CLASSIFYING SECTIONS OF DEL PEZZO FIBRATIONS, I

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ABSTRACT. We develop a strategy to classify the components of the space of sections of a del Pezzo fibration over $\mathbb{P}^1$. In particular, we prove the Movable Bend and Break lemma for del Pezzo fibrations. Our approach is motivated by Geometric Manin's Conjecture and proves upper bounds on the associated counting function. We also give applications to enumerativity of Gromov-Witten invariants and to the study of the Abel-Jacobi map.

CONTENTS

1. INTRODUCTION

We work over an algebraically closed field k of characteristic 0. A del Pezzo fibration over $\mathbb{P}^1$ is an algebraic fiber space $\pi : \mathcal{X} \to \mathbb{P}^1$ such that $\mathcal{X}$ is smooth and the general fiber is a del Pezzo surface. Our goal is to classify the components of the space $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing sections of π.

The key ingredient in our approach is Geometric Manin's Conjecture. The conjecture predicts that the Fujita invariant controls the behavior of the dimension and number of components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ as the anticanonical degree increases. Our classification scheme is also centered around the Fujita invariant, and our results imply upper bounds on the counting function in Geometric Manin's Conjecture.

In a companion paper, we will study the behavior of sections of del Pezzo surface fibrations over base curves of higher genus.

1.1. **Classifying sections.** Our first main theorem describes the components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ which parametrize a non-dominant family of sections.

Theorem 1.1. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively nef. Then there is a proper closed subset $V \subsetneq \mathcal{X}$ such that any component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing a non-dominant family of sections will parametrize sections contained in V.*

Our proof of Theorem 1.1 is constructive: we show that V is the union of the subvarieties of $\mathcal{X}$ which have larger generic Fujita invariant with a locus swept out by low height sections.

To understand dominant families of sections we will use the inductive approach of [HRS04] based on breaking and gluing rational curves. The main difficulty in this approach is to understand the possible outcomes of Mori's Bend-and-Break theorem. Our second main theorem shows that we can break curves while remaining in the setting of free curves.

Theorem 1.2 (Movable Bend-and-Break for sections). *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. There is a constant $Q(\mathcal{X})$ satisfying the following property. Suppose that $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ is a component that parametrizes a dominant family of sections C satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq Q(\mathcal{X})$. Then the closure of M in $\overline{M}_{0,0}(\mathcal{X})$ contains a point representing a stable map whose domain has exactly two components each mapping birationally onto a free curve.*

Here $Q(\mathcal{X})$ is an explicit constant determined by the behavior of the Fujita invariant and low degree sections. Based on the results above, we can use the following strategy to classify sections of a del Pezzo fibration with $-K_{\mathcal{X}/\mathbb{P}^1}$ relatively ample:

(1) Classify all non-dominant families of sections using Theorem 1.1.
(2) Classify all dominant families of sections of small height using the explicit geometry of the fibration.
(3) Classify all dominant families of sections of large height by induction using Theorem 1.2.

We illustrate this strategy in several examples:

Example 1.3. Let X_3 denote a smooth cubic threefold in $\mathbb{P}^4$. By taking a general pencil of hyperplane sections and resolving the base locus we obtain a del Pezzo fibration $\pi : \mathcal{X} \to \mathbb{P}^1$. In Example 8.3 we will use the strategy above to show that $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ admits a unique component representing sections of any height ≥ -1 and that each of these components has the expected dimension.

A similar argument works for other del Pezzo fibrations constructed by blowing up a Fano 3-fold, and we give several examples of this type.

Example 1.4. Consider the smooth threefold $\mathcal{X} \subset \mathbb{P}^1_{s,t} \times \mathbb{P}^3_{x,y,z,w}$ defined by the equation $sx^3 + ty^3 + (t+s)z^3 + (t+2s)w^3 = 0$ equipped with the cubic surface fibration $\pi : \mathcal{X} \to \mathbb{P}^1$. Using the above strategy one can show that for any integer $d \geq -1$ there is a unique component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ with height d. Since [Jah14, the Appendix] shows that $\mathcal{X}_\eta$ has Brauer group $\mathbb{Z}/3\mathbb{Z}$, this example demonstrates that it is the Brauer group of the total space and not the Brauer group of the generic fiber which plays a role in Geometric Manin's Conjecture over a complex curve (see Section 9.2).

1.2. **Upper bounds in Manin's Conjecture.** Let $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)_d$ denote the parameter space for sections of height d with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$. Geometric Manin's Conjecture is formulated

using a counting function $N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d)$ which measures the dimension and the number of components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)_d$. Loosely speaking the conjecture predicts that after removing the contributions of an exceptional set of curves we have an asymptotic growth rate of the form

$$N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d) \sim_{d\to\infty} Cq^d d^{\rho(\mathcal{X}_\eta)-1}$$

where $\mathcal{X}_\eta$ denotes the generic fiber of π. We will give a precise formulation of the conjecture in Section 9.

The exceptional set in Geometric Manin's Conjecture is contained in the closed set V of Theorem 1.1. Indeed, the theorem implies that after removing the sections in V every family has the expected dimension, giving the correct exponential term in the asymptotic formula for $N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d)$. In order to control the subexponential term, we apply ideas from [LT19a] to prove a special case of a conjecture of Batyrev:

Corollary 1.5. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. There is a polynomial $P(d)$ which is an upper bound for the number of components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ that parametrize sections C satisfying*

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \leq d.$$

Corollary 1.5 implies an upper bound on the counting function of the expected form

$$N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d) = O(q^d d^r)$$

for some positive integer r.

Of course, whenever one can completely classify sections one obtains a precise asymptotic formula for the counting function. In this way we can prove Geometric Manin's Conjecture for some low degree del Pezzo surfaces over $k(\mathbb{P}^1)$.

1.3. **Stabilization of the Abel-Jacobi map.** For a del Pezzo fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ over $\mathbb{C}$ we let $\mathrm{IJ}(\mathcal{X})$ denote the intermediate Jacobian of $\mathcal{X}$. For any reduced component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ the universal family of curves over M induces a rational map $\mathrm{AJ}_M : M \dashrightarrow \mathrm{IJ}(\mathcal{X})$. Using our main theorems we show that the Stein factorizations of the Abel-Jacobi maps "stabilize" as we increase the height.

Theorem 1.6. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample and the general fiber is a del Pezzo surface with degree ≥ 3. Consider the set of components $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ such that the Abel-Jacobi map $\mathrm{AJ}_M : M \dashrightarrow \mathrm{IJ}(\mathcal{X})$ is dominant. There are only finitely many morphisms $\{Z_i \to \mathrm{IJ}(\mathcal{X})\}$ which occur as a Stein factorization of a resolution of a projective compactification of one of these AJ_M.*

Example 1.7. Suppose that $\pi : \mathcal{X} \to \mathbb{P}^1$ is constructed by taking a general pencil of hyperplane sections of a Fano threefold of Picard rank 1, index 2, and degree 5. In this case $\mathrm{IJ}(\mathcal{X})$ is an elliptic curve. We prove that the Abel-Jacobi map for every family of sections of height ≥ 1 is dominant with connected fibers, so that the finite part of the Stein factorization is always the identity map. (We expect that in this example the Abel-Jacobi map coincides with the MRC fibration for every family of sections.)

Our original motivation for Theorem 1.6 was to analyze whether the Abel-Jacobi map relates to other possible types of stability results for components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$:

- Does the MRC fibration for components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ stabilize as the degree increases? ([HRS02], [dJS04], [Voi15])
- Do components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ exhibit cohomological stability? ([EV05], [CEF14])

However, Examples 10.5 and 10.7 show that for components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ neither the MRC fibration nor the Albanese map needs to coincide with the Abel-Jacobi map.

Example 1.8. Let S be a del Pezzo surface of degree 2 equipped with a conic fibration $g : S \to \mathbb{P}^1$. Let $\mathcal{X}$ be a conic bundle over S whose discriminant locus D is a very general element of $|-2K_S|$. By composing with g we obtain a degree 4 del Pezzo surface fibration $\pi : \mathcal{X} \to \mathbb{P}^1$. In this case $\mathrm{IJ}(\mathcal{X})$ is a 2-dimensional abelian variety.

We show that:
- If the Abel-Jacobi map for a component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ is dominant, then it is not birational to the MRC fibration. (This follows from results of [HKT16] and [Voi13].)
- There are components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ whose Albanese variety has dimension larger than 2.

In particular the dimension of the intermediate Jacobian does not provide an a priori bound on the dimension of $H^1(\overline{M}, \mathbb{C})$ for the resolution $\overline{M}$ of a projective closure of a component M of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$.

1.4. **Gromov-Witten theory.** The arguments we use to prove Theorem 1.2 also show that certain Gromov-Witten invariants for classes of sections are enumerative.

Proposition 1.9. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Let F be a general fiber of π. There is a constant $C(\mathcal{X})$ such that the following is true. Fix an integer $n \geq C(\mathcal{X})$. Let $\beta \in N_1(\mathcal{X})$ denote a curve class satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot \beta = 2n - 2$ and $F \cdot \beta = 1$ for a general fiber F of π. Then the GW invariant $\langle [pt]^n \rangle_{0,n}^{\mathcal{X}, \beta}$ is enumerative: it counts the number of rational curves of class β through n general points of $\mathcal{X}$.

Proposition 11.2 gives a similar result for classes of odd height and Proposition 11.3 gives conditions which guarantee that these GW-invariants do not vanish. These results complement [Tia12, Theorem 4.1] which proves the non-vanishing of other GW-invariants for del Pezzo fibrations over $\mathbb{P}^1$.

1.5. **Prior results.** Del Pezzo fibrations have been studied extensively due to their prominent role in the minimal model program for threefolds. [GHS03] guarantees that a del Pezzo fibration admits a section. There is a large body of literature on the Weak Approximation Conjecture for sections of del Pezzo fibrations, see e.g. [KMM92], [CTG04], [HT06], [HT08], [Xu12a], [Xu12b], [Kne13], [Tia15], and [STZ18]. The Abel-Jacobi map for sections has been studied for quadric surface fibrations by [HT12] and for degree 4 del Pezzo fibrations by [HT14]. [Tia12] has studied Gromov-Witten invariants for del Pezzo fibrations over $\mathbb{P}^1$.

2. Preliminaries

Fix a field k which is algebraically closed field and of characteristic 0. In this paper our ground field will usually be k or $k(\mathbb{P}^1)$. A variety is a reduced irreducible separated scheme of finite type over the ground field. When we take a component of a scheme, we always endow it with its reduced structure.

Throughout the paper we will use $\sim_{rat}$ to denote rational equivalence of cycles and $\sim_{alg}$ to denote algebraic equivalence of cycles.

Let X be a projective variety. We will let $N^1(X)$ denote the space of $\mathbb{R}$-Cartier divisors up to numerical equivalence on X and let $\overline{\mathrm{Eff}}^1(X)$ and $\mathrm{Nef}^1(X)$ denote respectively the pseudo-effective and nef cones of divisors. Dually, $N_1(X)$ denotes the space of $\mathbb{R}$-curves up to numerical equivalence and $\overline{\mathrm{Eff}}_1(X)$ and $\mathrm{Nef}_1(X)$ denote respectively the pseudo-effective and nef cones of curves. We denote the lattices generated by integral cycles by $N^1(X)_{\mathbb{Z}} \subset N^1(X)$ and $N_1(X)_{\mathbb{Z}} \subset N_1(X)$. We define $\mathrm{Nef}_1(X)_{\mathbb{Z}} := \mathrm{Nef}_1(X) \cap N_1(X)_{\mathbb{Z}}$.

We say that a reduced irreducible curve C is movable on X if C is a member of a family of curves which dominates X.

2.1. Fano fibrations.
In this paper, a Fano fibration will always be a smooth integral model for a Fano variety over $k(\mathbb{P}^1)$. More precisely:

Definition 2.1. A Fano fibration is a morphism $\pi : \mathcal{X} \to \mathbb{P}^1$ satisfying the following properties:

(1) $\mathcal{X}$ is smooth projective and f is an algebraic fiber space, and
(2) the general fiber of π is a smooth Fano variety.

We will always denote the generic point of $\mathbb{P}^1$ by η and the generic fiber of π by $\mathcal{X}_\eta$.

By [Kol96, IV.3.5 Corollary] Fano fibrations satisfy the following important properties:

- Every fiber of f is rationally chain connected.
- As we vary over all smooth fibers F the dimension of $N^1(F)$ is constant.

Furthermore, Fano fibrations have the nice property that the positivity of divisors on the generic fiber is related to positivity of divisors on a general fiber.

Lemma 2.2 ([Wiś09] Theorem 1, [dFH11] Theorem 6.8 and preceding discussion). *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a Fano fibration. Let L be a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$. Then the following are equivalent:*

(1) $L|_F$ *is ample for some smooth Fano fiber F.*
(2) $L|_F$ *is ample for all smooth Fano fibers F.*
(3) $L|_{\mathcal{X}_\eta}$ *is ample.*

The analogous statement is true for nefness, for bigness, and for pseudo-effectiveness.

2.2. Height functions.
Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a projective morphism of finite type separated schemes over the ground field. Then the choice of an ample $\mathbb{Q}$-Cartier divisor on the generic fiber $\mathcal{X}_\eta$ will define a height function on sections of π. It is well-known that a Northcott property holds in this setting:

Lemma 2.3. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a projective morphism of finite type separated schemes over the ground field and let L be a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ whose restriction to a general fiber is ample. For any fixed constant γ, the set of classes of sections C satisfying $L \cdot C \leq \gamma$ lies in a bounded subset of $N_1(\mathcal{X})$.*

One consequence of Lemma 2.3 is that there is a lower bound on the possible values of $L \cdot C$ as we vary C over all sections.

Definition 2.4. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a Fano fibration and let L be a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ whose restriction to the generic fiber is ample. We define $\mathrm{neg}(\mathcal{X}, L)$ to be the smallest value of $L \cdot C$ as we vary C over all sections of $\pi : \mathcal{X} \to \mathbb{P}^1$.

2.3. **Spaces of sections.** Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a Fano fibration. We let $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ denote the open subset of $\mathrm{Hilb}(\mathcal{X})$ parametrizing sections of π. Using the functorial definitions, we see that there is an inclusion $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1) \subset \overline{M}_{0,0}(\mathcal{X})$. In particular, any component M of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ embeds as a dense open subset of a unique component $\overline{M}$ of $\overline{M}_{0,0}(\mathcal{X})$.

The expected dimension of a component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ is

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C + \dim(\mathcal{X}) - 1$$

where C denotes a general section parametrized by M. The expected dimension is a lower bound for $\dim(M)$. If M parametrizes a dominant family of sections then the actual dimension agrees with the expected dimension.

Lemma 2.5. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a Fano fibration. Suppose that C is a section such that any n general points of $\mathcal{X}$ are contained in a deformation of C. If we let C' be a general deformation of C and write $N_{C'/X} = \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$ then each $a_i \geq n - 1$. Moreover we can find such a section C' through any n general points.*

Conversely, if C is a section of $\mathcal{X}$ with $N_{C/X} = \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$ and each $a_i \geq n - 1$, then any n general points of $\mathcal{X}$ are contained in some deformation of C.

Proof. This follows from the commutative diagram (2) in [She12, p. 240].

First suppose that some deformation of C contains n general points. Consider the sublocus of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing deformations of C through a fixed set of $n-1$ general points. We know that there is some component of this locus whose universal family maps dominantly onto $\mathcal{X}$. It follows that the differential map of tangent spaces for the universal family is generically full rank. Thus the commutative diagram (2) in [She12] implies that $a_i \geq n - 1$.

Conversely suppose that $a_i \geq n - 1$ for some C. We prove our assertion by induction on n. For the base case when $n = 1$, C is free so that our assertion is clear. For the induction step, suppose that the statement is true for n and $a_i \geq n$ for every i. By the induction hypothesis, a general deformation of C passes through n general points. We fix n general points and we consider the deformation space of curves passing through n general points. Shen shows that the differential of the map from the universal family to X is generically full rank. This means that this evaluation map must be dominant, so a general deformation of C will pass through $n + 1$ general points. □

2.4. **Rational curves on del Pezzo surfaces.** In this section we review some results of [Tes09] concerning the moduli spaces of rational curves on del Pezzo surfaces defined over an algebraically closed field of characteristic 0. When we discuss rational curves on del Pezzo surfaces we will always refer to the anticanonical polarization: lines mean $-K_S$-lines, conics

mean $-K_S$-conics, etc. We start by discussing the classification of rational curves of low anticanonical degree.

Lemma 2.6. *Let S be a del Pezzo surface of degree ≥ 2 defined over an algebraically closed field of characteristic 0. Then:*

- *The conics on S are either fibers of a conic fibration or, when S has degree 2, singular members of the anticanonical linear series.*
- *The cubics on S are pullbacks of elements of $|\mathcal{O}_{\mathbb{P}^2}(1)|$ under birational maps $\phi : S \to \mathbb{P}^2$ or, when S has degree 2 or 3, the pullback of singular members of the anticanonical linear series on a degree 3 del Pezzo surface S' via a birational map.*

In particular, suppose that C is a cubic curve on a del Pezzo surface S of degree between 2 and 8 that contains a point p_0 not lying on any line in S. Then we can deform C while keeping p_0 fixed to a stable map whose image is the union of a conic containing p_0 and a line.

By combining with [Tes09, Proposition 2.4] we obtain the following.

Lemma 2.7. *Let S be a del Pezzo surface of degree ≥ 3. Then the spaces of conics and cubics on S are rational.*

Testa's main theorem is the following.

Theorem 2.8 ([Tes09] Theorem 5.1)**.** *Let S be a del Pezzo surface of degree ≥ 2. Then for every numerical class α, there is at most one component of $\overline{M}_{0,0}(S,\alpha)$ that generically parametrizes stable maps that are birational onto their image.*

The following lemma is a key step in his proof and we will need it as well.

Lemma 2.9. *Let S be a del Pezzo surface. Let C be a general free curve on S such that*

$$-K_S \cdot C \geq 4.$$

Fix two general points x_0, x_1 of C. Then we can deform C as a stable map, keeping x_0 and x_1 fixed, so that the image of the resulting stable map is the union of two free curves $C_0 \cup C_1$ such that $x_0 \in C_0$ and $x_1 \in C_1$.

Moreover, by continuing this inductively we can deform C as a stable map into a chain of free curves of anticanonical degree ≤ 3 such that the end curves C_0 and C_1 contain x_0, x_1 respectively.

Proof. To see the first statement, consider the deformations of C which contain a fixed set of $-K_S \cdot C - 2$ general points on S. By Bend-and-Break C deforms into a broken curve through these points. By [Tes09, Lemma 1.14] this broken curve must be the union of two free curves, and we deduce the statement.

We prove the second statement by induction on $-K_S.C = r$. Let $\overline{M}$ denote the component of $\overline{M}_{0,0}(S)$ containing the stable map represented by the normalization $\mathbb{P}^1 \to C$. The first statement shows that $\overline{M}$ contains stable maps g whose domain has two components and whose image in S is $C_0 \cup C_1$ where C_0 and C_1 are free curves and $x_i \in C_i$. If both of the C_i have degree ≤ 3 then we are done.

