C}{\sum_i \operatorname{mult}_{x_i} C}.$$

Write $\alpha = B^{n-1}+\gamma$ for the Zariski decomposition. Then $sr_{\mathbf{x}}(A)(1+\delta) > sr_{\mathbf{x}}(B)$.

Proof. By Proposition 11.4 it suffices to show that

$$r^{1/n}\frac{\widehat{\operatorname{vol}}(\alpha)^{n-1/n}}{\sum_i \operatorname{mult}_{x_i} C} \geqslant sr_{\mathbf{x}}(B),$$

But this follows from the definition of Seshadri constants along with the fact that $B\cdot C = \widehat{\operatorname{vol}}(C)$. □

These results are of particular interest in the case when the points are very general, when it is easy to deduce the bigness of the class of C.

Certain geometric properties of Seshadri constants become very clear from this perspective. For example, following the notation of [Nag61] we say that a curve C on X is abnormal for a set of r points $\{x_i\}$ and a big and nef divisor A if C contains at least one x_i and

$$1 > \frac{r^{1/n}(A\cdot C)}{\operatorname{vol}(A)^{1/n}\sum_i \operatorname{mult}_{x_i} C}.$$

Corollary 11.6. *Let X be a smooth projective variety of dimension n and let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Fix r very general points $x_1,\ldots,x_r$. Then no abnormal curve goes through a very general point of X aside from the x_i.*

Proof. Since the x_i are very general, any curve going through at least one more very general point deforms to cover the whole space, so its class is big and nef. Then combine Proposition 11.4 and Proposition 11.1 to deduce that if the Seshadri constant of the $\{x_i\}$ is computed by a curve through an additional very general point then $sr_{\mathbf{x}}(A) = 1$. □

11.2 Rationally connected varieties

Given a rationally connected variety X of dimension n, it is interesting to ask for the possible volumes of curve classes representing rational curves. In particular, one would like to know if one can find classes whose volumes satisfy a uniform upper bound depending only on the dimension. There are four natural options:

1. Consider all classes of rational curves.
2. Consider all classes of chains of rational curves which connect two general points.
3. Consider all classes of irreducible rational curves which connect two general points.
4. Consider all classes of very free rational curves.

Note that each criterion is more special than the previous ones. We call a class of the second kind an RCC class and a class of the fourth kind a VF class. Every one of the classes (2), (3), (4) has positive volume; indeed, [BCE$^+$02] shows that if two general points of X can be connected via a chain of curves of class α, then α is a big class.

On a Fano variety of Picard rank 1, the minimal volume of an RCC class is determined by the degree and the minimal degree of an RCC class against the ample generator (or equivalently, the degree, the index, and the length of an RCC class). The minimum volume is thus related to these well studied invariants.

In higher dimensions, the work of [KMM92] and [Cam92] shows that there are constants $C(n)$, $C'(n)$ such that any n-dimensional smooth Fano variety carries an RCC class satisfying $-K_X \cdot \alpha \leqslant C(n)$, and a VF class satisfying $-K_X \cdot \beta \leqslant C'(n)$. We then also obtain explicit bounds on the minimal volume of an RCC or VF class on X. It is interesting to ask what happens for arbitrary rationally connected varieties.

Example 11.7. We briefly review bounds on the volumes of such classes for smooth surfaces. Consider first the Hirzebruch surfaces $\mathbb{F}_e$. It is clear that on a Hirzebruch surface a curve class is RCC if and only if it is big, and one easily sees that the minimum volume for an RCC class is $\frac{1}{e}$. Thus there is no non-trivial universal lower bound for the minimum volume of an RCC class.

In terms of upper bounds, note that if $\pi : Y \to X$ is a birational map and α is an RCC class, then $\pi_*\alpha$ is an RCC class as well. Conversely, given any RCC class β on X, there is some preimage β' on Y which is also an RCC class. Thus by Proposition 5.16, we see that any rational surface carries an RCC class of volume no greater than that of an RCC class on a minimal surface. This shows that any smooth rational surface has an RCC class of volume at most 1.

On a surface any VF class is necessarily nef, so the universal lower bound on the volume is 1. In the other direction, consider again the Hirzebruch surface $\mathbb{F}_e$. Any VF class will have the form $aC_0 + bF$ where C_0 is the section of negative self-intersection and F is the class of a fiber. Note that the self intersection is $2ab - a^2e$. For a VF class we clearly must have $a \geqslant 1$, so that $b \geqslant ea$ to ensure nefness. Thus the smallest possible volume of a VF class is e, and this is achieved by the class $C_0 + eF$. Note that there is no uniform upper bound on the minimum volume of a VF class.