If exactly one of the C_i has degree ≤ 3, we may suppose without loss of generality that this component is C_0. Choose a general point x_2 on C_0. By choosing x_0 general in S, we may assume that x_2 is also general in S. By deforming C_1 (and deforming the attachment

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point to C_0), we may suppose that C_1 contains both x_1 and x_2 so that the attachment is x_2. Then we apply the induction hypothesis to C_1 with x_1 and x_2.

Finally, suppose that both of the C_i have degree > 3. Let $\overline{M}_0$ and $\overline{M}_1$ denote the components of $\overline{M}_{0,0}(S)$ containing C_0 and C_1 and let $\overline{M}_0^{(1)}$ and $\overline{M}_1^{(1)}$ denote the corresponding components of $\overline{M}_{0,1}(S)$. Since g is a smooth point of $\overline{M}_{0,0}(S)$, we see that $\overline{M}$ contains a component of $\overline{M}_0^{(1)} \times_S \overline{M}_1^{(1)}$ representing gluings of deformations of C_0 and C_1. Since C_0 and C_1 are free, a dimension count shows that this component of $\overline{M}_0^{(1)} \times_S \overline{M}_1^{(1)}$ contains a stable map onto two free curves $\widetilde{C}_0$ and $\widetilde{C}_1$ such that $\widetilde{C}_0$ contains x_0 but not x_1 and $\widetilde{C}_1$ contains x_1 but not x_0. We may furthermore assume that the image of the node is a general point $x_2 \in S$. Then we may apply the induction hypothesis to $\widetilde{C}_0$ with x_0, x_2 and to $\widetilde{C}_1$ with x_1, x_2 and our assertion follows. $\qquad \square$

We will also need the following lemma:

Lemma 2.10. *Let S be a smooth del Pezzo surface of degree ≥ 2. Then every element of $\mathrm{Nef}_1(S)_{\mathbb{Z}}$ is represented by a sum of free rational curves.*

Proof. By [Tes09, Corollary 2.3], it suffices to show that for every del Pezzo surface S of degree ≥ 2 there is a free rational curve in the anticanonical linear series $|-K_S|$. This is clear for $\mathbb{P}^1 \times \mathbb{P}^1$; for any other del Pezzo surface, we can find a birational map $\phi : S \to \mathbb{P}^2$. Then the strict transform C of a rational cubic on $\mathbb{P}^2$ which contains the ϕ-exceptional locus will be a rational curve in $|-K_S|$. Using deformation theory we see that C must deform in at least a 1-dimensional family on S, showing that a general deformation will be free. $\qquad \square$

3. Fujita invariants

In this section we work over an arbitrary field of characteristic 0.

Definition 3.1. Let X be a smooth projective variety over a field of characteristic 0. Let L be a big and nef $\mathbb{Q}$-Cartier divisor on X. We define the Fujita invariant, or the a-invariant, to be
$$a(X, L) = \min\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\}. \tag{3.1}$$
When L is nef but not big, we formally set $a(X, L) = +\infty$.

When X is singular, we define the Fujita invariant as the Fujita invariant of the pullback of L to any smooth model. This is well-defined because of [HTT15, Proposition 2.7].

Note that the a-invariant is geometric: it does not change under field extension. We will be interested in how the Fujita invariant behaves over $k(\mathbb{P}^1)$.

Definition 3.2. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a morphism with irreducible generic fiber and let L be a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ whose restriction to the generic fiber is big and nef. The generic a-invariant of $\mathcal{X}$ with respect to L is $a(\mathcal{X}_\eta, L|_{\mathcal{X}_\eta})$.

When π is a Fano fibration, we obtain a geometric interpretation of the generic a-invariant using Lemma 2.2.

Lemma 3.3. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a Fano fibration and let L be a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ whose restriction to a general fiber is big and nef. Then for any smooth Fano fiber F we have*
$$a(\mathcal{X}_\eta, L|_{\mathcal{X}_\eta}) = a(F, L|_F).$$

For del Pezzo surfaces, it is easy to work out the behavior of the a-invariant of the anti-canonical divisor when restricted to subvarieties. This leads to the following description:

Lemma 3.4. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration. Then:*

- *A subvariety Y will have $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}) > 1$ if and only if its intersection with a general fiber F is a union of curves of the following types: (-1)-curves, or rational curves in $|-K_F|$ when F has degree 1.*
- *A subvariety Y will have $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}) = 1$ if and only if its intersection with a general fiber F is a union of curves of the following types: irreducible fibers of a conic fibration on F, the rational curves in $|-K_F|$ if F has degree 2, and the rational curves which lie in $|-2K_F|$ or the pullback of the anticanonical linear series on a degree 2 del Pezzo surface if F has degree 1.*

In particular, the subvarieties Y with larger generic a-invariant than $\mathcal{X}$ form a closed subset. The subvarieties Y with the same generic a-invariant as $\mathcal{X}$ are a little more complicated; note that they need not form a bounded family on $\mathcal{X}$ (even though the corresponding subvarieties of $\mathcal{X}_\eta$ do form a bounded family).

4. Bend-and-Break

We will frequently use Bend-and-Break (as in [Mor82] and [KM98, Lemma 1.9]) for families of rational curves through two fixed points, but we will need a slightly more precise version than is usually stated in the literature. In order to explain this precise version we review the following lemma (used in the proof of Bend-and-Break) which identifies the exact criterion we will need.

Lemma 4.1. *Let $\pi : Y \to C$ be a fibration with general fiber $\mathbb{P}^1$ from a smooth projective surface Y to a smooth projective curve C. Let X be a projective variety and suppose $f : Y \to X$ is a morphism such that $\dim(f(Y)) = 2$ and two sections C_1, C_2 are contracted to different points x_1, x_2 in X. Then there is some fiber F of π admitting (at least) two different components F_1, F_2 which satisfy:*

- *each F_i is not contracted by f, and*
- *each F_i contains either x_1 or x_2.*

Proof. We assume otherwise for a contradiction. Let D_1 denote the connected component of the f-fiber over x_1 that contains C_1. Using the Hodge index theorem, we can assign coefficients to the components of D_1 so that the resulting divisor $\widetilde{D}_1$ satisfies the following properties:

- C_1 occurs in $\widetilde{D}_1$ with coefficient 1,
- every other π-horizontal component of D_1 occurs with coefficient 0 in $\widetilde{D}_1$, and
- $\widetilde{D}_1 \cdot E = 0$ for every π-vertical component E of D_1.

In the same way we use C_2 to construct D_2 and $\widetilde{D}_2$ in the fiber over x_2. Note that $\widetilde{D}_1$ and $\widetilde{D}_2$ do not intersect.

By our assumption, every fiber F has a unique component F_0 such that

- F_0 is not contracted by f,
- F_0 is the unique component of F that meets D_1 but is not contained in D_1, and
- F_0 is the unique component of F that meets D_2 but is not contained in D_2.

If F_0 occurs with coefficient m_0 in F, then by construction
$$m_0 F_0 \cdot \widetilde{D}_1 = F \cdot \widetilde{D}_1 = 1.$$
Similarly $m_0 F_0 \cdot \widetilde{D}_2 = 1$. We deduce that $\widetilde{D}_1 - \widetilde{D}_2$ has vanishing intersection against every π-vertical curve. Since the general fiber of π is a rational curve, this condition implies that $\widetilde{D}_1 - \widetilde{D}_2$ is numerically equivalent to a curve whose support is π-vertical. By [Mat02, Lemma 1-2-10] we deduce that $\widetilde{D}_1 - \widetilde{D}_2$ is numerically equivalent to a multiple of the fiber F. Thus
$$0 = (\widetilde{D}_1 - \widetilde{D}_2)^2 = \widetilde{D}_1^2 + \widetilde{D}_2^2 - 2\widetilde{D}_1 \cdot \widetilde{D}_2.$$
But since $\widetilde{D}_1$ and $\widetilde{D}_2$ are contracted by f they both have negative self-intersection. Furthermore, $\widetilde{D}_1 \cdot \widetilde{D}_2 = 0$. This yields a contradiction. $\qquad\square$

Following through the proof of Bend-and-Break, we obtain the following corollary.

Corollary 4.2. *Let X be a projective variety. Let $f : \mathbb{P}^1 \to X$ be a birational map onto a rational curve in X. Suppose that that there are two different points $p_1, p_2 \in \mathbb{P}^1$ and a curve $B^\circ \subset \mathrm{Mor}(\mathbb{P}^1, X, f|_{p_1, p_2})$ such that the images of the maps parametrized by B° sweep out a surface in X. Then f deforms as a stable map to a morphism $g : C \to X$ such that C has (at least) two components C_1, C_2 which are not contracted by g and which satisfy $f(p_i) \in g(C_i)$ for $i = 1, 2$.*

Proof. By construction, there is some curve $B^\circ \subset \mathrm{Mor}(\mathbb{P}^1, X, f|_{p_1, p_2})$ containing f such that we get a morphism $s : \mathbb{P}^1 \times B^\circ \to X$ that contracts two sections. If we let B denote a projective closure of a normalization of B° then s defines a rational map $\mathbb{P}^1 \times B \dashrightarrow X$. Let $\phi : S \to \mathbb{P}^1 \times B$ be a birational map of smooth varieties resolving s. Note that any fiber of $S \to B$ is a union of rational curves. By applying Lemma 4.1 to s, we find a reducible fiber C of $S \to B$ such that $s|_C$ has the desired properties. Although $s|_C$ may not be a stable map, after a stabilization procedure we obtain the desired deformation of stable maps. $\quad\square$

4.1. **Breaking curves on surfaces.** It will also be helpful to have more precise breaking results for sections of a fibration $f : S \to \mathbb{P}^1$ whose generic fiber is $\mathbb{P}^1$. We first need a lemma:

Lemma 4.3. *Let Y be a smooth projective surface with a morphism $\pi : Y \to \mathbb{P}^1$ such that a general fiber of π is isomorphic to $\mathbb{P}^1$. Let F be a singular fiber of π with components $\{E_i\}_{i=1}^r$. Suppose that E_1 is a (-1)-curve that has multiplicity 1 in the fiber F. Then there is another (-1)-curve in the fiber F.*

Proof. The argument is by induction on the number of components of F. When F has two components the claim is clear.

We next prove the inductive step. There is a sequence of contractions of (-1)-curves
$$Y = Y_1 \xrightarrow{\phi_1} Y_2 \xrightarrow{\phi_2} Y_3 \xrightarrow{\phi_3} \dots \xrightarrow{\phi_{r-1}} Y_{r-1}$$
which contract all the components of F but one. We may suppose that E_1 is the exceptional divisor for ϕ_1 since otherwise the statement is clear. Let E_2 denote the exceptional divisor for ϕ_2 and also (by abuse of notation) its strict transform on Y. If E_2 does not intersect E_1 on Y then it is another (-1)-curve, proving the claim. If E_2 does intersect E_1 on Y, then on Y_2 the curve E_2 is a (-1)-curve and it must have multiplicity 1 (since the multiplicity of E_2 on Y_2 is a lower bound for the multiplicity of E_1 on Y). By applying the induction hypothesis

to Y_2 we find another (-1) curve E_j. The strict transform of E_j to Y can not intersect E_1 (since this would make the multiplicity of E_1 in the fiber F larger than 1), showing that E_j is a (-1)-curve on Y in the fiber F. $\qquad\square$

Corollary 4.4. *Suppose that $\pi : Y \to \mathbb{P}^1$ is a Fano fibration of relative dimension 1. Fix a section C of π. There is a birational map ρ from Y to a Hirzebruch surface $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ that is an isomorphism on an open neighborhood of C.*

Proof. Suppose that π admits a reducible fiber F_0. Then F_0 must contain a (-1)-curve. This curve may or may not be disjoint from C. However, if it is not disjoint from C then it must have multiplicity 1 in F_0 and Lemma 4.3 guarantees that there is a different (-1)-curve in this fiber. Thus either way we are guaranteed to have a (-1)-curve in F_0 disjoint from C. Contracting such (-1)-curves inductively yields the desired Hirzebruch surface. $\qquad\square$

The following theorem gives us two breaking results for sections on surfaces, depending on whether or not we want the resulting section to be movable.

Theorem 4.5. *Let $\pi : Y \to \mathbb{P}^1$ be a Fano fibration of relative dimension 1 and let F denote a general fiber of π. Fix a dominant family of sections C on Y and set $q = -K_{Y/\mathbb{P}^1} \cdot C$. Let $\rho : Y \to \mathbb{F}_e$ be the birational map constructed by Corollary 4.4 applied to C. Then $e \leq q$ and*
  (1) *There exists a section C_0 such that $C \sim_{rat} C_0 + \frac{q+e}{2}F + T$ for some π-vertical effective cycle T.*
  (2) *There exists a dominant family of sections C_1 such that $C \sim_{rat} C_1 + \frac{q-e}{2}F$.*
In both situations the coefficient of F is an integer.

Proof. Pushing forward general deformations of C to $\mathbb{F}_e$, we obtain a dominant family of sections $\widetilde{C}$ on $\mathbb{F}_e$ of height q. Since the minimal height of a moving section of $\mathbb{F}_e$ is e, we have $e \leq q$.

We have $\widetilde{C} \sim_{rat} \widetilde{C}_0 + \frac{q+e}{2}\widetilde{F}$ where $\widetilde{C}_0$ is the rigid section and $\widetilde{F}$ is a fiber of the projective bundle structure. We also have $\widetilde{C} \sim_{rat} \widetilde{C}_1 + \frac{q-e}{2}\widetilde{F}$ where $\widetilde{C}_1$ is the minimal free section of $\mathbb{F}_e$. Pulling back general deformations to Y gives us the two desired expressions. $\qquad\square$

5. Expected dimension of sections

[LT19a] proves that if X is a smooth Fano variety and M is a component of $\mathrm{Mor}(\mathbb{P}^1, X)$ with dimension larger than expected then the curves parametrized by M will all be contained in a subvariety $Y \subset X$ with $a(Y, -K_X) > a(X, -K_X)$. We would like to formulate an analogous result for Fano fibrations using the generic a-invariant with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$. However, the analogy fails in two different ways.

First, there can be components with higher than expected dimension that are not observed by the a-invariant at all.

Example 5.1. Consider the Fano fibration $f : \mathbb{F}_e \to \mathbb{P}^1$ where $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$. Since $\mathbb{F}_{e,\eta} = \mathbb{P}^1_{k(\mathbb{P}^1)}$ there are no subvarieties with larger generic a-invariant. However, the rigid section has higher than expected dimension as soon as $e > 1$.

Examples of this type indicate that our proposed analogy should only address sections with large anticanonical degree. (Note that this setting is sufficient for applications to Manin's Conjecture.)

Second, even sections which have large anticanonical degree and which deform more than expected need not be contained in subvarieties with higher generic a-invariant. The following example shows that such sections can sweep out a subvariety Y whose generic a-invariant is the same as the generic a-invariant of $\mathcal{X}$. As above we will let $\mathbb{F}_e$ denote the Hirzebruch surface defined by $\mathcal{O} \oplus \mathcal{O}(-e)$.

Example 5.2. Let $h : \mathbb{F}_2 \to \mathbb{P}^1$ denote the projection map, let C_0 denote the rigid section of h and let D denote a fiber of h. We define $g : \mathcal{X} \to \mathbb{F}_2$ to be the $\mathbb{P}^1$-bundle defined by $\mathcal{O} \oplus \mathcal{O}(-C_0 - D)$. The composed map $\pi : \mathcal{X} \to \mathbb{P}^1$ realizes X as an $\mathbb{F}_1$-fibration over $\mathbb{P}^1$. We let ζ denote the unique effective section of $\mathcal{O}_{\mathcal{X}/\mathbb{F}_2}(1)$. Note that $K_{\mathcal{X}/\mathbb{P}^1} = -2\zeta + g^*(-3C_0 - 3D)$.

Let $Y \subset \mathcal{X}$ denote the g-preimage of C_0. Then Y is isomorphic to $\mathbb{F}_1$ and we have $K_{\mathcal{X}/\mathbb{P}^1}|_Y = K_{Y/\mathbb{P}^1} + 2T$ where T is the fiber of $Y \to \mathbb{P}^1$. Note that the generic a-invariants of $\mathcal{X}$ and Y with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$ are equal.

Let C be any section of $Y \to \mathbb{P}^1$ whose deformations cover Y. The calculation above shows that the expected dimension of the moduli space of deformations of C on $\mathcal{X}$ is always one less than the actual dimension. Furthermore we can find such sections C with arbitrarily large height.

In light of the examples above, the following conjectural statements should be seen as the correct analogues of [LT19a, Theorem 1.1]:

(1) Any section with sufficiently large anticanonical degree that deforms more than expected will sweep out a subvariety $Y \subset \mathcal{X}$ whose generic a-invariant with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$ is at least as large as that of $\mathcal{X}$.

(2) If Y is an "accumulating subvariety" (in the sense that sections on Y will dominate the expected exponential term in Geometric Manin's Conjecture), then the generic a-invariant of Y with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$ is strictly larger than that of $\mathcal{X}$.

Remark 5.3. Another distinction with the case of trivial families is that one does not expect any sort of converse to hold. That is, there can be subvarieties Y satisfying $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}) > a(X_\eta, -K_{\mathcal{X}/\mathbb{P}^1})$ which do not admit any families of sections with higher than the expected dimension. This is a consequence of the fact that the resolution of the structure map $Y \to \mathbb{P}^1$ need not have connected fibers.

In this section we will prove statements of this type for del Pezzo fibrations.

Theorem 5.4. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively nef. Let M denote a component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing sections C satisfying

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \sup\{-2\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 1, 1\}$$

Let Y denote the closure of the locus swept out by the corresponding sections. Suppose that either

- $Y \subsetneq \mathcal{X}$, or
- $\dim(M) > -K_{\mathcal{X}/\mathbb{P}^1} \cdot C + 2$.

Then $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}|_Y) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1})$.

Proof. Note that every rigid section C will satisfy $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \leq -2$. Due to our restriction on height we see that in either of the two cases of the theorem we must have $\dim(Y) = 2$. Let $\phi : \widetilde{Y} \to Y$ denote a resolution of singularities. Let $\widetilde{C}$ denote the strict transform of a

general deformation of the section C. By assumption the deformations of $\widetilde{C}$ are Zariski dense on $\widetilde{Y}$; thus the natural map $\psi : \widetilde{Y} \to \mathbb{P}^1$ is an algebraic fiber space. Moreover our height bound guarantees that there is at least a 1-parameter family of deformations of $\widetilde{C}$ through two general points of $\widetilde{Y}$. By Bend-and-Break we see that $\widetilde{Y}$ is generically a $\mathbb{P}^1$-bundle over the base. Since the $\widetilde{C}$ dominate $\widetilde{Y}$, we have

$$\dim(M) = -K_{\widetilde{Y}/\mathbb{P}^1} \cdot \widetilde{C} + 1.$$

Thus in either of the two cases in the statement of the theorem we are guaranteed to have

$$(K_{\widetilde{Y}/\mathbb{P}^1} - \phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y) \cdot \widetilde{C} < 0.$$

By applying Theorem 4.5 to $\widetilde{Y}$, we find a Hirzebruch surface $\mathbb{F}_e$ and a birational map $\rho : \widetilde{Y} \to \mathbb{F}_e$ which is an isomorphism on a neighborhood of $\widetilde{C}$. As in the statement of Theorem 4.5 there exists a section $\widetilde{C}_0$ such that $\widetilde{C} \sim_{rat} \widetilde{C}_0 + \frac{q+e}{2} F + T$ for some π-vertical effective cycle T, where

$$q = -K_{Y/\mathbb{P}^1} \cdot \widetilde{C}$$
$$> -K_{\mathcal{X}/\mathbb{P}^1} \cdot \widetilde{C}$$
$$\geq -2\text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 1$$

the last inequality following from our height bounds. In particular, since the coefficient of F is an integer we have $\frac{q+e}{2} \geq -\text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})$. Note that

$$0 > (K_{\widetilde{Y}/\mathbb{P}^1} - \phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y) \cdot \widetilde{C}$$
$$= -q - \phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y \cdot \left(\widetilde{C}_0 + \frac{q+e}{2} F + T \right)$$
$$= e + \frac{q+e}{2} (-\phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y \cdot F - 2) - \phi^* K_{\mathcal{X}/\mathbb{P}^1} \cdot (\widetilde{C}_0 + T)$$
$$\geq e + \frac{q+e}{2} (-\phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y \cdot F - 2) - K_{\mathcal{X}/\mathbb{P}^1} \cdot \phi_* \widetilde{C}_0 \qquad (5.1)$$

where we use the relative nefness of $-K_{\mathcal{X}/\mathbb{P}^1}$ at the last step. Suppose for a contradiction that there is an inequality $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}|_Y) < a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1})$. This is equivalent to saying that $-\phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y \cdot F \geq 3$. Then by combining the two inequalities above we obtain

$$0 > e + \frac{q+e}{2} + \text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) \geq e \geq 0$$

yielding a contradiction. $\qquad\qquad\square$

The most interesting situation in Theorem 5.4 is when $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}|_Y) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1})$. In this case the same argument gives a little more:

Theorem 5.5. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively nef. Let M denote a component of $\text{Sec}(\mathcal{X}/\mathbb{P}^1)$. Suppose that the sections C parametrized by M satisfy*

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \sup\{-2\text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 1, 1\}$$

and sweep out a surface $Y \subsetneq X$ satisfying

$$a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}|_Y) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1}).$$

Then there is a component M' of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing sections C_1 satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C_1 < -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 1$ such that the corresponding sections sweep out Y. Furthermore, the difference between the expected and actual dimension of M is the same as the difference between the expected and actual dimension of M'.

Proof. We return to the setting of the proof of Theorem 5.4, keeping the constructions and notation established there. In this situation we have $-\phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y \cdot F = 2$ and

$$(K_{\widetilde{Y}/\mathbb{P}^1} - \phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y) \cdot F = 0.$$

Let $\widetilde{C}_1$ denote the strict transform on $\widetilde{Y}$ of a general moving section C_1 of $\mathbb{F}_e$ of minimal height. Since ρ is an isomorphism on a neighborhood of $\widetilde{C}_1$, we have

$$0 > (K_{\widetilde{Y}/\mathbb{P}^1} - \phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y) \cdot \widetilde{C} = (K_{\widetilde{Y}/\mathbb{P}^1} - \phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y) \cdot \widetilde{C}_1 = -e - \phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y \cdot \widetilde{C}_1.$$

On the other hand by Equation (5.1) in the proof of Theorem 5.4 we also have

$$0 > e + \mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}).$$

Altogether it follows that

$$-\phi^* K_{\mathcal{X}/\mathbb{P}^1}|_Y \cdot \widetilde{C}_1 < -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 1.$$

Let C_1 be the image in $\mathcal{X}$ of $\widetilde{C}_1$. We let M'_Y denote the component of $\mathrm{Sec}(\widetilde{Y}/\mathbb{P}^1)$ parametrizing deformations of $\widetilde{C}_1$ and let M' denote the component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ which parametrizes deformations of C_1. Since $\widetilde{C}$ and $\widetilde{C}_1$ differ by mF for some integer m, we have

$$\dim(M) = -K_{\widetilde{Y}/\mathbb{P}^1} \cdot \widetilde{C} + 1 = -K_{\widetilde{Y}/\mathbb{P}^1} \cdot \widetilde{C}_1 + 2m + 1 = \dim(M'_Y) + 2m$$
$$\mathrm{expdim}(M) = -K_{\mathcal{X}/\mathbb{P}^1} \cdot C + 2 = -K_{\mathcal{X}/\mathbb{P}^1} \cdot C' + 2m + 2 = \mathrm{expdim}(M') + 2m$$

Note that there is a natural pushforward map $M'_Y \to M'$. We claim that in fact this map is birational. In other words, we must show that Y is the closure of the locus swept out by the sections defined by M'. It is clear that this locus contains Y, and we only must verify that it is not a larger set. We know that $\dim(M) - \mathrm{expdim}(M) \geq 0$, since any family of curves on $\mathcal{X}$ deforms at least as much as the expected dimension. By the equations above $\dim(M'_Y) - \mathrm{expdim}(M') \geq 0$ as well. Suppose for a contradiction that M' parametrizes a dominant family of sections. Then $\dim(M') = \mathrm{expdim}(M')$ so that $\dim(M'_Y) - \dim(M') \geq 0$. Since the reverse inequality is also true, we deduce $\dim(M'_Y) = \dim(M')$. But then a general deformation of C_1 in $\mathcal{X}$ should be coming from a deformation of $\widetilde{C}_1$ in Y, contradicting the fact that M' parametrizes a dominant family. Thus the curves parametrized by M' cannot sweep out all of $\mathcal{X}$ and must be contained in Y. The final statement is a consequence of the identification $\dim(M'_Y) = \dim(M')$ and the earlier equations. $\square$

Example 5.6. We return to Example 5.2, keeping the notation established there. Recall that the fibers T of $Y \to \mathbb{P}^1$ satisfy $-K_{\mathcal{X}/\mathbb{P}^1} \cdot T = -K_{Y/\mathbb{P}^1} \cdot T$. Thus if we take any fixed section C in Y, glue on copies of the free curve T, and smooth the resulting curve, the difference between the expected dimension of deformations in $\mathcal{X}$ and the actual dimension of deformations in Y stays constant.

Conversely, starting from any movable section in Y, we can repeatedly break off copies of T. Eventually we will obtain a movable section of smallest degree, namely $\zeta|_Y$. This section

satisfies $-K_{\mathcal{X}/\mathbb{P}^1} \cdot \zeta|_Y = -1$. This verifies Theorem 5.5 which claims that there is a section of height less than $-\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 1 = 2$ whose deformations cover Y.

Remark 5.7. Note that in the situation of Theorem 5.5 the difference between the actual and expected dimension is constant for the sections which sweep out Y. This means that the contributions of Y to the counting function in Geometric Manin's Conjecture have the same exponential term as the expected value. In other words, the "accumulating subvarieties" which contribute a larger exponential term are exactly the subvarieties with larger generic a-value.

5.1. **Constructing a closed set.** Combining previous results, we prove Theorem 1.1.

Proof of Theorem 1.1: By Theorem 5.4, any component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing a non-dominant family will satisfy one of the following properties:

(1) M will parametrize sections C satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C < \sup\{-2\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 2, 1\}$, or

(2) M will parametrize sections C satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \sup\{-2\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 2, 1\}$ which sweep out a 2-dimensional subvariety Y satisfying $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}|_Y) > a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1})$, or

(3) M will parametrize sections C satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \sup\{-2\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 2, 1\}$ which sweep out a 2-dimensional subvariety Y satisfying $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}|_Y) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1})$.

Lemma 2.3 shows that curves of the first type lie in a bounded family. Thus, the union of the subvarieties swept out by the non-dominant families will be a proper closed subset of $\mathcal{X}$. Lemma 3.4 shows that the subvarieties defined by the components of the second type will lie in a proper closed subset of $\mathcal{X}$. We still need to address the third type.

Suppose that Y is a 2-dimensional subvariety satisfying $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}|_Y) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1})$ that is swept out by the sections parametrized by M. By Theorem 5.5, we know that there will be a component $M' \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing a non-dominant family of sections C which satisfy $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C < -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 1$ and which sweep out Y. By Lemma 2.3 we conclude that the union of all such subvarieties Y is a proper closed subset of $\mathcal{X}$. $\qquad\square$

6. STABLE MAPS THROUGH GENERAL POINTS

Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration. Suppose that $f : C \to \mathcal{X}$ is a genus 0 stable map whose image contains n general points of $\mathcal{X}$. In this section we study what restrictions this condition imposes on C and f. We first discuss the bound for irreducible curves:

Lemma 6.1. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration. Let $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ denote a component such that for any n general points of $\mathcal{X}$ there exists a member of M containing those points. Then the curves C parametrized by M have height $\geq 2n - 2$ with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$. Furthermore, if the height is exactly $2n - 2$ then there are only finitely many curves parametrized by M through n fixed general points and if the height is $2n - 1$ then there is sublocus of M of dimension at most 1 parametrizing curves through n fixed general points.*

Proof. Since the points are general they will impose independent conditions on M, so that $\dim(M) \geq 2n$. Since M parametrizes a dominant family of curves, it has the expected dimension, showing that

$$2n \leq -K_{\mathcal{X}/\mathbb{P}^1} \cdot C + 2.$$

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The last two statements are clear. $\qquad\square$

To understand reducible curves takes more preparation.

Definition 6.2. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration. Fix an integer d. We let $\mathrm{maxdef}(d)$ denote the maximum dimension of any component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing sections of height d. When there is no section of height d, we simply set $\mathrm{maxdef}(d) = -\infty$. We also define
$$\mathrm{maxdef}(\mathcal{X}) = \sup_{d<0} \mathrm{maxdef}(d).$$

Lemma 6.3. *Let* $\pi : \mathcal{X} \to \mathbb{P}^1$ *be a del Pezzo fibration such that* $-K_{\mathcal{X}/\mathbb{P}^1}$ *is relatively ample. Fix an integer* $d < 0$ *and a positive integer* $n \geq \mathrm{maxdef}(d)$. *Suppose that we have a connected effective curve* $C = C_0 + \sum a_i T_i$ *where* C_0 *is a section of height* d *and each* T_i *is a* π-*vertical curve. If* C *contains* n *general points* $\{x_j\}_{j=1}^n$ *of* $\mathcal{X}$ *then* $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq d + 3n - \mathrm{maxdef}(d)$.

Proof. Let M be the component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ containing C_0. Since $d < 0$ by assumption we know that M does not parametrize a dominant family. First suppose that the curves parametrized by M sweep out a surface $Y \subset \mathcal{X}$. Note that deformations of C_0 can not contain any general points of $\mathcal{X}$. Thus each general point x_j is contained in a different free vertical curve T_j.

Suppose that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot T_j = 2$. There is only a finite set of conics in a del Pezzo surface through a fixed general point x_j. By generality of x_j the intersection points of these conics with Y form a finite set. To ensure that C is connected, C_0 must contain one of this finite set of points. By generality, the conditions imposed on C_0 by insisting that C_0 contain the attachment points for conics T_j will be independent for different j. Thus we see that there can be at most $\mathrm{maxdef}(d)$ such conics T_j in the curve C. This proves that
$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq d + 2\mathrm{maxdef}(d) + 3(n - \mathrm{maxdef}(d))$$

When C_0 is a rigid curve, the argument is similar but easier since every component T_j containing a general point must satisfy $-K_{\mathcal{X}/\mathbb{P}^1} \cdot T_j \geq 3$. $\qquad\square$

The two following propositions describe which sections can pass through the maximal number of general points of $\mathcal{X}$. The proof of [Tia12, Theorem 4.1] establishes a related statement for a particular type of curves of minimal height. The first proposition handles the case of even height.

Proposition 6.4. *Let* $\pi : \mathcal{X} \to \mathbb{P}^1$ *be a del Pezzo fibration such that* $-K_{\mathcal{X}/\mathbb{P}^1}$ *is relatively ample. Fix a positive integer*
$$n \geq \mathrm{maxdef}(\mathcal{X}) + 2 + \sup\{0, -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})\}.$$
Suppose that $f : C \to \mathcal{X}$ *is a genus* 0 *stable map whose image has anticanonical height* $2n - 2$ *such that the unique component of* C *whose image is not* π-*vertical maps birationally to a section. Then:*

(1) *Suppose the image of* C *contains* n *general points of* $\mathcal{X}$. *Then* f *is a birational map to a free section.*

(2) *Fix a general curve* Z *in a basepoint free linear series in a general fiber of* π. *Suppose the image of* C *contains* $n - 1$ *general points of* $\mathcal{X}$ *and intersects* Z. *Suppose also that the image of* C *is reducible and at least one of the general points is contained*

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in a π-vertical component of C. Then C has exactly two components and f maps one component birationally onto a free section and the other birationally onto either a free conic or a free cubic in a general fiber of π.

Construction 6.5. Before giving the proof, we clarify what "general" means in the statement of the proposition. We may ensure that n general points and a general member Z of a basepoint free linear series in a general fiber of π satisfy the following conditions:

(1) We may ensure that any section of height at most $2n - 2$ that contains one of our points is free.
(2) Suppose we fix any subset of our set of points of size d. Then we may ensure that there are only finitely many sections of height $2d - 2$ which contain this subset of points, and there are finitely many loci of dimension ≤ 1 in $\overline{M}_{0,0}(X)$ that parametrize sections of height $2d - 1$ which contain this set of points.
(3) We may ensure that every fiber of π containing one of our points is a smooth del Pezzo surface F and that the point does not lie on any line in F. Furthermore, we may ensure that the finite set of conics in F through the point have the expected dimension of intersection against the sections described in (2); namely, for any 0-dimensional component of the parameter space the corresponding section is disjoint from each conic, and for any 1-dimensional component of the parameter space there are only finitely many sections intersecting each conic. Furthermore, we may ensure that there are only finitely many cubics in F through the general point that intersect any section from (2) which is parametrized by a 0-dimensional subset.
(4) We may ensure that the sections in (2) meet Z in the expected dimension: for any 0-dimensional component of the parameter space the corresponding section is disjoint from Z, and for any 1-dimensional component of the parameter space there are only finitely many sections intersecting Z.
(5) We may ensure that there is no section as in (2) intersecting conics through two different points in our set. Furthermore, we may ensure that if a section as in (2) intersects a conic through a general point then this section does not intersect Z or a line meeting Z.
(6) Since $n \geq \mathrm{maxdef}(\mathcal{X})$, we may ensure that for any $d < 0$ and any subset of our set of points of size $\geq \mathrm{maxdef}(d)$ the points are general in the sense of Lemma 6.3.

It is straightforward to ensure that these conditions can be satisfied by constructing a suitable incidence correspondence and doing a dimension count. The key point is that for any family of free curves it is a codimension 1 condition to intersect any fixed curve.

Proof of Proposition 6.4: (1). It suffices to show that the image of f is irreducible. Suppose otherwise, so that $f(C) = C_0 + \sum_{i \in I} a_i T_i$ for some π-vertical curves T_i. Let d denote the height of C_0 and set $t_i = -K_{\mathcal{X}/\mathbb{P}^1} \cdot T_i$, so that

$$2n - 2 = d + \sum_{i \in I} a_i t_i.$$

Note that C_0 can contain at most $\sup\{0, \lfloor \frac{d}{2} \rfloor + 1\}$ general points of $\mathcal{X}$. An irreducible π-vertical curve T_i can contain at most 1 general point and if it does then $t_i \geq 2$. Let $I' \subset I$ denote the set of vertical curves that contain one of the general points.

We now break the argument into several cases.

Case 1: $d \geq 0$. Then the number of general points contained in $f(C)$ is bounded above by the number of general points contained in C_0 and in the T_i. Thus:

$$\sup\left\{0, \left\lfloor \frac{d}{2} + 1 \right\rfloor\right\} + |I'| \geq n$$
$$= \frac{d}{2} + 1 + \sum_{i \in I} a_i t_i / 2.$$

Since $d \geq 0$ the RHS is an upper bound for the LHS. Thus the inequality above must be an equality. This means that d is even and C_0 goes through the maximum number of points possible, that each $a_i = 1$, and that each component of T is a free vertical curve through one of the general points with $t_i = 2$. In particular, the set of $\frac{d}{2} + 1$ general points determines a finite number of possibilities for C_0, and each vertical curve is also determined by a general point up to a finite set of possibilities. If there are any vertical components, then for general choices $f(C)$ will not be connected, an impossibility. Thus $f(C)$ is irreducible.

Case 2: $d < 0$. Due to our lower bound on n we may apply Lemma 6.3. It shows that C must have height

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \operatorname{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) + 3n - \operatorname{maxdef}(\mathcal{X})$$
$$> 2n - 2$$

contradicting our assumption on the height of C.

(2). Write $f(C) = C_0 + \sum_{i \in I} a_i T_i$ for some π-vertical curves T_i. Let d denote the height of C_0 and set $t_i = -K_{\mathcal{X}/\mathbb{P}^1} \cdot T_i$, so that

$$2n - 2 = d + \sum_{i \in I} a_i t_i.$$

Let $I' \subset I$ denote the set of vertical curves that contain one of the general points. Again we separate into cases:

Case 1: $d \geq 0$ and C_0 intersects Z. Since C_0 intersects Z it can contain at most $\sup\{0, \lfloor \frac{d+1}{2} \rfloor\}$ general points of $\mathcal{X}$. The number of general points contained in $f(C)$ is bounded above by the number of general points contained in C_0 and in the T_i. Thus:

$$\sup\left\{0, \left\lfloor \frac{d+1}{2} \right\rfloor\right\} + |I'| \geq n - 1$$
$$\geq \frac{d}{2} + \sum_{i \in I} a_i t_i / 2.$$

If d is even, then we must have equality everywhere. This means that set of deformations of C_0 which meet Z and go through the maximal number of points is at most 1-dimensional, that every $a_i = 1$, and that each component of T is a free vertical curve through one of the general points with $t_i = 2$. However, for C_0 to meet a vertical conic through a general point is a codimension 1 condition, so by generality there can be at most one vertical component and we obtain the desired expression.

If d is odd, then C_0 must contain the maximal number of points $\lfloor \frac{d+1}{2} \rfloor$. This means that the difference between $|I'|$ and $\sum a_i t_i / 2$ can be at most $1/2$. Thus there are only three options for the vertical components:

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(1) every component of T has anticanonical degree 2 and contains a general point, or
(2) every component of T but one has anticanonical degree 2 and contains a general point, and the last one has anticanonical degree 3 and contains a general point, or,
(3) every component of T but one has anticanonical degree 2 and contains a general point, and the last component has anticanonical degree 1.