As indicated in the previous example, it is most interesting to look for upper bounds on the minimum volume of an RCC class. Indeed, by taking products with projective spaces, one sees that in any dimension the only uniform lower bound for volumes of RCC classes is 0. Furthermore, there is no uniform upper bound for the minimum volume of a VF class. The crucial distinction is that VF classes are nef, while RCC classes need not be, so that a uniform bound on the volume of a VF class can only be expected for bounded families of varieties.

The following question gives a "birational" version of the well-known results of [KMM92].

Question 11.8. Let X be a smooth rationally connected variety of dimension n. Is there a bound $d(n)$, depending only on n, such that X admits an RCC class of volume at most $d(n)$?

It is also interesting to ask for optimal bounds on volumes. The first situation to consider are the "extremes" in the examples above. Note that the lower bound of the volume of a VF class is 1 by Proposition 11.1, so it is interesting to ask when the minimum is achieved.

Question 11.9. For which varieties X is the smallest volume of an RCC class equal to 1?

For which varieties X is the smallest volume of a VF class equal to 1?

12 Appendix A

12.1 Reverse Khovanskii-Teissier inequalities

An important step in the analysis of the Morse inequality is the "reverse" Khovanskii-Teissier inequality for big and nef divisors A, B, and a movable curve class β:

$$n(A \cdot B^{n-1})(B \cdot \beta) \geqslant B^n (A \cdot \beta).$$

We prove a more general statement on "reverse" Khovanskii-Teissier inequalities in the analytic setting. Some related work has appeared independently in the recent preprint [Pop15].

Theorem 12.1. *Let X be a compact Kähler manifold of dimension n. Let $\omega, \beta, \gamma \in \overline{\mathcal{K}}$ be three nef classes on X. Then we have*

$$(\beta^k \cdot \alpha^{n-k}) \cdot (\alpha^k \cdot \gamma^{n-k}) \geqslant \frac{k!(n-k)!}{n!} \alpha^n \cdot (\beta^k \cdot \gamma^{n-k}).$$

Proof. The proof depends on solving Monge-Ampère equations and the method of [Pop14]. Without loss of generality, we can assume γ is normalised such that $\beta^k \cdot \gamma^{n-k} = 1$. Then we need to show

$$(\beta^k \cdot \alpha^{n-k}) \cdot (\alpha^k \cdot \gamma^{n-k}) \geqslant \frac{k!(n-k)!}{n!} \alpha^n. \tag{1}$$

We first assume α, β, γ are all Kähler classes. We will use the same symbols to denote the Kähler metrics in corresponding Kähler classes. By the Calabi-Yau theorem [Yau78], we can solve the following Monge-Ampère equation:

$$(\alpha + i\partial\bar{\partial}\psi)^n = \left(\int \alpha^n\right) \beta^k \wedge \gamma^{n-k}. \tag{2}$$

Denote by α_ψ the Kähler metric $\alpha + i\partial\bar{\partial}\psi$. Then we have

$$
\begin{aligned}
(\beta^k \cdot \alpha^{n-k}) \cdot (\alpha^k \cdot \gamma^{n-k}) &= \int \beta^k \wedge \alpha_\psi^{n-k} \cdot \int \alpha_\psi^k \wedge \gamma^{n-k} \\
&= \int \frac{\beta^k \wedge \alpha_\psi^{n-k}}{\alpha_\psi^n} \alpha_\psi^n \cdot \int \frac{\alpha_\psi^k \wedge \gamma^{n-k}}{\alpha_\psi^n} \alpha_\psi^n \\
&\geqslant \left(\int \left(\frac{\beta^k \wedge \alpha_\psi^{n-k}}{\alpha_\psi^n} \cdot \frac{\alpha_\psi^k \wedge \gamma^{n-k}}{\alpha_\psi^n} \right)^{1/2} \alpha_\psi^n \right)^2 .
\end{aligned}
$$

The last line follows because of the Cauchy-Schwarz inequality. We claim that the following pointwise inequality holds:

$$
\frac{\beta^k \wedge \alpha_\psi^{n-k}}{\alpha_\psi^n} \cdot \frac{\alpha_\psi^k \wedge \gamma^{n-k}}{\beta^k \wedge \gamma^{n-k}} \geqslant \frac{k!(n-k)!}{n!}.
$$

Then by (2) it is clear the above pointwise inequality implies the desired inequality (1). For any fixed point $p \in X$, we can choose some coordinates such that at the point p:

$$
\alpha_\psi = i \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j, \quad \beta = i \sum_{j=1}^{n} \mu_j dz^j \wedge d\bar{z}^j,
$$

and

$$
\gamma^{n-k} = i^{n-k} \sum_{|I|=|J|=n-k} \Gamma_{IJ} dz_I \wedge d\bar{z}_J.
$$

Denote by μ_J the product $\mu_{j_1}...\mu_{j_k}$ with index $J = (j_1 < ... < j_k)$ and denote by J^c the complement index of J. Then it is easy to see at the point p we have

$$
\frac{\beta^k \wedge \alpha_\psi^{n-k}}{\alpha_\psi^n} \cdot \frac{\alpha_\psi^k \wedge \gamma^{n-k}}{\beta^k \wedge \gamma^{n-k}} = \frac{k!(n-k)!}{n!} \frac{(\sum_J \mu_J)(\sum_K \Gamma_{KK})}{\sum_J \mu_J \Gamma_{J^c J^c}} \geqslant \frac{k!(n-k)!}{n!}.
$$

This finishes the proof of the case when α, β, γ are all Kähler classes. If they are just nef classes, by taking limits, then we get the desired inequality. □

Remark 12.2. By [Xia15, Section 2.1.1], for $k = 1$ we can always replace γ^{n-1} in Theorem 12.1 by an arbitrary movable class.

Remark 12.3. It would be interesting to find an algebraic approach to Theorem 12.1, thus generalizing it to projective varieties defined over arbitrary fields.

12.2 Towards the transcendental holomorphic Morse inequality

Recall that the (weak) transcendental holomorphic Morse inequality over compact Kähler manifolds conjectured by Demailly is stated as follows:

Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in \overline{\mathcal{K}}$ be two nef classes. Then we have $\mathrm{vol}(\alpha - \beta) \geqslant \alpha^n - n\alpha^{n-1} \cdot \beta$. In particular, if $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$ then there exists a Kähler current in the class $\alpha - \beta$.

Indeed, the last statement has been proved in the recent work [Xia13, Pop14]. The missing part is how to bound the volume $\mathrm{vol}(\alpha - \beta)$ by $\alpha^n - n\alpha^{n-1} \cdot \beta$.

By [Xia15, Theorem 2.1 and Remark 2.3] the volume for transcendental pseudo-effective $(1,1)$-classes is conjectured to be characterized as following:

$$\mathrm{vol}(\alpha) = \inf_{\gamma \in \mathcal{M}, \mathfrak{M}(\gamma)=1} (\alpha \cdot \gamma)^n \tag{3}$$

For the definition of $\mathfrak{M}$ in the Kähler setting, see [Xia15, Definition 2.2]. If we denote the right hand side of (3) by $\overline{\mathrm{vol}}(\alpha)$, then we can prove the following:

Theorem 12.4. *Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in \overline{\mathcal{K}}$ be two nef classes. Then we have*

$$\overline{\mathrm{vol}}(\alpha - \beta)^{1/n} \, \mathrm{vol}(\alpha)^{n-1/n} \geqslant \alpha^n - n\alpha^{n-1} \cdot \beta.$$

Proof. We only need to consider the case when $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$. And [Pop14] implies the class $\alpha - \beta$ is big. By the definition of $\overline{\mathrm{vol}}$, we have

$$\overline{\mathrm{vol}}(\alpha - \beta)^{1/n} = \inf_{\gamma \in \mathcal{M}, \mathfrak{M}(\gamma)=1} (\alpha - \beta) \cdot \gamma.$$

So we need to estimate $(\alpha - \beta) \cdot \gamma$ with $\mathfrak{M}(\gamma) = 1$:

$$\begin{aligned}
(\alpha - \beta) \cdot \gamma &= \alpha \cdot \gamma - \beta \cdot \gamma \\
&\geqslant \alpha \cdot \gamma - \frac{n(\alpha^{n-1} \cdot \beta) \cdot (\alpha \cdot \gamma)}{\alpha^n} \\
&= \frac{\alpha \cdot \gamma}{\alpha^n} (\alpha^n - n\alpha^{n-1} \cdot \beta) \\
&\geqslant \mathrm{vol}(\alpha)^{1-n/n} (\alpha^n - n\alpha^{n-1} \cdot \beta),
\end{aligned}$$

where the second line follows from Theorem 12.1 and Remark 12.2, and the last line follows the definition of $\mathfrak{M}$ and $\mathfrak{M}(\gamma) = 1$.

By the arbitrariness of γ we get

$$\overline{\mathrm{vol}}(\alpha - \beta)^{1/n} \, \mathrm{vol}(\alpha)^{n-1/n} \geqslant \alpha^n - n\alpha^{n-1} \cdot \beta.$$

□

Remark 12.5. Without using the conjectured equality (3), it is observed independently by [Tos15] and [Pop15] that one can replace $\overline{\mathrm{vol}}$ by the volume function vol in Theorem 12.4.