Note that there are only finitely many deformations of C_0 which meet Z and go through $\frac{d+1}{2}$ points. Recall that by assumption there is a vertical component of $f(C)$ through a general point. However, since there are only finitely many vertical conics through a general point, by generality no such conic can intersect C_0. This rules out the first and third situations, showing that C must be the union of a free section and a cubic in a fiber.

Case 2: $d \geq 0$ and C_0 does not intersect Z. Just as before, the number of general points contained in $f(C)$ is bounded above by the number of general points contained in C_0 and in the T_i. Thus:

$$\sup\left\{0, \left\lfloor \frac{d}{2} \right\rfloor + 1\right\} + |I'| \geq n - 1$$

$$\geq \frac{d}{2} + \sum_{i \in I} a_i t_i / 2.$$

Since C_0 does not intersect Z, there must be a vertical curve that does intersect Z but does not contain any general points, so that

$$\sum_{i \in I} a_i t_i / 2 - |I'| \geq \sum_{i \in I} a_i t_i / 2 - \sum_{i \in I'} a_i t_i / 2 \geq 1/2.$$

Thus when d is even, C_0 must contain the maximal number of points $\frac{d}{2} + 1$. In this situation the difference between $|I'|$ and $\sum_{i \in I'} a_i t_i / 2$ is at most $1/2$. There are four options for the vertical components:

(1) every component of T but one has anticanonical degree 2 and contains a general point, and the last has anticanonical degree 1 and meets Z.
(2) every component of T but two has anticanonical degree 2 and contains a general point, one has anticanonical degree 1 and meets Z, and the last one has anticanonical degree 3 and contains a general point,
(3) every component of T but two has anticanonical degree 2 and contains a general point, and the last two components have anticanonical degree 1, one of which meets Z,
(4) every component of T has anticanonical degree 2, all but one contain a general point, and the last component meets Z.

Since by generality there are only finitely many deformations of C_0 through the required number of points, such C_0 can not intersect a vertical line meeting Z or a vertical conic through a general point. This rules out the first three cases immediately, and the fourth is also ruled out since by assumption there exists at least one vertical curve which contains some general point.

When d is odd then C_0 must contain the maximal number of points $\lfloor \frac{d}{2} \rfloor + 1$. In this case there is only one option: every component of T but one has anticanonical degree 2 and contains a general point, and the last has anticanonical degree 1 and meets Z. However, since by generality C_0 can only deform in a one-parameter family while containing the maximal

number of points, it is impossible for C_0 to meet both a line intersecting Z and a conic through a general point. Since by assumption $f(C)$ contains a vertical component through a general point, this case is also ruled out.

Case 3: $d < 0$. In this case Lemma 6.3 shows that C must have height

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) + 3(n-1) - \mathrm{maxdef}(\mathcal{X})$$
$$> 2n - 2$$

proving the impossibility of this case. □

The next proposition is the analogue of Proposition 6.4 for sections of odd height.

Proposition 6.6. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Fix a positive integer*

$$n \geq \mathrm{maxdef}(\mathcal{X}) + \sup\{0, -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})\}.$$

Suppose that $f : C \to \mathcal{X}$ is a genus 0 stable map whose image has anticanonical height $2n-1$ such that the unique component of C whose image is not π-vertical maps birationally to a section. Then:

(1) *Fix a general curve Z in a basepoint free linear series in a general fiber of π. Suppose the image of C contains n general points of $\mathcal{X}$ and intersects Z. Then f is a birational map to a free section.*

(2) *Suppose the image of C contains n general points of $\mathcal{X}$. Suppose also that the image of C is reducible and at least one of the general points is contained in a π-vertical component of C. Then C has exactly two components and f maps one component birationally onto a free section and the other birationally onto a free conic or a free cubic in a general fiber of π.*

The proof is essentially the same as the proof of Proposition 6.4, but slightly easier.

7. Movable Bend-and-Break for del Pezzo fibrations

The following conjecture is essential for understanding sections of Fano fibrations.

Conjecture 7.1 (Movable Bend-and-Break for sections). Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a Fano fibration. There is a constant $Q = Q(\mathcal{X})$ such that the following holds. Suppose that C is a movable section of π satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C > Q(\mathcal{X})$. Then C deforms (as a stable map) to a chain of free curves with at least 2 components.

We establish Movable Bend-and-Break for sections of del Pezzo fibrations such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. The proof is based on techniques from [She12] and [LT19b].

Theorem 7.2. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Define*

$$Q(\mathcal{X}) = \sup\{3, -2\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 5,$$
$$- \mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) + 3,$$
$$2\mathrm{maxdef}(\mathcal{X}) - 5\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 5,$$
$$2\mathrm{maxdef}(\mathcal{X}) - \mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 3,$$
$$2\mathrm{maxdef}(\mathcal{X}) + 2 + 2\sup\{0, -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})\}\}.$$

Suppose that $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ is a component that parametrizes a dominant family of sections C satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq Q(\mathcal{X})$. Then the closure of M in $\overline{M}_{0,0}(\mathcal{X})$ contains a point representing a stable map whose domain has exactly two components each mapping birationally onto a free curve.

For later applications it will be convenient to introduce the following definition:

Definition 7.3. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. We define $\mathrm{MBBbound}(\mathcal{X})$ to be the smallest non-negative integer r such that any component of $\overline{M}_{0,0}(\mathcal{X})$ which generically parametrizes free sections of height at least r will also contain a stable map whose domain has exactly two components each mapping birationally to a free curve.

Thus Theorem 7.2 establishes that $\mathrm{MBBbound}(\mathcal{X})$ exists and gives an explicit upper bound $\mathrm{MBBbound}(\mathcal{X}) \leq Q(\mathcal{X})$.

Proof of Theorem 7.2: Suppose that M is a component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing a dominant family of sections C that satisfy

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq Q(\mathcal{X}).$$

In particular, the general section in the family is free. Since each section is smooth we can consider its normal bundle, and we separate into several cases based on the normal bundle of the general curve C. Write $N_{C/\mathcal{X}} = \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $0 \leq a \leq b$. Our height bound implies that if $a = 0$ then $b \geq 3$ and if $a = 1$ then $b \geq 2$.

Case 1: $a \geq 1$ and $b \geq a + 2$.

Suppose we fix $a + 1$ general points of X and choose a curve C with the generic normal bundle containing these points. According to [She12, Remark 2.2], the locus parametrizing curves with the generic normal bundle that contain all $a + 1$ points has a unique smooth irreducible component that contains C. The closure of the locus swept out by these curves is an irreducible surface Σ. Let $\Sigma' \to \Sigma$ denote the normalization map. [She12, Corollary 2.7] shows that if C is a general curve through these points then Σ' is smooth along the strict transform C' of C and the normal bundle of C' in Σ' is $\mathcal{O}(b)$. These properties are preserved by passing to the resolution $\nu : \widetilde{\Sigma} \to \Sigma$ and the strict transform $\widetilde{C}$ in $\widetilde{\Sigma}$. Note that $\widetilde{\Sigma}$ is covered by curves with degree one over the base $\mathbb{P}^1$, so that the map $\psi : \widetilde{\Sigma} \to \mathbb{P}^1$ is an algebraic fiber space. Furthermore, since $b \geq 3$ by the usual Bend-and-Break we know that the general fiber of ψ must be $\mathbb{P}^1$. Since Σ contains a general point, we may assume that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot F \geq 2$ where F is a general fiber of ψ.

As in the statement of Theorem 4.5 there exists a birational map $\rho : \widetilde{Y} \to \mathbb{F}_e$, a moving section $\widetilde{C}_1$ of $\widetilde{Y}$ such that $\widetilde{C} \sim_{rat} \widetilde{C}_1 + \frac{b-e}{2}F$, and a section $\widetilde{C}_0$ such that $\widetilde{C} \sim_{rat} \widetilde{C}_0 + \frac{b+e}{2}F + T$ for some π-vertical curve T. Thus

$$a = -b - \nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot \widetilde{C}$$

$$= -b + \frac{b+e}{2}\left(-\nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot F\right) - \nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot (\widetilde{C}_0 + T)$$

$$= e + \frac{b+e}{2}\left(-\nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot F - 2\right) - \nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot (\widetilde{C}_0 + T)$$

$$\geq e + \frac{b+e}{2}\left(-\nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot F - 2\right) - \nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot \widetilde{C}_0 \qquad (7.1)$$

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where the last inequality follows from the fact that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample.

Claim 7.4. We have $b > e$.

The statement of Theorem 4.5 guarantees that $b \geq e$; suppose for a contradiction that $b = e$. We separate the argument into two cases. First suppose that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot F \geq 3$. Then the inequality (7.1) yields

$$a \geq 2b - \nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot \widetilde{C}_0 \geq 2b + \mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}).$$

Since also $b - 2 \geq a$, we deduce that $b \leq -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 2$. Thus

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C = a + b \leq 2b - 2 \leq -2\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 6$$

which contradicts with our height bounds on C.

Now suppose that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot F = 2$. Then the inequality (7.1) yields

$$a - b \geq -\nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot \widetilde{C}_0 \tag{7.2}$$

and in particular, $-2 \geq -\nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot \widetilde{C}_0$. Recall that Σ contains $a + 1$ general points of $\mathcal{X}$, and that their preimages give a set of $a + 1$ distinct points in $\widetilde{\Sigma}$. Since $b > a + 1$, we can deform the b fibers F in our broken curve to ensure that they meet these $a + 1$ points. By taking images in $\mathcal{X}$, we find a broken curve which is a deformation of C through $a+1$ general points of $\mathcal{X}$ such that the section $\nu_* \widetilde{C}_0$ in this broken curve has height ≤ -2. According to Lemma 6.3, this implies that either

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) + 3(a + 1) - \mathrm{maxdef}(\mathcal{X})$$

or $a + 1 < \mathrm{maxdef}(\mathcal{X})$. Furthermore by the inequality (7.2)

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C = a + b$$
$$\leq 2a - \mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})$$

Comparing the two bounds we see that

$$a \leq \sup\{\mathrm{maxdef}(\mathcal{X}) - 2\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 3, \mathrm{maxdef}(\mathcal{X}) - 2\}$$

which in turn implies that

$$-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \leq \sup\{2\mathrm{maxdef}(\mathcal{X}) - 5\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 6, 2\mathrm{maxdef}(\mathcal{X}) - \mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) - 4\}.$$

This possibility is ruled out by our degree bounds and finishes the proof of the claim.

We now return to the main argument. Since $\frac{b-e}{2}$ is an integer, Claim 7.4 shows that it is at least 1. If $\frac{b-e}{2} > 1$, then we smooth $\widetilde{C}_1 + (\frac{b-e}{2} - 1)F$ to obtain a section $\widetilde{C}_2$. If $\frac{b-e}{2} = 1$, we set $\widetilde{C}_2 = \widetilde{C}_1$. In either case we have $\widetilde{C} \sim_{rat} \widetilde{C}_2 + F$. We claim that the ν-images of $\widetilde{C}_2$ and F are movable curves in $\mathcal{X}$. Recall that Σ contains a general point of $\mathcal{X}$, and in particular, Σ is not contained in the closed set constructed by Theorem 1.1. Since $\nu_* \widetilde{C}_2$ deforms to cover Σ, its image in $\mathcal{X}$ must deform to cover $\mathcal{X}$. Similarly, F deforms to cover Σ, and thus contains a general point of $\mathcal{X}$. Since the $-K_{\mathcal{X}/\mathbb{P}^1}$ degree of F admits an a priori upper bound in terms of the height of C and in terms of $\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})$, this implies that F is a movable curve.

Case 2: $a = 0$ and $b \geq 3$.

The argument is very similar to Case 1. Fix a general point x and consider a component Σ of the locus swept out by the curves in M through x with generic normal bundle. Lemma

2.5 shows that $\Sigma \subsetneq \mathcal{X}$. Since there is a b-dimensional family of curves through x with generic normal bundle, Σ must be a surface.

Let $\nu : \widetilde{\Sigma} \to \Sigma$ be a resolution; by possibly blowing up further we may suppose that the preimage of x is divisorial, so that the condition for a curve on $\widetilde{\Sigma}$ to go through $x \in \Sigma$ is a numerical condition. The strict transforms $\widetilde{C}$ of the general sections sweeping out Σ will define a dominant family of sections of $\widetilde{\Sigma}$ that deforms in dimension b. Thus we have $-K_{\widetilde{\Sigma}/\mathbb{P}^1} \cdot \widetilde{C} = b - 1$. Note that the induced map $\psi : \widetilde{\Sigma} \to \mathbb{P}^1$ has connected fibers since $\widetilde{\Sigma}$ is dominated by sections of ψ. Also, since $b - 1 \geq 2$ by the usual Bend-and-Break we know that the general fiber of ψ is $\mathbb{P}^1$.

As in the statement of Theorem 4.5 there exists a birational map $\rho : \widetilde{Y} \to \mathbb{F}_e$, a moving section $\widetilde{C}_1$ of $\widetilde{Y}$ such that $\widetilde{C} \sim_{rat} \widetilde{C}_1 + \frac{b-e-1}{2}F$, and a section $\widetilde{C}_0$ such that $\widetilde{C} \sim_{rat} \widetilde{C}_0 + \frac{b+e-1}{2}F + T$ for some π-vertical curve T. The analogue of inequality (7.1) is

$$0 \geq e - 1 + \frac{b + e - 1}{2}\left(-\nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot F - 2\right) - \nu^* K_{\mathcal{X}/\mathbb{P}^1} \cdot \widetilde{C}_0. \qquad (7.3)$$

We claim that $b - 1 > e$. Assume for a contradiction that $b - 1 = e$. First suppose that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot F \geq 3$. By inequality (7.3) we deduce that $2e \leq 1 - \text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})$ and thus

$$-K_{\mathcal{X}/\mathbb{P}^1}.\widetilde{C} \leq -\frac{1}{2}\text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) + \frac{3}{2}$$

contradicting our height bound. Next suppose that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot F = 2$. Then by inequality (7.3) we deduce $e \leq 1 - \text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})$ and thus

$$-K_{\mathcal{X}/\mathbb{P}^1}.\widetilde{C} \leq -\text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) + 2$$

contradicting our height bound.

Since $\frac{b-e-1}{2}$ must be an integer our claim implies that it is at least 1, and we conclude the argument in the same way as before.

Case 3: $a \geq 1$ and $b - a \leq 1$.

First suppose that $a = b$; in this case our height bounds imply that

$$a \geq \text{maxdef}(\mathcal{X}) + 1 + \sup\{0, -\text{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})\}.$$

Fix a general points of $\mathcal{X}$ and fix a general curve Z in a basepoint free linear series in a general fiber of π. There is a union of one-parameter family of deformations of C containing a general points and meeting Z. Fix one component of this parameter space.

By applying the Bend-and-Break theorem (as in Corollary 4.2), we can deform $C \sim_{alg} C_0 + T$, where:

- C_0 is a section,
- T is an effective π-vertical curve and at least one component contains one of our a general points.

By Proposition 6.4 we see that the stable map corresponding to $C_0 + T$ has the desired form.

The case when $b - a = 1$ is similar. In this case there is a one-parameter family of sections containing $a + 1$ general points. By considering the broken curve $C_0 + T$ and applying Proposition 6.6, we can repeat the argument to obtain the same conclusion. $\qquad \square$

7.1. Consequences of Movable Bend-and-Break.

Definition 7.5. A comb is a nodal curve consisting of one "central curve" C_0 and of $r \geq 0$ "teeth" $T_1, \ldots, T_r$ such that each T_i intersects C_0 at a single point and there are no other intersections. For the purposes of this paper every tooth of a comb will be a rational curve.

By repeating the breaking argument of Theorem 1.2 inductively, we obtain:

Corollary 7.6. *Let* $\pi : \mathcal{X} \to \mathbb{P}^1$ *be a del Pezzo fibration such that* $-K_{\mathcal{X}/\mathbb{P}^1}$ *is relatively ample. Suppose that* C *is a section of* π *satisfying* $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \mathrm{MBBbound}(\mathcal{X})$ *whose deformations dominate* $\mathcal{X}$. *Then* C *deforms as a stable map to a morphism* $f : C \to \mathcal{X}$ *whose domain* C *is a comb* $C_0 \cup T_1 \cup \ldots \cup T_r$ *where*
- *f maps each component of C birationally onto a free curve,*
- *the image of C_0 is a section satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C_0 < \mathrm{MBBbound}(\mathcal{X})$, and*
- *the image of each T_i is a free π-vertical curve.*

Proof. Let $\overline{M}$ be a component of $\overline{M}_{0,0}(\mathcal{X})$ containing a stable map $f : C \to X$ such that C is a comb, the central curve C_0 is mapped birationally to a free section of π, and the teeth $T_1, \ldots, T_r$ are mapped birationally to free π-vertical curves. (Our original assumption is that $\overline{M}$ contains such a curve with zero teeth.) Suppose that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C_0 \geq \mathrm{MBBbound}(\mathcal{X})$. We show that $\overline{M}$ also contains a comb C' of the same form such that the central curve C_0' has strictly smaller degree against $-K_{\mathcal{X}/\mathbb{P}^1}$.

Let $\overline{M}_i$ denote the component of $\overline{M}_{0,0}(\mathcal{X})$ containing the curve C_0 (when $i = 0$) or T_i (when $i \geq 1$). Each such component generically parametrizes irreducible free curves. Let $\overline{M}_i^{(k)}$ denote the unique component of $\overline{M}_{0,k}(\mathcal{X})$ lying over $\overline{M}_i$. Since the comb $f : C \to \mathcal{X}$ is a smooth point of the moduli space, there is a component $\widetilde{M}$ of

$$\overline{M}_0^{(r)} \times_{\mathcal{X}^{\times r}} \left(\prod_{i \geq 1} \overline{M}_i^{(1)} \right)$$

that maps into $\overline{M}$. Note that $\widetilde{M}$ dominates each component of the product under the corresponding projection map: indeed, since the evaluation map

$$\prod_{i \geq 1} \overline{M}_i^{(1)} \to \mathcal{X}^{\times r}$$

is flat on the locus parametrizing free curves, every component of the product containing a union of free curves will have the expected dimension and will thus map dominantly onto every factor. By Theorem 1.2, the component $\overline{M}_0^{(r)}$ contains in its smooth locus a general point of a component of $\overline{N}_0^{(r+1)} \times_{\mathcal{X}} \overline{N}_1^{(1)}$ where $\overline{N}_0$ generically parametrizes free sections of smaller $-K_{\mathcal{X}/\mathbb{P}^1}$-degree and $\overline{N}_1$ generically parametrizes free π-vertical curves. Since the locus $\overline{N}_0^{(r+1)} \times_{\mathcal{X}} \overline{N}_1^{(1)}$ has codimension 1 in $\overline{M}_0^{(r)}$ and the projection map to $\overline{M}_0^{(r)}$ is flat on the locus of free curves, a dimension count shows that a general point in the preimage of $\overline{N}_0^{(r+1)} \times_{\mathcal{X}} \overline{N}_1^{(1)}$ in $\widetilde{M}$ represents a comb of free curves. $\square$

The next statement is similar to Theorem 1.2, but we additionally impose an upper bound on the height of C_0.

Theorem 7.7. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Suppose that C is a section of π whose deformations dominate $\mathcal{X}$ that satisfies $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \mathrm{MBBbound}(\mathcal{X})$. Then C deforms (as a stable map) to a union of a free section C_0 satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C_0 < \mathrm{MBBbound}(\mathcal{X})$ with a free π-vertical moving curve.*

We prove this by regluing the π-vertical curves in the setting of Corollary 7.6.

Proof. Let $\overline{M} \subset \overline{M}_{0,0}(\mathcal{X})$ be a component containing a point $f : \mathbb{P}^1 \to \mathcal{X}$ which is a birational map onto a section as in the statement of Theorem 1.2. Corollary 7.6 shows that $\overline{M}$ contains a smooth point representing a comb of $r + 1$ free curves such that the section C_0 satisfies the desired bound. We prove by induction on r that $\overline{M}$ contains a point representing a union of two free curves such that the section C_0 satisfies the desired bound. In the base case when $r = 1$ there is nothing to prove.

For the induction step, label the π-vertical free curves as $T_1, \ldots, T_r$. Since all the components are free, we may deform the comb so that C_0 and T_1 intersect at a general point of X. Suppose we then deform T_2 along C_0 until its attaching point coincides with that of T_1. Due to the generality of $C_0 \cap T_1$, we may ensure that the corresponding deformation of T_2 through this point is an irreducible free curve T_2'. Thus, the resulting stable map $f' : C' \to X$ will have $r + 2$ components: there will be one curve $E \subset C'$ which is contracted by f and which is attached to the components mapping to C_0, T_1, T_2'. Note that f' represents a smooth point of $\overline{M}_{0,0}(X)$. Thus the component M also contains a point representing a stable map whose image is a comb $C_0 \cup T_1 \cup T_2'' \cup T_3 \cup \ldots \cup T_r$ where we attach an irreducible free deformation T_2'' of T_2 to a general point of T_1. Note that this is again a smooth point of $\overline{M}_{0,0}(X)$. Since T_1 and T_2'' are free curves in a fiber, we can glue them while fixing the attaching point with C_0 by [Kol96, II.7.6.1 Theorem], yielding a comb of the form $C_0 \cup T_* \cup T_3 \cup \ldots \cup T_r$ where T_* is a free curve in a fiber. This point is also contained in $\overline{M}$, and we conclude by induction on the number of components of the comb. $\qquad\square$

Another version of the result is:

Proposition 7.8. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Suppose that C is a section of π whose deformations dominate $\mathcal{X}$ that satisfies $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \mathrm{MBBbound}(\mathcal{X})$. Then C deforms (as a stable map) to a union of a free section C_0 with a free π-vertical moving curve T such that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot T \leq 3$.*

Proof. By Theorem 7.7, $\overline{M}$ contains a point representing the union of a free section C_0 satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C_0 \leq \mathrm{MBBbound}(\mathcal{X}) - 1$ with a free π-vertical curve. We may assume that the gluing point of the two curves is a general point of the fiber containing it. By Lemma 2.9 we can further deform the free π-vertical curve to a chain of free π-vertical curves, each of which has anticanonical degree at most 3, while preserving the attachment point with C_0. After smoothing all but one of the π-vertical components into the section, we obtain the desired breaking. $\qquad\square$

Finally, by breaking again we can obtain a polynomial upper bound on the number of height d components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ as predicted by Batyrev.

Proof of Corollary 1.5: It suffices to prove the existence of a polynomial upper bound on the number of components of rigid families, the number of components of families of sections

which sweep out surfaces, and the number of components of dominant families of sections. The first claim is clear since there are only finitely many rigid curves.

Suppose that $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizes sections of height d which sweep out a closed surface Y. Let $\widetilde{Y}$ denote a resolution of Y. If $d \geq 0$ then there is a one-parameter family of rational curves through a general point on $\widetilde{Y}$. Thus $\widetilde{Y}$ is rationally connected so that two sections on $\widetilde{Y}$ will be numerically equivalent if and only if they are linearly equivalent. Thus, the number of components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ that sweep out Y and have height at most d is bounded above by the number of numerical classes of sections of $\widetilde{Y}$ of height at most d. This latter number is a polynomial in d. By Theorem 1.1 only finitely many surfaces Y can be obtained in this way, giving a universal polynomial bound on all components of sections of this type.

Finally, suppose that M is a component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ that parametrizes a dominant family of sections. Let $\overline{M}$ denote the corresponding component of $\overline{M}_{0,0}(\mathcal{X})$. By Theorem 7.7, if the height of the sections parametrized by M is at least $\mathrm{MBBbound}(\mathcal{X})$ then $\overline{M}$ contains a point representing the union of a free section C_0 satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C_0 \leq \mathrm{MBBbound}(\mathcal{X}) - 1$ with a free π-vertical curve. We may assume that the gluing point of the two curves is a general point of the fiber containing it. By Lemma 2.9 we can further deform the free π-vertical curve to a chain of free π-vertical curves, each of which has anticanonical degree at most 3, while preserving the attachment point with C_0.

Choose a positive integer m such that $A := -K_{\mathcal{X}/\mathbb{P}^1} + mF$ is an ample divisor where F denotes a general fiber of π. We have shown that each component M parametrizing a dominant family of sections yields a component $\overline{M}$ which contains a chain of free curves where each component has A-degree no more than $\sup\{3, m + \mathrm{MBBbound}(\mathcal{X})\}$. [LT19a, Theorem 5.13] proves that there is a polynomial upper bound $P(d)$ on the number of components $\overline{M} \subset \overline{M}_{0,0}(X)$ of A-degree d which contain a chain of free curves where each component has A-degree $\leq \sup\{3, m + \mathrm{MBBbound}(\mathcal{X})\}$. This yields a polynomial upper bound on the number of components M as desired. $\qquad\square$

7.2. Fibers of evaluation maps.
We now apply Movable Bend-and-Break to show that for sections with large height the evaluation map for the universal family has connected fibers. This is the analogue in our setting of [LT19a, Proposition 5.15].

Lemma 7.9. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Let $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ denote a component that parametrizes a dominant family of sections. Let $p : \mathcal{U} \to M$ denote the universal family and let $s : \mathcal{U} \to \mathcal{X}$ denote the evaluation map.*

Suppose that there is a component Q of $\overline{M}_{0,0}(\mathcal{X})$ parametrizing a dominant family of reduced π-vertical curves C with $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq 3$ such that a general curve parametrized by Q appears as a component of some stable map in the closure $\overline{M}$ of M in $\overline{M}_{0,0}(\mathcal{X})$. For any birational model $\mathcal{C}'$ of $\mathcal{C}$ the induced map $s' : \mathcal{C}' \to \mathcal{X}$ has irreducible general fiber.

Proof. By taking projective closures and taking resolutions we find smooth projective varieties W, M_W which admit maps $\overline{s} : W \to \mathcal{X}$ and $\overline{p} : W \to M_W$ that agree with s and p on open subsets. Let $f : Y \to \mathcal{X}$ denote a resolution of the Stein resolution of $\overline{s}$. Our goal is to show that f is a birational map.

By construction Y is dominated by sections of the natural map $\psi = \pi \circ f : Y \to \mathbb{P}^1$, so that Y has the structure of an algebraic fiber space over $\mathbb{P}^1$. Let $\overline{N}$ denote the component

of $\overline{M}_{0,0}(Y)$ whose general point is the image of a general fiber of $\overline{p}$. We know that $\overline{N}$ admits a birational morphism to the corresponding component $\overline{M}$ of $\overline{M}_{0,0}(\mathcal{X})$. In particular, there is a dominant family of ψ-vertical curves P mapping onto the corresponding curves in Q. Since the induced map from P to Q is dominant, the expected dimensions of P and Q agree. This implies that the curves C parametrized by P satisfy $(K_{Y/\mathbb{P}^1} - f^*K_{\mathcal{X}/\mathbb{P}^1}) \cdot C = 0$. By the Riemann-Hurwitz formula $K_{Y/\mathbb{P}^1} - f^*K_{\mathcal{X}/\mathbb{P}^1}$ is relatively pseudo-effective, and since C is a movable ψ-vertical curve we see that this divisor is not relatively big. This means that

$$a(Y_\eta, -f^*K_{\mathcal{X}/\mathbb{P}^1}) = 1 = a(X_\eta, -K_{\mathcal{X}/\mathbb{P}^1}).$$

By [LT17, Theorem 6.2] this implies that either f_η is birational or the curves C map to conics in the fibers of $\mathcal{X}$. However, the latter contradicts the fact that $-K_{\mathcal{X}/\mathbb{P}^1} \cdot f_*C \geq 3$. We deduce that f_η is birational, hence f is birational. $\qquad\square$

Corollary 7.10. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Let $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ denote a component that parametrizes a dominant family of sections. Suppose that the curves C parametrized by M satisfy $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \geq \mathrm{MBBbound}(\mathcal{X}) + 2$. Then any resolution of the universal family map over M has irreducible general fiber.*

Proof. Let $\overline{M}$ denote the corresponding component of $\overline{M}_{0,0}(\mathcal{X})$. Applying Theorem 7.7 to $\overline{M}$ on $\mathcal{X}$, we see that it contains points which are the union of a free π-vertical curve of $-K_{\mathcal{X}/\mathbb{P}^1}$-degree at least 3 and a horizontal moving section. Since such a stable map is a smooth point of the moduli space, a general deformation of this π-vertical curve must appear as a component of some curve parametrized by $\overline{M}$. We then apply Lemma 7.9. $\qquad\square$

8. Examples

In this section we analyze several explicit examples of del Pezzo fibrations.

8.1. **Improving previous bounds.** In many situations the upper bound on MBBbound given in Theorem 7.2 is much larger than the actual value. In order to compute examples it will be useful to give more precise bounds for del Pezzo fibrations satisfying certain properties.

- Suppose that we have classified the sections on $\mathcal{X}$ of negative height. Then we can hope to prove a more precise statement in place of Lemma 6.3. This in turn will improve the bounds in Cases 1 and 2 of Theorem 7.2 in the $d < 0$ situation, and in Case 3 of Theorem 7.2 via Propositions 6.4 and 6.6.
- Suppose that $\mathcal{X}_\eta$ does not admit any rational curves of anticanonical degree 2 defined over the ground field $k(\mathbb{P}^1)$. This implies that the surface $\widetilde{\Sigma}$ constructed in the proof of Theorem 7.2 is not ruled by $-K_{\mathcal{X}/\mathbb{P}^1}$-conics. In particular, this allows us to give more precise bounds in inequalities (7.1) and (7.3).
- When the height is 3 or 4 and the conditions above hold we can mimic the proof of Theorem 7.2 using more explicit constructions to prove a Movable Bend-and-Break theorem by hand.

By implementing these improvements we have:

Lemma 8.1. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Suppose that $\mathrm{maxdef}(d) - d \leq 2$ for every $d < 0$ such that there is a section of height d. If $\mathcal{X}_\eta$ does not admit any rational curves defined over $k(\mathbb{P}^1)$ of anticanonical degree 2 then*

$$\mathrm{MBBbound}(\mathcal{X}) \leq 3.$$

The following theorem improves Proposition 7.8 by giving conditions where we are guaranteed to be able to break off a π-vertical conic. This simplifies inductive arguments, since it means that we can understand dominant families of sections of height n using only the dominant families of sections of height $n - 2$.

Theorem 8.2. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Assume that a general fiber of this fibration is a del Pezzo surface with $2 \leq (-K_S)^2 \leq 8$ and that every surface swept out by the π-vertical lines contained in smooth fibers is a big divisor. Suppose that C is a section of π of height d whose deformations dominate $\mathcal{X}$ and such that $d \geq \mathrm{MBBbound}(\mathcal{X})$. Then C deforms (as a stable map) to a union of a free section C_0 and a free π-vertical conic.*

Proof. By Proposition 7.8 we know that we can find a breaking $C \sim_{alg} C_0 + T$ where the vertical curve T is either a free π-vertical conic or a free π-vertical cubic. In the first case we are done. In the second case, let $\overline{P}$ denote the component of $\overline{M}_{0,0}(\mathcal{X})$ parametrizing the deformations of T. By Lemma 2.6 we see that every general fiber of π contains a curve parametrized by $\overline{P}$ which is the union of a conic and a line. Since such curves are smooth points of $\overline{P}$, in fact there is an entire component $\overline{Q}$ of the parameter space of π-vertical lines contained in smooth fibers such that every stable map parametrized by $\overline{Q}$ occurs as the restriction of a stable map parametrized by $\overline{P}$ to a subcurve.

By assumption the surface swept out by the curves parametrized by $\overline{Q}$ is big. Thus every deformation of C_0 intersects this surface. Consider the point of $\overline{P}$ obtained by gluing a general π-vertical line parametrized by $\overline{Q}$ to a π-vertical conic. Then there is a free deformation C_1 of C_0 that intersects the image of this curve along the line. By gluing, we obtain a stable map whose image has the form $C_1 \cup \ell \cup T_1$ where C_1 is a free deformation of C_0, ℓ is a π-vertical line, and T_1 is a free π-vertical conic. This is a smooth point of $\overline{M}_{0,0}(\mathcal{X})$ that lies in the same component of $\overline{M}_{0,0}(\mathcal{X})$ as $C_0 \cup T$. We can then glue C_1 to ℓ to conclude the desired statement. $\qquad\square$

8.2. Blow-ups of Fano threefolds.
One of the most common ways to construct a del Pezzo fibration is by blowing up a Fano threefold. In this section we analyze several examples of this type.

Example 8.3. Let X_3 be a smooth cubic 3-fold in $\mathbb{P}^4$. Let Z denote a smooth elliptic curve which is the base locus of a general pencil of hyperplane sections. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ denote the resolution of the pencil. We have $\mathrm{Pic}(\mathcal{X}) = \mathbb{Z}H \oplus \mathbb{Z}E$ where H is the pullback of the hyperplane class from X_3 and E is the exceptional divisor of the blow-up $\phi : \mathcal{X} \to X_3$. In this notation $-K_{\mathcal{X}/\mathbb{P}^1} = E$, so that $\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}) = -1$ and the sections achieving this bound will exactly be the sections contained in $E \cong Z \times \mathbb{P}^1$.

Computation of π-vertical curves: Since the fibers of $\mathcal{X}$ form a Lefschetz pencil in the sense of [Voi03, Definition 2.6], the monodromy action on the Néron-Severi space of the general fiber is the full Weyl group $W(E_6)$. This implies that for π-vertical curves of anticanonical degree 1 or 2 there is a unique family of rational curves on $\mathcal{X}$ that dominates the base $\mathbb{P}^1$.

Computation of the closed set: Since any rigid section will satisfy $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C \leq -2$, there are no rigid sections of π.

By Lemma 3.4, if $Y \subset \mathcal{X}$ is a subvariety with $a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}) > a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1})$ then the intersection of Y with a general fiber is a union of (-1)-curves. Since the monodromy action

is maximal, there is a unique surface V of this type. If $\widetilde{V}$ denotes a resolution of V then the map $\widetilde{V} \to \mathbb{P}^1$ is not an algebraic fiber space, showing that V can not be covered by sections.

On a cubic del Pezzo surface the subvarieties with the same a-value are exactly the conics. Since the monodromy action is maximal, the set of all conics in general fibers is parametrized by an irreducible surface. In particular, the Stein factorization of the evaluation map for the universal family of π-vertical conics is non-trivial. This implies that any 1-parameter family of conics that dominates the base $\mathbb{P}^1$ will fail to be an algebraic fiber space over $\mathbb{P}^1$ and thus can not be dominated by sections.

By combining the results above with Theorem 5.4 we see that the only possible non-dominant families of sections will have height ≤ 0. We analyze sections in this range below and we deduce that there is only one non-dominant family consisting of the sections contained in E.

Computation of low degree sections: We analyze the sections of low height with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$:

- Height -1: as discussed above these are exactly the rational curves in $E \cong Z \times \mathbb{P}^1$.
- Height 0: these will be the strict transforms of lines in X_3 not meeting Z. Thus this space is irreducible.
- Height 1: these will be the strict transforms of conics in X_3 which meet Z once. [LT19a, Theorem 7.6] shows that for the universal family of conics $\mathcal{U}$ on X_3 the evaluation map $s : \mathcal{U} \to X_3$ has equidimensional irreducible fibers over an open neighborhood of Z. Thus the space of height 1 sections is irreducible.
- Height 2: these will be the strict transforms of twisted cubics in X_3 which meet Z in two points. Any twisted cubic C in X_3 is contained in a unique hyperplane section $H \subset \mathbb{P}^4$ and C is a smooth rational curve of degree 3 in the cubic surface $S = H \cap X_3$. Such a curve is the strict transform of a line via a birational morphism $S \to \mathbb{P}^2$. Since the space of twisted cubics in X_3 is irreducible (see [HRS05]), we know that the monodromy action on classes of twisted cubics in S is transitive as H varies in an open set $U \subset (\mathbb{P}^4)^*$. Choose two points p_1, p_2 on Z and consider the hyperplanes $H \subset \mathbb{P}^4$ containing $\ell = \mathrm{span}(p_1, p_2)$. For each H there are only finitely many twisted cubics passing through p_1, p_2. By the Lefschetz hyperplane theorem the monodromy action of the hyperplanes containing ℓ is still transitive, so these curves are parametrized by an irreducible variety. The irreducibility of the space of sections of height 2 follows.

Classification of sections: Lemma 8.1 shows that $\mathrm{MBBbound}(\mathcal{X}) \leq 3$. We can now classify all families of sections. As explained above, there is a unique family of sections of height $-1, 0, 1$, and 2. We prove the irreducibility of the moduli space of sections of height ≥ 3 by induction. As discussed earlier we know that any component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ in this height range will parametrize a dominant family of sections. By Lemma 8.1 and Theorem 8.2 we know that for $d \geq 3$ any free section of height d will deform as a stable map to the union of a free section of height $d - 2$ and a free π-vertical conic. So it suffices to show that if M denotes the unique component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)_{d-2}$ then the space of curves obtained by gluing free π-vertical conics to curves in M is also irreducible. By the fact that the monodromy is maximal, we know that the space of conics which meet a fixed section is irreducible, and the desired conclusion follows immediately.

An analogous argument holds for other similar constructions.

Example 8.4. Let X_5 be a general codimension 3 linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$. Let Z denote a smooth elliptic curve which is the base locus of a general pencil of hyperplane sections. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ denote the resolution of the pencil. We have $\mathrm{Pic}(\mathcal{X}) = \mathbb{Z}H \oplus \mathbb{Z}E$ where H is the pullback of the hyperplane class and E is the exceptional divisor of the blow-up $\phi : \mathcal{X} \to X_5$. In this notation $-K_{\mathcal{X}/\mathbb{P}^1} = E$.