13 Appendix B

13.1 Non-convexity of the complete intersection cone

We give an example explicitly verifying the non-convexity of $\mathrm{CI}_1(X)$. Undoubtedly there are simpler examples, but this is the first one we wrote down.

Example 13.1. [FS09] gives an example of a smooth toric threefold X such that every nef divisor is big. We show that for this toric variety $\mathrm{CI}_1(X)$ is not convex.

Let X be the toric variety defined by a fan in $N = \mathbb{Z}^3$ on the rays

$$\begin{array}{llll} v_1 = (1,0,0) & v_2 = (0,1,0) & v_3 = (0,0,1) & v_4 = (-1,-1,-1) \\ v_5 = (1,-1,-2) & v_6 = (1,0,-1) & v_7 = (0,-1,-2) & v_8 = (0,0,-1) \end{array}$$

with maximal cones

$$\begin{gathered} \langle v_1,v_2,v_3\rangle,\ \langle v_1,v_2,v_6\rangle,\ \langle v_1,v_3,v_4\rangle,\ \langle v_1,v_4,v_5\rangle, \\ \langle v_1,v_5,v_6\rangle,\ \langle v_2,v_3,v_4\rangle,\ \langle v_2,v_4,v_8\rangle,\ \langle v_2,v_5,v_6\rangle, \\ \langle v_2,v_5,v_8\rangle,\ \langle v_4,v_5,v_7\rangle,\ \langle v_4,v_7,v_8\rangle,\ \langle v_5,v_7,v_8\rangle. \end{gathered}$$

Since X is the blow-up of $\mathbb{P}^3$ along 4 rays, it has Picard rank 5. Let D_i be the divisor corresponding to the ray v_i and C_{ij} denote the curve corresponding to the face generated by v_i and v_j. Standard toric computations show that the pseudo-effective cone of divisors is simplicial and is generated by D_1, D_5, D_6, D_7, D_8. The pseudo-effective cone of curves is also simplicial and is generated by $C_{14}, C_{16}, C_{25}, C_{47}, C_{48}$. From now on we will write divisor or curve classes as vectors in these (ordered) bases.

The intersection matrix is:

	D_1	D_5	D_6	D_7	D_8
C_{14}	-2	1	0	0	0
C_{16}	1	1	-2	0	0
C_{25}	0	-1	1	0	1
C_{47}	0	1	0	-2	1
C_{48}	0	0	0	1	-2

The nef cone of divisors is dual to the pseudo-effective cone of curves. Thus it is simplicial and has generators $A_1, \ldots, A_5$ determined by the columns of the inverse of the matrix above:

$$\begin{aligned} A_1 &= (1,3,2,2,1) \\ A_2 &= (3,6,4,4,2) \\ A_3 &= (6,12,9,8,4) \\ A_4 &= (2,4,3,2,1) \\ A_5 &= (4,8,6,5,2) \end{aligned}$$

A computation shows that for real numbers $x_1, \dots, x_5$,

$$
\begin{aligned}
\left(\sum_{i=1}^{5} x_i A_i\right)^2 =&(1,3,6,2,4)(x_1^2 + 6x_1x_2 + 12x_1x_3 + 4x_1x_4 + 8x_1x_5)+\\
&(9,22,45,15,30)x_2^2+\\
&(12,30,60,20,40)(x_2x_4 + 2x_2x_5 + 3x_2x_3 + 3x_3^2 + 2x_3x_4 + 4x_3x_5)+\\
&(4,10,20,6,13)x_4^2+\\
&(16,40,80,26,52)(x_4x_5 + x_5^2).
\end{aligned}
$$

Note that the five vectors above form a basis of $N_1(X)$ and each one is proportional to one of the A_i^2.

It is clear from this explicit description that the cone is not convex. For example, the vector

$$v = (9,22,45,15,30) + (4,10,20,6,13)$$

can not be approximated by curves of the form H^2 for an ample divisor H. Indeed, if we have a sequence of ample divisors $H_j = \sum x_{i,j} A_i$ with $x_{i,j} > 0$ such that H_j^2 converges to v, then

$$\lim_{j\to\infty} x_{2,j} = 1 \qquad \text{and} \qquad \lim_{j\to\infty} x_{4,j} = 1.$$

But then the limit of the coefficient of $(12,30,60,20,40)$ is at least 1, a contradiction. Exactly the same argument shows that the closure of the set of all products of 2 (possibly different) ample divisors is not convex.

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