Since the fibration is constructed by resolving a Lefschetz pencil, the monodromy action will again be full. Thus, by arguing just as in Example 8.3, we can classify all components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ by analyzing the behavior of sections of height ≤ 2.

Computation of low degree sections: We analyze the sections of low height with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$:

- Height -1: these are the rational curves in $E \cong Z \times \mathbb{P}^1$.
- Height 0: these will be the strict transforms of lines in X_5 not meeting Z. As described by [Isk79], this space is irreducible.
- Height 1: these will be the strict transforms of conics in X_5 which meet Z once. [LT19a, Theorem 7.6] shows that for the universal family of conics $\mathcal{U}$ on X_5 the evaluation map $s : \mathcal{U} \to X_5$ has equidimensional irreducible fibers over an open neighborhood of Z. Thus the space of height 1 sections is irreducible.
- Height 2: these will be the strict transforms of twisted cubics in X_5 which meet Z in two points. Consider the incidence correspondence I of pairs (H, C) where H is a hyperplane such that $S = H \cap X_5$ is smooth and C is a twisted cubic in S. Note that each S is a degree 5 del Pezzo surface and that C is the pullback of a line under a birational map $S \to \mathbb{P}^2$. The correspondence I admits a map to an open subset $U \subset \mathbb{P}^{6*}$ and the monodromy action of U on the classes of twisted cubics in a fixed surface S is transitive. Choose two points p_1, p_2 on Z. They are parametrized by $\mathrm{Sym}^2(Z)$. Then we consider hyperplanes $H \subset \mathbb{P}^6$ containing $\ell = \mathrm{span}(p_1, p_2)$. For each H there are only finitely many twisted cubics contained in $H \cap X_5$ passing through p_1, p_2. By the Lefschetz hyperplane theorem the monodromy action of the hyperplanes containing ℓ is still transitive on the set of cubics in each surface S, so the set of twisted cubics meeting Z in two points are parametrized by an irreducible variety. The irreducibility of the space of sections of height 2 follows.

By mimicking the argument in Example 8.3, we see that there is a unique irreducible family of sections of every height ≥ -1.

Similar computations hold for

- the blow up of the intersection of two quadrics along the base locus of a general pencil of hyperplanes, and
- the blow up of a quadric along the base locus of a general pencil of quadrics.

In these cases the behavior of low degree sections is described by [HT14].

8.3. Subvarieties of products of projective spaces.

Another common way of constructing a del Pezzo fibration is by taking a complete intersection inside a product of a threefold with a curve. The following example illustrates how one can compute sections in this situation.

Example 8.5. Let $\mathcal{X}$ be a general hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ of degree $(2, 3)$. The projection $\pi : \mathcal{X} \to \mathbb{P}^1$ is a fibration by cubic surfaces. We denote the pullbacks of the hyperplane class from the two projections by H_1 and H_2 respectively. Then $-K_{\mathcal{X}/\mathbb{P}^1} = H_2 - 2H_1$.

Computation of π-vertical curves: As explained in [Bea86] the monodromy action on the space of all cubic surfaces in $\mathbb{P}^3$ is full. Since $\mathcal{X}$ is constructed by taking a general conic in this parameter space, by [Kol15, Corollary 6] the monodromy action for $\mathcal{X}$ is also full. This implies that the generic fiber does not admit a curve of anticanonical degree 1 or 2 that is defined over the ground field.

Computation of the closed set: The minimal possible height of a section is -2, and any rigid section will have this height. We will verify below that there is a finite set of sections of height -2 and all are rigid.

The union of the subvarieties of $\mathcal{X}$ with larger generic a-invariant is swept out by the π-vertical lines. Since the monodromy action is full, this space is irreducible. Similarly, the union of the subvarieties with the same generic a-invariant is swept out by the π-vertical conics. Again this is an irreducible subset. Since the Stein factorization of the map to $\mathbb{P}^1$ does not have connected fibers, this surface cannot admit a dominant family of sections.

By combining the results above with Theorem 5.4 we see that the only possible non-dominant families of sections will have height ≤ 2. We analyze sections in this range below.

Computation of low degree sections: We analyze the sections of low height with respect to $-K_{\mathcal{X}/\mathbb{P}^1}$. We claim that every component of the moduli space for the space of sections of height ≤ 2 has the expected dimension and, if the height is between -1 and 2, is also irreducible. An easy dimension count proves that there are only finitely many sections of height -2. A lengthier dimension count proves that a general section of height d with $-1 \leq d \leq 2$ will map birationally onto a general rational curve of degree $d + 2$ in $\mathbb{P}^3$, and that every component parametrizing such sections has the expected dimension.

It only remains to verify that for each $d \leq 2$ the space of sections of height d which map birationally onto a general rational curve of degree $d + 2$ in $\mathbb{P}^3$ is irreducible. For some fixed d let N° denote the space of such sections in $\mathbb{P}^1 \times \mathbb{P}^3$ and let $\mathcal{C}$ denote the universal curve over N°. Consider the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ s\ } & \mathbb{P}^1 \times \mathbb{P}^3 \\
{\scriptstyle p}\downarrow & & \\
N^\circ & &
\end{array}
$$

Consider $\mathcal{E} := p_* s^* \mathcal{O}(2,3)$. Since s has connected fibers we can identify sections of $s^* \mathcal{O}(2,3)$ with sections of $\mathcal{O}(2,3)$. Thus a general section of $\mathcal{E}$ will parametrize the sublocus of curves parametrized by N° that are contained in a general section of $\mathcal{O}(2,3)$. We claim that $\mathcal{E}$ is a globally generated vector bundle; this can be proved by writing down the ideal sheaf of a general curve C parametrized by N° and computing that $H^1(\mathbb{P}^1 \times \mathbb{P}^3, I_C(2,3)) = 0$. This means that a general section of $\mathcal{E}$ is irreducible. In other words, the open dense locus of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)_d$ parametrizing sections which map onto a general curve in $\mathbb{P}^3$ is irreducible for this degree range.

Classification of sections: By Lemma 8.1 MBBbound$(\mathcal{X}) \leq 3$. As explained earlier, we can deduce that every component of the space of sections has the expected dimension, and that there is a unique component of any height > -2.

Example 8.6 (Diagonal cubic surface (1,t,t+1,t+2))**.** Consider the smooth threefold $\mathcal{X} \subset \mathbb{P}^1_{s,t} \times \mathbb{P}^3_{x,y,z,w}$ defined by the equation $sx^3 + ty^3 + (t+s)z^3 + (t+2s)w^3 = 0$. The projection map to $\mathbb{P}^3$ realizes $\mathcal{X}$ as the blow-up of the base locus of a pencil of cubic surfaces on $\mathbb{P}^3$. We let Z denote the smooth genus 10 curve which is the base-locus of this pencil and let

E denote the exceptional divisor. The projection $\pi : \mathcal{X} \to \mathbb{P}^1$ is a cubic surface fibration. [Jah14, the Appendix] shows that $\mathcal{X}_\eta$ has Brauer group $\mathbb{Z}/3\mathbb{Z}$. However, since $\mathcal{X}$ is birational to $\mathbb{P}^3$ its Brauer group is trivial. Let H_1 and H_2 denote the pullbacks of the hyperplane class under the two projection maps. Then $-K_{\mathcal{X}/\mathbb{P}^1} = -H_1 + H_2$ is relatively ample.

Computation of π-vertical curves: The Galois group acts on $\mathrm{Pic}(\mathcal{X}_{\overline{\eta}})$ through $(\mathbb{Z}/3\mathbb{Z})^3$ and the rank of $\mathrm{Pic}(\mathcal{X}_\eta)$ is 1. There are three orbits of lines, each of size 9, and three orbits of classes of conics, each of size 9. This implies that $\mathcal{X}_\eta$ does not admit a curve of anticanonical degree 1 or 2 that is defined over the ground field.

The computation above shows that $\overline{X}$ admits three families of π-vertical lines which meet a general fiber. There are also four singular fibers of π, each of which is a cone over a cubic curve. There is a one-parameter family of π-vertical lines contained in every singular fiber of π. In all seven families of π-vertical lines a general line has restricted tangent bundle $\mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O}(-1)$.

Computation of the closed set: A section must have intersection number 1 against H_1. So its degree under the pushforward to $\mathbb{P}^3$ will just be 1 more than the height. The lowest possible height is -1, and each such section is contracted by the map $\mathcal{X} \to \mathbb{P}^3$. As explained above, we know that the generic fiber of π has no line or conic defined over the ground field, so by Theorem 5.4 we see that the only possible non-dominant families of sections will have height ≤ 0. We analyze sections in this range below.

Computation of low degree sections: We next classify sections of low height.

Height -1: These will be the sections contained in E. There is an irreducible one parameter family of such sections.

Height 0: These will be strict transforms of lines in $\mathbb{P}^3$. Such a line must meet every hyperplane in our family with multiplicity 2 along Z, so these lines are the bisecants to Z. They are parametrized by an irreducible space birational to $\mathrm{Sym}^2(Z)$.

Height 1: These will be strict transforms of smooth conics which meet each cubic in our family with multiplicity 5 along Z. Each generic plane P contains a finite number of such conics. Indeed, $P \cap Z$ will be nine points, and any conic which contains 5 of these points will yield a section. We claim there is a unique component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ whose generic section has this form. Indeed, by [Kad19, Proposition 1.1] as we vary the plane P the monodromy action on $P \cap Z$ is the full symmetric group, which proves the claim.

We show there are no other components using a dimension count. Suppose that P is a plane containing a family of height 1 sections which has dimension ≥ 1. This means that there is a length ≤ 4 dimension 0 subscheme of P such that every conic containing this subscheme meets Z with multiplicity 5 along these points. In particular the restriction of every cubic in our family to P must be singular. However, there is no such plane P, since the tangent planes of our cubics at each point $z \in Z$ vary.

Height 2: These will be strict transforms of cubics in $\mathbb{P}^3$. We first show that there is a unique component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ whose general curve C corresponds to a twisted cubic C' in $\mathbb{P}^3$. Our strategy is to show that any such component M will contain broken curves of a certain type. Recall that there are two irreducible components M_0, M_1 of the moduli space of rational curves which parametrize free sections of height 0 and height 1 respectively. We let L_0, L_1, L_2 denote the three components which parametrize a family of π-vertical lines which dominate $\mathbb{P}^1$, and we let S_1, S_2, S_3, S_4 denote the four components which parametrize vertical

lines in the 4 singular fibers of π. Finally are three components N_0, N_1, N_2 parametrizing vertical free conics.

Let M be any component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ consisting of the strict transforms of twisted cubics C'. Then the general C' meets with Z at 8 points. Fix one general point x on $\mathcal{X}$ and one general point z on Z; the set of sections C passing through x and meeting the exceptional curve in $\mathcal{X}$ corresponding to z in Z form a family of dimension at least 1. The pushforwards of deformations of C to $\mathbb{P}^3$ are twisted cubics C' meeting the 2 fixed points x, z. Applying Bend and Break to this family of twisted cubics we obtain a breaking curve $\sum_{i=1}^{k} C_i'$ in $\mathbb{P}^3$. The breaking curve must be either the union of a line and a conic or the union of three lines. However a connected chain of three lines cannot meet x and z while meeting Z at 7 other points. So the broken curve must be a line plus a conic. There are two possibilities: (i) the line meets with Z at 3 points and the conic meets with Z at 5 points; (ii) the line meets with Z at 2 points and the conic meets with Z at 6 points. Going back to $\mathcal{X}$, this means that we break a curve C to either (i) a vertical line and a section of height 1, or (ii) a vertical conic and a section of height 0. Moreover by construction our breaking curve must meet with x and with 1 general section of height -1, and in fact by Lemma 4.1 each component of the broken curve must meet either x or the general section.

Next we show that both types of breaking curve are smooth points of $\overline{M}_{0,0}(\mathcal{X})$. In case (i), there are only finitely many π-vertical lines meeting a fixed general section of height -1. If the section is general then the line must also be general so its normal bundle has the form $\mathcal{O} \oplus \mathcal{O}(-1)$. Furthermore, the sections of height 1 passing through x form a 1 dimensional locus. Among them only finitely many meet with a fixed general vertical line. This implies that our section of height 1 must be free. We conclude that the broken curve is a smooth point of the moduli space. In case (ii), x meets with either a section of height 0 or a vertical conic. If the former holds, there are finitely many sections of height 0 passing through x and by generality any such conic must be free. Then there are only finitely many choices of a π-vertical conic meeting fixed general sections of height 0 and height -1, so the conic must also be free. If the latter holds, there are only finitely many choices for a vertical conic and by generality it must be free. Then there are only finitely many sections of height 0 meeting with 1 general section of height -1 and with one of our chosen conics, so the section must also be free.

Before continuing, we note that for any $j = 1, 2, 3, 4$ there is a unique component of $M_1^{(1)} \times_{\mathcal{X}} S_j^{(1)}$ which generically parametrizes the union of a free section and a general line in S_j. Indeed, since any section meets the corresponding singular fiber of π in one point, the parameter space of such glued curves is birational to M_1.

Suppose that M contains the union of a general vertical line in some L_i and a free section of height 1. Then the locus of broken curves in M of this type must admit a dominant map to M_1. Since the space of height 1 sections is irreducible, its closure in $\overline{M}_{0,0}(\mathcal{X})$ contains a two-dimensional locus parametrizing unions of a free height 0 section and a general line in S_1. Correspondingly, the closure of M in $\overline{M}_{0,0}(\mathcal{X})$ parametrizes a curve which is a chain $\ell_1 \cup C_0 \cup \ell_2$ where ℓ_1 is a general line in S_1, C_0 is a free height 0 section, and ℓ_2 is a line in L_i which must be general by a dimension count. Note that this is a smooth point of $\overline{M}_{0,0}(\mathcal{X})$. We can smooth $C_0 \cup \ell_2$, and furthermore we can deform ℓ_1 and the intersection point along with the smoothing. Thus we see that the closure of M in $\overline{M}_{0,0}(\mathcal{X})$ must contain $M_1^{(1)} \times_{\mathcal{X}} S_1^{(1)}$. Using analogous reasoning, it also contains $M_1^{(1)} \times_{\mathcal{X}} S_j^{(1)}$ for any $j = 1, 2, 3, 4$.

Suppose that M contains the union of a general vertical line in some S_j and a free section of height 1. Since this is a smooth point of $\overline{M}_{0,0}(\mathcal{X})$, M must contain the entire locus $M_1^{(1)} \times_{\mathcal{X}} S_j^{(1)}$.

Finally, suppose that M contains the union of a general vertical conic in some N_i and a free section of height 0. Then the locus of broken curves in M of this type must admit a dominant map to M_0. Since the space of height 0 sections is irreducible, its closure in $\overline{M}_{0,0}(\mathcal{X})$ contains a 1-dimensional locus parametrizing unions of a general height -1 section and a general line in S_1. Since this locus is 1-dimensional, an incidence correspondence argument shows that the closure of M in $\overline{M}_{0,0}(\mathcal{X})$ parametrizes a chain $\ell \cup C_0 \cup T$ where ℓ is a general line in S_1, C_0 is a general height -1 section, and T is a free conic. This is a smooth point of $\overline{M}_{0,0}(\mathcal{X})$. Note that we can smooth $C_0 \cup T$, and furthermore we can deform ℓ and the intersection point along with the smoothing. Thus we see that the closure of M in $\overline{M}_{0,0}(\mathcal{X})$ must contain $M_1^{(1)} \times_{\mathcal{X}} S_1^{(1)}$. Using analogous reasoning, it also contains $M_1^{(1)} \times_{\mathcal{X}} S_j^{(1)}$ for any $j = 1, 2, 3, 4$.

We have shown that any component M parametrizing the strict transform of twisted cubics must contain $M_1^{(1)} \times_{\mathcal{X}} S_j^{(1)}$ for some $j = 1, 2, 3, 4$. Furthermore, there must be at least one component which contains a smoothing of a general height 1 section and a general line in L_i. This particular component contains $M_1^{(1)} \times_{\mathcal{X}} S_j^{(1)}$ for every j. Since for each j the space $M_1^{(1)} \times_{\mathcal{X}} S_j^{(1)}$ is irreducible and contains smooth points of $\overline{M}_{0,0}(\mathcal{X})$, altogether we deduce that there is a unique such component M.

We next show that there is no component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ whose general point corresponds to a planar cubic C in $\mathbb{P}^3$. Let P be the plane containing C. If P is generic, then either C contains 8 points of $P \cap Z$ or the singular point of C lies on $P \cap Z$ and C must contain 6 other points of $P \cap Z$. Note that there are only finitely many possible such C in any fixed P. Thus a dimension count shows that the collection of such curves cannot be dense in a component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$. Similar dimension counts work for non-generic P to show that we get no new components in this way.

Classification of sections: By Lemma 8.1 MBBbound$(\mathcal{X}) \leq 3$. Let $d \geq 3$ and assume that for every $-1 \leq k < d$ there is a unique component M_k parametrizing free sections of height k. Suppose we fix a component M of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ generically parametrizing free sections of height $d \geq 3$. Then $\overline{M}$ contains a stable map consisting of a free section C_{d-2} of height $d - 2$ and a free conic T. Fix some $j \in \{1, 2, 3\}$. Since the space of height $d - 2$ sections is irreducible, a free section of height $d - 2$ can be deformed into the union of a general section C_{d-4} of height $d - 4$ and a general vertical free conic T' in the component N_j parametrizing vertical conics. Furthermore since T is free we can deform T and the attachment point to C_{d-2} along with the deformation above to obtain a chain $T \cup C_{d-4} \cup T'$ where T is still a free vertical conic. Then we smooth $T \cup C_{d-4}$ and obtain an element of $M_{d-2}^{(1)} \times_{\mathcal{X}} N_j^{(1)}$ which is a smooth point of $\overline{M}_{0,0}(\mathcal{X})$. Since $M_{d-2}^{(1)} \times_{\mathcal{X}} N_j^{(1)}$ is irreducible, we conclude that M is unique.

9. Bounds in Geometric Manin's Conjecture

9.1. Algebraic and numerical equivalence for curves. We start by recalling some facts about algebraic and numerical equivalence for curve classes on del Pezzo fibrations.

9.1.1. *Numerical equivalence.* By dualizing the surjective restriction map $N^1(\mathcal{X}) \to N^1(\mathcal{X}_\eta)$ we obtain an inclusion $N_1(\mathcal{X}_\eta) \subset N_1(\mathcal{X})$. Let $i : F \hookrightarrow \mathcal{X}$ denote the inclusion of a general fiber of π. By applying the Invariant Cycle Theorem to divisors and dualizing, we see that

$$N_1(\mathcal{X}_\eta) = i_*(N_1(F)) = i_*(N_1(F)^{mon})$$

where $N_1(F)^{mon}$ denotes the monodromy invariant part of $N_1(F)$. Under this identification we have $\mathrm{Nef}_1(\mathcal{X}_\eta) = i_*\mathrm{Nef}_1(F)$.

The behavior of lattices of curves in these spaces is more subtle. We have inclusions

$$N_1(\mathcal{X}_\eta)_\mathbb{Z} \subset i_*(N_1(F)_\mathbb{Z}) \subset N_1(\mathcal{X})_\mathbb{Z} \cap N_1(\mathcal{X}_\eta)$$

as full-rank sublattices. While the first inclusion can be strict, the second inclusion is an equality for del Pezzo fibrations.

Proposition 9.1. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a Fano fibration. Let F denote a general fiber of π. Suppose that the pairing of $N^1(F)_\mathbb{Z}$ and $N_1(F)_\mathbb{Z}$ is unimodular. Then $i_*(N_1(F)_\mathbb{Z}) = N_1(\mathcal{X})_\mathbb{Z} \cap N_1(\mathcal{X}_\eta)$.*

Proof. First note that the restriction map $N^1(\mathcal{X})_\mathbb{Z} \twoheadrightarrow N^1(\mathcal{X}_\eta)_\mathbb{Z}$ is surjective, since any Cartier divisor on $\mathcal{X}_\eta$ can be extended to a Weil divisor on $\mathcal{X}$ and $\mathcal{X}$ is smooth. Furthermore, since the Brauer group $\mathrm{Br}(k(\mathbb{P}^1))$ is equal to 0, the Hochschild-Serre spectral sequence shows that $N^1(\mathcal{X}_\eta)_\mathbb{Z} \cong N^1(F)_\mathbb{Z}^{mon}$. Combining, we have maps

$$N^1(\mathcal{X})_\mathbb{Z} \twoheadrightarrow N^1(\mathcal{X}_\eta)_\mathbb{Z} \cong N^1(F)_\mathbb{Z}^{mon} \hookrightarrow N^1(F)_\mathbb{Z}$$

where the first map is the pullback and the third is the inclusion. Taking the dual we obtain

$$N_1(F)_\mathbb{Z} \twoheadrightarrow (N^1(F)_\mathbb{Z}^{mon})^\vee \cong N^1(\mathcal{X}_\eta)_\mathbb{Z}^\vee \hookrightarrow N^1(\mathcal{X})_\mathbb{Z}^\vee$$

Finally we note that $N^1(\mathcal{X}_\eta)_\mathbb{Z}^\vee = N_1(\mathcal{X}_\eta) \cap N^1(\mathcal{X})_\mathbb{Z}^\vee$. Indeed, since $N^1(\mathcal{X})_\mathbb{Z}$ is torsion free, the kernel of $N^1(\mathcal{X})_\mathbb{Z} \twoheadrightarrow N^1(\mathcal{X}_\eta)_\mathbb{Z}$ is also a free abelian group. Thus the cokernel of $N^1(\mathcal{X}_\eta)_\mathbb{Z}^\vee \hookrightarrow N^1(\mathcal{X})_\mathbb{Z}^\vee$ is torsion free. This proves that $N_1(\mathcal{X}_\eta) \cap N_1(\mathcal{X})_\mathbb{Z} \subset i_*(N_1(F)_\mathbb{Z})$ as desired. $\square$

9.1.2. *Algebraic equivalence.* For a smooth rationally connected threefold $\mathcal{X}$ over $\mathbb{C}$ we have

$$\mathrm{Br}(\mathcal{X}) \cong H^3(\mathcal{X}, \mathbb{Z})_{\mathrm{tors}}.$$

According to the universal coefficient theorem, we can equally well think of $\mathrm{Br}(\mathcal{X})$ as the torsion classes of $H_2(\mathcal{X}, \mathbb{Z})$. Let $Q_1(\mathcal{X})$ denote the set of algebraic equivalence classes of curves of $\mathcal{X}$. [BS83, Theorem 1] shows that algebraic and homological equivalence coincide for curve classes on $\mathcal{X}$ and [Voi06, Theorem 2] proves the integral Hodge conjecture for $\mathcal{X}$. Together these show:

Theorem 9.2 ([BS83], [Voi06])**.** *Let $\mathcal{X}$ be a smooth rationally connected threefold over $\mathbb{C}$. Then $|\mathrm{Br}(\mathcal{X})|$ is the size of the kernel of the quotient map $q : Q_1(\mathcal{X}) \to N_1(\mathcal{X})_\mathbb{Z}$.*

9.2. **Counting components.** The key to Geometric Manin's Conjecture is to count the number of components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$. We separate this into two questions:

(1) Which numerical classes are represented by a section?
(2) How many components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ represent a fixed numerical class? How many such components parametrize a dominant family of curves?

In this section we analyze these questions using heuristic arguments. The goal is to develop a precise conjecture describing what we will need for Geometric Manin's Conjecture.

For Question (1) the key input is the Weak Approximation Conjecture. A section can only intersect components of fibers which occur with multiplicity 1. Conversely, since the Brauer group of $k(\mathbb{P}^1)$ is trivial the Weak Approximation Conjecture predicts that this is the only restriction on the possible intersection numbers with components of fibers (see e.g. [HT06, Conjecture 2]). Let $\Gamma_{\mathcal{X}}$ be the (finite) set of all possible combinations of intersection numbers of a section against π-vertical divisors, i.e., the set parametrizing the ways of choosing one component of multiplicity 1 in every fiber of π. We call an element λ of $\Gamma_{\mathcal{X}}$ an "intersection profile". We will let $N_\lambda \subset N_1(\mathcal{X})$ denote the affine linear subspace of classes with a given intersection profile λ. Note that once we have one section with a given intersection profile, we can obtain many more by gluing on π-vertical free curves.

We next turn to Question (2). Each component of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ naturally determines an algebraic equivalence class. Conversely, we expect that each "sufficiently positive" algebraic equivalence class is represented by at most one dominant family of sections. By combining with Theorem 9.2, we expect that each "sufficiently positive" numerical equivalence class is represented by $|\mathrm{Br}(\mathcal{X})|$ different dominant families of sections. Precisely:

Conjecture 9.3. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration. Fix an intersection profile λ and let $\mathrm{Nef}_\lambda = \mathrm{Nef}_1(\mathcal{X}) \cap N_\lambda$. There is some translate $\mathcal{T}$ of Nef_λ in N_λ such that every numerical class in $\mathcal{T}_{\mathbb{Z}}$ is represented by exactly $|\mathrm{Br}(\mathcal{X})|$ different dominant families of sections.

The following example shows that when $|\mathrm{Br}(\mathcal{X})| > 1$ we can indeed find multiple dominant components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ representing some sufficiently positive numerical classes.

Example 9.4. Consider the Artin-Mumford example of a unirational threefold X with non-trivial Brauer group from [AM72]. The construction starts with a linear subspace $\mathbb{P}^3$ in the parameter space for quadrics on $\mathbb{P}^3$. We then take a double cover $Y \to \mathbb{P}^3$ ramified over the locus of singular quadrics. Y has 10 double point singularities, and by resolving them we obtain our threefold X.

Let $p \in \mathbb{P}^3$ denote the image of one of the singular points on Y. By composing the map $X \to \mathbb{P}^3$ with projection from p, we obtain a morphism $g : X \to \mathbb{P}^2$. This is generically a conic bundle, and [AM72, Section 2] shows that the discriminant locus consists of two elliptic curves $D_1, D_2 \subset \mathbb{P}^2$. Furthermore, if we take one component T_1 of a reducible conic over D_1 and one component T_2 of a reducible conic over D_2, then $T_1 - T_2$ is a class that is numerically equivalent to 0 but not algebraically equivalent to 0 on X.

Now compose the map $X \to \mathbb{P}^3$ with projection from a general line. By resolving this map, we obtain $\pi : \mathcal{X} \to \mathbb{P}^1$. The general fiber is a double cover of $\mathbb{P}^2$ ramified along a smooth quartic, so that π gives $\mathcal{X}$ the structure of a degree 2 del Pezzo fibration. The curves T_1, T_2 above yield two sections of π that are numerically equivalent but not algebraically equivalent on $\mathcal{X}$. If we glue these two curves to any sufficiently positive π-vertical curve and smooth, we will find two different dominant components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ representing different algebraic classes but the same numerical class.

9.3. **Geometric Manin's Conjecture.** Throughout this section $\pi : \mathcal{X} \to \mathbb{P}^1$ denotes a del Pezzo fibration polarized by $-K_{\mathcal{X}/\mathbb{P}^1}$. For simplicity, we will assume that the general fibers are not isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. (In these two cases one must adjust the counting

function slightly to reflect the fact that the general fiber does not contain any curve class with anticanonical degree 1.)

Just as in [LST18], we expect the "exceptional set" of sections in Manin's Conjecture to be controlled by the generic a-invariant. The following definition describes the sections which are allowed to contribute to the counting function.

Definition 9.5. We say that a component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ is a Manin component if for the universal family $\mathcal{U} \to M$ the evaluation map $\mathcal{U} \to \mathcal{X}$ does not factor rationally through any proper subvariety Y satisfying

$$a(Y_\eta, -K_{\mathcal{X}/\mathbb{P}^1}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1}).$$

We let Manin_i denote the set of Manin components that parametrize sections C satisfying $-K_{\mathcal{X}/\mathbb{P}^1} \cdot C = i$.

Remark 9.6. The above formulation is slightly different from the formulation in [LST18] which predicts the exceptional set for rational points in the general setting. (The difference is in how we handle subvarieties with the same a-value.) However, in our situation the difference is negligible for the asymptotic formula of the counting function in Geometric Manin's Conjecture, so for the sake of simplicity we will use the definition of a Manin Component given above.

The counting function encodes the number and dimension of Manin components representing sections of height at most d.

Definition 9.7. Fix a real number $q > 1$. For any positive integer d define

$$N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d) := \sum_{i=1}^{d} \sum_{M \in \mathrm{Manin}_i} q^{\dim M}.$$

For the remainder of the section, we study the asymptotic growth rate of $N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d)$ by modifying Batyrev's heuristic for the relative setting. We will need the following constants:

Definition 9.8. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration. We equip $N_1(\mathcal{X}_\eta)$ with the Lebesgue measure μ such that the fundamental domain for the lattice $N_1(\mathcal{X}_\eta)_\mathbb{Z}$ has volume 1. Then we define the alpha-constant of $\mathcal{X}_\eta$ by

$$\alpha(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1}) := \dim N_1(\mathcal{X}_\eta) \cdot \mu(\mathrm{Nef}_1(\mathcal{X}_\eta) \cap \{\gamma \in N_1(\mathcal{X}_\eta) \,|\, -K_{\mathcal{X}_\eta} \cdot \gamma \leq 1\}).$$

Definition 9.9. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration. We define

$$\tau_{\mathcal{X}} = |\Gamma_{\mathcal{X}}| \cdot [N_1(\mathcal{X})_\mathbb{Z} \cap N_1(\mathcal{X}_\eta) : N_1(\mathcal{X}_\eta)_\mathbb{Z}]$$

where Γ_X denotes the set of allowable intersection profiles for $\mathcal{X}$ as in Section 9.2.

Putting everything together, Theorem 1.1 and Theorem 1.2 imply the following asymptotic formula.

Theorem 9.10. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample and the general fiber is a del Pezzo surface of degree ≥ 2 that is not $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Assume that Conjecture 9.3 holds for every intersection profile λ. Then*

$$N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d) \underset{d\to\infty}{\sim} \left(\tau_X \cdot \alpha(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1}) \cdot |\mathrm{Br}(\mathcal{X})| \cdot \frac{q}{q-1} \right) q^d d^{\rho(\mathcal{X}_\eta)-1}.$$

The restriction on the degree of the general fiber is not necessary: we impose this restriction in order to apply Lemma 2.10, but with more care one can prove a similar statement for degree 1 del Pezzo fibrations. (After appropriate modifications we can also find a formula for $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ fibrations.)

We expect the same asymptotic formula to hold even when the relative canonical divisor is not relatively ample.

Proof. For convenience we let D denote the discriminant of $N_1(\mathcal{X}_\eta)_\mathbb{Z}$ in $N_1(\mathcal{X})_\mathbb{Z} \cap N_1(\mathcal{X}_\eta)$.

By Theorem 1.1 each Manin component of sufficiently high degree parametrizes a dominant family of sections. Thus we can estimate $N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d)$ by counting how many dominant families of sections parametrize curve classes of height at most d. We execute this plan for each intersection profile λ separately and then add the contributions at the end.

Recall that for each intersection profile $\lambda \in \Gamma_X$ there is a unique translate N_λ of $N_1(\mathcal{X}_\eta)$ in $N_1(\mathcal{X})$ parametrizing classes with this intersection profile. For each λ we will fix an isomorphism ψ between $N_1(\mathcal{X}_\eta)$ and N_λ by mapping 0 to a some fixed $\mathbb{Z}$-class $v_\lambda \in N_\lambda$ and extending linearly. Under this isomorphism $N_1(\mathcal{X}_\eta)_\mathbb{Z}$ is identified with a full-rank sublattice of $N_{\lambda,\mathbb{Z}} := N_\lambda \cap N_1(\mathcal{X})_\mathbb{Z}$ of index D.

Let Σ_λ denote the set of dominant components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ parametrizing sections with intersection profile λ. By composing the map to $N_1(\mathcal{X})$ with ψ^{-1}, we obtain

$$\phi : \Sigma_\lambda \to N_1(\mathcal{X}_\eta).$$

Fix any coset Ξ of $N_1(\mathcal{X}_\eta)_\mathbb{Z}$ in $N_1(\mathcal{X}_\eta) \cap N_1(\mathcal{X})_\mathbb{Z}$. We claim that there is a movable section with class in $\phi^{-1}\Xi$. Indeed since a general fiber F is a del Pezzo surface $N_1(F)_\mathbb{Z}$ is generated by curves that are movable in F. By [Kho92, §3 Proposition 3] there is a set $\mathcal{T} \subset N_1(F)$ which is a translate of a full dimensional cone such that every lattice point $\mathcal{T}_\mathbb{Z}$ can be written as a non-negative sum of elements in this generating set. By Conjecture 9.3 we know that there is a movable section C with intersection profile λ. Since $i_*(N_1(F)_\mathbb{Z}) = N_1(\mathcal{X}_\eta) \cap N_1(\mathcal{X})_\mathbb{Z}$ there is an element $\beta \in \mathcal{T}_\mathbb{Z}$ such that the sum of β and the class of C lies in $\phi^{-1}\Xi$. By gluing C to a union of free curves representing β we find a movable section in $\phi^{-1}\Xi$.

We next claim that there are classes $v_1, v_2 \in \psi^{-1}(\Xi)$ such that

$$v_1 + \mathrm{Nef}_1(\mathcal{X}_\eta)_\mathbb{Z} \subset \phi(\Sigma_\lambda) \cap \Xi \subset v_2 + \mathrm{Nef}_1(\mathcal{X}_\eta)_\mathbb{Z}.$$

To see the first inclusion, let v_1 be the class of a movable section in $\phi(\Sigma_\lambda) \cap \Xi$. Lemma 2.10 implies that for a general fiber F every element in $\mathrm{Nef}_1(F)_\mathbb{Z}$ is represented by a union of free curves. Using a gluing argument, we see that every element of $v_1 + \mathrm{Nef}_1(\mathcal{X}_\eta)_\mathbb{Z}$ is represented by a dominant family of free sections.

To see the second inclusion, note that Movable Bend and Break (more precisely Theorem 7.7) shows that $\phi(\Sigma_\lambda)$ is contained in the sum of $\mathrm{Nef}_1(\mathcal{X}_\eta) \cap N_1(\mathcal{X})_\mathbb{Z}$ with a finite set of classes. This implies that $\phi(\Sigma_\lambda) \cap \Xi$ is contained in the sum of $\mathrm{Nef}_1(\mathcal{X}_\eta)_\mathbb{Z}$ with a finite set of classes. Thus $\phi(\Sigma_\lambda) \cap \Xi$ is contained in a translate of $\mathrm{Nef}_1(\mathcal{X}_\eta)_\mathbb{Z}$.

For each $\lambda \in \Gamma_X$, we can determine the contribution of λ to the counting function $N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d)$ by taking a sum over the lattice points $\phi(\Sigma_\lambda) \subset N_1(\mathcal{X}_\eta)$, which can further be subdivided as a sum over cosets of $N_1(\mathcal{X}_\eta)_\mathbb{Z}$. As we determined above, for any coset Ξ the set $\phi(\Sigma_\lambda) \cap \Xi$ is sandwiched between two translates of $\mathrm{Nef}_1(\mathcal{X}_\eta)_\mathbb{Z}$. Furthermore, Conjecture 9.3 implies that away from a set that is asymptotically negligible the fibers of ϕ have size exactly $|\mathrm{Br}(\mathcal{X})|$. In other words, each element of $\phi(\Sigma_\lambda)$ contributes $|\mathrm{Br}(\mathcal{X})|q^{\dim M}$ to the sum.

We can now estimate the contribution of sections of intersection type λ to the asymptotic formula using the usual Tauberian methods for linear functionals on the cone $\mathrm{Nef}_1(\mathcal{X}_\eta)$. Using [CLT10, Corollary A.13], each intersection profile λ gives an asymptotic contribution of

$$D \cdot \alpha(\mathcal{X}_\eta, -K_{\mathcal{X}/\mathbb{P}^1}) \cdot |\mathrm{Br}(\mathcal{X})| \cdot \frac{q}{q-1} q^d d^{\rho(\mathcal{X}_\eta)-1}.$$

We then multiply by $|\Gamma_X|$ to add up the contributions from every $\lambda \in \Gamma_X$, yielding the formula in the statement of the theorem. $\qquad\square$

9.4. Upper bounds.
Our results allow us to prove upper bounds on the counting function which approximate the expected form.

Theorem 9.11. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration such that $-K_{\mathcal{X}/\mathbb{P}^1}$ is relatively ample. Then there is a positive integer r such that*

$$N(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1}, q, d) = \mathcal{O}(q^d d^r).$$

Proof. By Corollary 1.5 we can choose r sufficiently large so that the number of components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ with $-K_{\mathcal{X}/\mathbb{P}^1}$-degree at most d is $\mathcal{O}(d^r)$. By arguing as in the proof of Theorem 9.10 but using Corollary 1.5 in place of Conjecture 9.3 one obtains the desired statement. $\qquad\square$

10. Stabilization of the Abel-Jacobi map

In this section $\pi : \mathcal{X} \to \mathbb{P}^1$ will denote a del Pezzo fibration over $\mathbb{C}$. We let $\mathrm{IJ}(\mathcal{X})$ denote the intermediate Jacobian of $\mathcal{X}$.

Let W be a smooth variety and let $Z \subset W \times X$ be a family of homologically equivalent effective 1-cycles on $\mathcal{X}$. After subtracting a constant 1-cycle we obtain an Abel-Jacobi map $\mathrm{AJ}_W : W \to \mathrm{IJ}(\mathcal{X})$. The choice of constant cycle only affects the map up to translation, and thus we frequently omit it from the discussion. More generally, if W is any projective variety and Z is a family of homologically equivalent effective 1-cycles on $\mathcal{X}$ over W, we first define the Abel-Jacobi map on the open locus where W is smooth and then identify this with a rational map $\mathrm{AJ}_W : W \dashrightarrow \mathrm{IJ}(\mathcal{X})$.

Our goal is to analyze the Abel-Jacobi map for families of sections. Recall that any component M of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ is a reduced subscheme of $\mathrm{Hilb}(\mathcal{X})$. Thus there is a natural rational map $M \dashrightarrow \mathrm{Chow}(X)$ defined over the (semi)normal locus of M. By pulling back the universal family over $\mathrm{Chow}(X)$, we obtain a family of cycles over the smooth locus of M, yielding $\mathrm{AJ}_M : M \dashrightarrow \mathrm{IJ}(\mathcal{X})$ that is well-defined on the smooth locus.

We need to analyze how the Abel-Jacobi map behaves under gluing of free curves.

Proposition 10.1. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration whose general fiber is a del Pezzo surface with degree ≥ 3. Suppose that $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ is a component parametrizing a dominant family of sections such that AJ_M is dominant. Suppose that we construct a family of sections M' by gluing a family of π-vertical free conics or cubics to M and smoothing. Then $\mathrm{AJ}_{M'}$ is also dominant.*

Furthermore, let $Z \to \mathrm{IJ}(\mathcal{X})$ denote the Stein factorization of the resolution of the Abel-Jacobi map for a projective closure of M, and let $Z' \to \mathrm{IJ}(\mathcal{X})$ denote the similar construction for M'. Then we have a factorization $Z \to Z' \to \mathrm{IJ}(\mathcal{X})$.

Proof. Choose a general fiber F of $\pi : \mathcal{X} \to \mathbb{P}^1$. Fix a component W of the parameter space of conics or cubics in F. By Lemma 2.7 W is rational; thus the Abel-Jacobi map for W is

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constant. Note that since M parametrizes free curves a general section in M will intersect a general curve parametrized by W. Let $G \subset \overline{M}_{0,0}(\mathcal{X})$ denote the locus parametrizing the stable maps obtained by gluing curves parametrized by M to curves parametrized by W. Note that G admits a dominant morphism $\psi : G \to \overline{M}$; from now on we will replace G by a component for which ψ is dominant. Since G intersects the smooth locus of $\overline{M}_{0,0}(\mathcal{X})$ the universal family induces an Abel-Jacobi map $\mathrm{AJ}_G : G \dashrightarrow \mathrm{IJ}(\mathcal{X})$. Since the Abel-Jacobi map is constant on W it is in particular constant on the fibers of ψ, so we see that AJ_G coincides with $\mathrm{AJ}_{\overline{M}} \circ \psi$ as a rational map. Since the forgetful map $G \to M$ has connected fibers the Stein factorization of the resolution of a projective closure of AJ_G is the same as Z.

Let M' denote the family of sections obtained by smoothing the curves parameterized by G and let $\overline{M}'$ denote the corresponding component of $\overline{M}_{0,0}(X)$. Let $Z' \to \mathrm{IJ}(\mathcal{X})$ denote the Stein factorization of a resolution of the Abel-Jacobi map $\mathrm{AJ}_{\overline{M}'}$. Since G is contained in the smooth locus of $\overline{M}'$, the restriction of $\mathrm{AJ}_{\overline{M}'}$ to G is just AJ_G. Using the universal property of Stein factorizations, we see that the map $Z \to \mathrm{IJ}(\mathcal{X})$ must factor through the map $Z' \to \mathrm{IJ}(\mathcal{X})$. $\qquad\square$

By combining Proposition 10.1 with Lemma 2.9 and an inductive argument, we obtain a similar statement when gluing π-vertical free curves of arbitrary degree.

Corollary 10.2. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration whose general fiber is a del Pezzo surface with degree ≥ 3. Suppose that M is a dominant family of sections such that the induced Abel-Jacobi map is also dominant. Suppose that we construct a family of sections M' by gluing a family of π-vertical free curves to M and smoothing. Then $\mathrm{AJ}_{M'}$ is also dominant.*

Furthermore, let $Z \to \mathrm{IJ}(\mathcal{X})$ denote the Stein factorization of the resolution of the Abel-Jacobi map for a projective closure of M, and let $Z' \to \mathrm{IJ}(\mathcal{X})$ denote the similar construction for M'. Then we have a factorization $Z \to Z' \to \mathrm{IJ}(\mathcal{X})$.

By a combinatorial argument we can deduce a stabilization result for the Abel-Jacobi maps of the spaces of sections.

Fact 10.3. Equip $\mathbb{Z}_{\geq 0}^m$ with the partial order $\succeq$ such that $\vec{w} \succeq \vec{w}'$ when every coordinate of $\vec{w}$ is at least as large as the corresponding coordinate of $\vec{w}'$. Suppose we have a function $h : \mathbb{Z}_{\geq 0}^m \to \mathbb{Z}$ such that h preserves the partial order, i.e. $\vec{w} \succeq \vec{w}' \implies h(\vec{w}) \geq h(\vec{w}')$.

Then for any integer c, the set $h^{-1}(\mathbb{Z}_{\geq c})$ is either empty or a finite union of translates of $\mathbb{Z}_{\geq 0}^m$. By a c-corner of h we will mean any element of the minimal finite set of vectors $\{\vec{v}_i\}$ such that

$$h^{-1}(\mathbb{Z}_{\geq c}) = \cup_i \left(\vec{v}_i + \mathbb{Z}_{\geq 0}^m \right).$$

Proof of Theorem 1.6: Let $\mathcal{F} \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ be the set of components which generically parametrize free curves of height $\geq \mathrm{MBBbound}(\mathcal{X})$. Since there are only finitely many dominant components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ not contained in $\mathcal{F}$, it suffices to restrict our attention to this set.

By applying Proposition 7.8 repeatedly, we see that every family in $\mathcal{F}$ of height $\geq \mathrm{MBBbound}(\mathcal{X}) + 3$ can be obtained by gluing free π-vertical conics and cubics to a family of free sections of height at most $\mathrm{MBBbound}(\mathcal{X}) + 2$ and smoothing. We let $\{M_i\} \subset \mathcal{F}$ denote the finite set of sections of height at most $\mathrm{MBBbound}(\mathcal{X}) + 2$. Let $\{R_k\}_{k=1}^r$ denote the families of free π-vertical conics on $\mathcal{X}$ and let $\{S_l\}_{l=1}^s$ denote the families of free π-vertical cubics

on $\mathcal{X}$. We let $\vec{a}$ denote a vector with non-negative integer components whose kth coordinate records how many times we glue conics from R_k to M_i; we let $\vec{b}$ denote the similar vector for the S_l. By combining Corollary 7.10 with [LT19a, Lemma 5.11] which allows us to reorder components, we see that if we fix M_i, $\vec{a}$, and $\vec{b}$ then by gluing the corresponding curves and smoothing we can only obtain one component of $\mathcal{F}$. In sum, we have a well-defined surjective map ϕ from the set of triples $\{(M_i, \vec{a}, \vec{b}) | \, \vec{a} \in \mathbb{Z}_{\geq 0}^r, \vec{b} \in \mathbb{Z}_{\geq 0}^s\}$ to $\mathcal{F}$.

Let $\mathcal{F}_{dom}$ denote the subset of $\mathcal{F}$ such that AJ_M is dominant. For each M_i, we define the function $h_i : \mathbb{Z}_{\geq 0}^{r+s} \to \{0,1\}$ where $h_i(\vec{a}, \vec{b}) = 1$ exactly when $\phi((M_i, \vec{a}, \vec{b})) \in \mathcal{F}_{dom}$. By Proposition 10.1 h_i preserves the partial order. By Fact 10.3, for each fixed M_i the set of pairs $(\vec{a}, \vec{b})$ such that $\phi((M_i, \vec{a}, \vec{b})) \in \mathcal{F}_{dom}$ is a finite union of translates of $\mathbb{Z}_{\geq 0}^{r+s}$. We conclude there is a finite subset $\{N_j\} \subset \mathcal{F}_{dom}$ such that every element of $\mathcal{F}_{dom}$ can be obtained by gluing vertical conics and cubics onto the N_j. Just as before, we represent this fact by a surjective map $\psi : \{(N_j, \vec{a}, \vec{b})\} \to \mathcal{F}_{dom}$.

For each fixed j, let d_j denote the degree of the Stein factorization of a resolution of AJ_{N_j}. We define a function $g_j : \mathbb{Z}_{\geq 0}^{r+s} \to \mathbb{Z}$ sending a pair $(\vec{a}, \vec{b})$ to $d_j - d(\vec{a}, \vec{b})$ where $d(\vec{a}, \vec{b})$ denotes the degree of the Stein factorization of a resolution of $\mathrm{AJ}_{\psi(N_j, \vec{a}, \vec{b})}$. Fact 10.3 guarantees that there are only finitely many c-corners of g_j as we vary $c \in \{0, \ldots, d_j\}$. Furthermore, due to the factorization structure in Proposition 10.1 we know that the Stein factorization of the resolution of any Abel-Jacobi map $\mathrm{AJ}_{\psi(N_j, \vec{a}, \vec{b})}$ will be dominated by the Stein factorization coming from one of these corners. Since there are only finitely many N_j, we deduce the statement. $\qquad\square$

Example 10.4. Consider the del Pezzo fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ consisting of a general pencil of hyperplane sections of the degree 5 index 2 Fano threefold $X_5 \subset \mathbb{P}^6$. As discussed in Example 8.4, the spaces of sections of every height are irreducible.

Let $\phi : \mathcal{X} \to X_5$ be the birational map and let E denote the exceptional divisor. There is a $\mathbb{P}^1$-fibration $\pi : E \to Z$ where Z is an elliptic curve in X_5. We have isomorphisms

$$H^3(\mathcal{X}, \mathbb{Z})^\vee \underset{g}{\to} H^1(E, \mathbb{Z})^\vee \underset{\pi^{*\vee}}{\to} H^1(Z, \mathbb{Z})^\vee$$

where g is the sequence of isomorphisms

$$H^3(\mathcal{X}, \mathbb{Z})^\vee \underset{PD}{\to} H_3(\mathcal{X}, \mathbb{Z}) \underset{\cap E}{\to} H_1(E, \mathbb{Z}) \underset{PD}{\to} H^1(E, \mathbb{Z})^\vee.$$

These operations all respect the Hodge structure, giving isomorphisms

$$H^{2,1}(\mathcal{X})^\vee \underset{g}{\to} H^{1,0}(E)^\vee \underset{\pi^{*\vee}}{\to} H^{1,0}(Z)^\vee.$$

Suppose M_d is the family of sections of π of height d. Fix a general curve C_0 parametrized by M_d. For a general point $b \in M_d$, we can identify a real 3-cycle σ on $\mathcal{X}$ such that $[C_b] - [C_0] = \partial\sigma$. The Abel-Jacobi map sends b to $[\sigma] \in H^{2,1}(X)^\vee/H_3(X, \mathbb{Z})$. Using the isomorphism g we obtain an element of $H^{1,0}(E)^\vee/H_1(E, \mathbb{Z})$ which is represented by integration against $[\sigma \cap E]$. Note that $\sigma \cap E$ is a 1-cycle whose boundary is exactly $[C_b \cap E] - [C_0 \cap E]$. We now translate once more under the isomorphism $\pi^{*\vee}$; the corresponding element of $\mathrm{Jac}(Z)$ is given by integration against the 1-cycle $\pi(\sigma \cap E)$ whose boundary is $[\phi(C_b) \cap Z] - [\phi(C_0) \cap Z]$. In sum, under the chain of isomorphisms above, a general point $b \in M_d$ is mapped under the Abel-Jacobi map to the point $\phi(C_b) \cap Z - \phi(C_0) \cap Z$ in $\mathrm{Jac}(Z)$.

This map can be realized geometrically as follows. Note that for any $d \geq 0$ a general height d section on $\mathcal{X}$ is the same thing as a curve in X_5 of degree $d+1$ which meets Z in d distinct points. This equips M_d with a rational map to $\mathrm{Sym}^d(Z)$. Since Z is an elliptic curve $\mathrm{Sym}^d(Z)$ is a projective bundle over $\mathrm{Jac}(Z)$, and the composition $M_d \dashrightarrow \mathrm{Jac}(Z)$ is the Abel-Jacobi map (up to translation).

We can now analyze the Abel-Jacobi map for low height sections:

Height 0: The Abel-Jacobi map contracts M_0 to a point. Note that since M_0 is birational to the space of lines in X_5 a projective closure is rationally connected by [FN89].

Height 1: We claim that the Abel-Jacobi map is the MRC fibration for M_1. Recall that an open subset of M_1 parametrizes conics on X_5 which meet Z once. By [Ili94] the space of conics on X_5 is isomorphic to $\mathbb{P}^{4\vee}$; the isomorphism sends a general conic $T \subset X_5 \subset G(2,5)$ to the hyperplane in $\mathbb{P}^4$ spanned by the corresponding lines.

Suppose we now fix a quintic elliptic Z in X_5. Note that there is a morphism from $G(2,5)$ to the parameter space P of Schubert varieties $\sigma_{1,1}$ in $\mathbb{P}^{4\vee}$ sending a line to the set of hyperplanes containing it. The image of Z under this map is an elliptic curve in P, and the corresponding Schubert varieties sweep out a $\mathbb{P}^2$-bundle over Z. We claim that M_1 is birational to this $\mathbb{P}^2$ bundle. Indeed, we know that M_1 has dimension 3 and is contained in this locus, thus it must contain an open subset.

The Abel-Jacobi map for M_1 sends a curve to its intersection point with Z. This agrees with the projection map from the $\mathbb{P}^2$-bundle to Z. This proves our claim.

Height 2: The Abel-Jacobi map is a MRC fibration. Indeed, $\mathrm{Sym}^2(Z) \to \mathrm{Jac}(Z)$ is a projective bundle. We claim that a general fiber of $M_2 \dashrightarrow \mathrm{Sym}^2(Z)$ is rationally connected. Fix two general points x_1, x_2 on Z. [FGP19, Theorem 4.9] shows that the general fiber of the map from the 2-pointed cubic curves on X_5 to $X_5 \times X_5$ is rationally connected. (Note that this does not yet imply our desired statement since the fiber of interest in our situation is not general.) Their argument uses the birational map $\phi : X_5 \dashrightarrow Q$ to the three-dimensional quadric Q given by projection from a line through x_1. Tracing through the proof, it suffices to show that the space of conics in the three-dimensional quadric Q which:
  − meet the twisted cubic where ϕ^{-1} is not defined,
  − meet the line in Q defined by as the fiber over x_1, and
  − meet $\phi(x_2)$
is rationally connected. This is the same as the space of planes in $\mathbb{P}^4$ meeting these loci, and it is not difficult to see directly that this surface admits a rational map to $\mathbb{P}^1$ whose fibers are rational curves.

By the argument of Theorem 1.6 we deduce that the Abel-Jacobi map has connected fibers for the family of sections of any height ≥ 1. We conjecture that in this situation the Abel-Jacobi map always agrees with the MRC fibration for a family of sections.

10.1. **Conic bundles over del Pezzo surfaces.** In this section we discuss a couple examples showing that the Abel-Jacobi map does not need to interact well with other possible stabilization results. These examples were suggested to us by Hassett.

The first example shows that there might be no components of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ whose Abel-Jacobi map $M \dashrightarrow \mathrm{IJ}(\mathcal{X})$ is dominant and birationally equivalent to the MRC fibration.

Example 10.5. Let S be a general del Pezzo surface of degree 2 equipped with a conic fibration $g : S \to \mathbb{P}^1$. Let $q : \mathcal{X} \to S$ be a conic bundle over S such that its discriminant divisor D is a very general element in $|-2K_S|$ (as constructed by [AM72]). Then $\pi = g \circ q :$ $\mathcal{X} \to \mathbb{P}^1$ is a del Pezzo fibration of degree 4. [HKT16, Theorem 1] shows that $\mathcal{X}$ does not admit an integral Chow decomposition of the diagonal.

The variety $\mathcal{X}$ also does not admit an integral homological decomposition of the diagonal. To see this, let $\mathcal{M}$ be a component of the parameter space for the set of conic bundles over S whose discriminant divisor lies in $|-2K_S|$. Since $|-2K_S|$ is basepoint free and contains a union of two smooth elliptic curves meeting transversally in 2 points, [HKT16, Proof of Theorem 1] shows that $\mathcal{M}$ parametrizes a conic bundle which has only ordinary double point singularities and whose resolution has non-trivial Brauer group. [Voi13, Corollary 4.5] shows that this resolution does not admit an integral homological decomposition of the diagonal. Finally, [Voi15, Theorem 2.1.(ii)] (combined with a Hilbert scheme argument) applies in our situation to show that a very general $\mathcal{X}$ parametrized by $\mathcal{M}$ also does not admit an integral homological decomposition of the diagonal.

We next verify that the hypotheses of [Voi13, Theorem 4.9] hold in this situation. Since $\mathcal{X}$ is rationally connected, all the higher cohomology of the structure sheaf vanishes verifying Hypothesis (i). [Voi13, Theorem 1.3] shows that $Z^4(\mathcal{X}) = 0$ verifying Hypothesis (ii). Hypothesis (iii) has two parts. First, since IJ$(\mathcal{X})$ is isomorphic to the Jacobian of a genus 2 curve, there is a one-cycle Γ of class $[\Theta]^{g-1}/(g-1)!$ on IJ$(\mathcal{X})$. Second, since $\mathcal{X} \to S$ has irreducible smooth discriminant locus, [AM72, Theorem 2] shows that the Brauer group of $\mathcal{X}$ is trivial.

Since $\mathcal{X}$ does not admit an integral homological decomposition of the diagonal, [Voi13, Theorem 4.9] implies that if $M \subset \operatorname{Sec}(\mathcal{X}/\mathbb{P}^1)$ is a component then the Abel-Jacobi map is either non-dominant or the projective closure of a general fiber is not rationally connected. In particular the MRC fibration for a resolution of a projective closure of M cannot coincide with the Abel-Jacobi map whenever the Abel-Jacobi map is dominant.

Remark 10.6. We expect that in Example 10.5 the dimension of the base of the MRC fibration is unbounded. Indeed, consider the map $h : \operatorname{Sec}(\mathcal{X}/\mathbb{P}^1) \to \operatorname{Sec}(S/\mathbb{P}^1)$ induced by pushforward. Let M be a component of $\operatorname{Sec}(\mathcal{X}/\mathbb{P}^1)$. Every component of a general fiber of $h|_M$ is rationally connected and the finite part of the Stein factorization of $h|_M$ can have very large degree. Thus it seems quite likely that the Stein factorization of $h|_M$ will often coincide with the MRC fibration for M.

The following example shows that there can be an infinite set of components of $\operatorname{Sec}(\mathcal{X}/\mathbb{P}^1)$ whose Albanese varieties have larger dimension than IJ$(\mathcal{X})$.

Example 10.7. Let S be a general del Pezzo surface of degree 2 equipped with a conic fibration $g : S \to \mathbb{P}^1$. Let $q : \mathcal{X} \to S$ be a conic bundle over S whose discriminant locus D is a general element of $|-2K_S|$ (as constructed by [AM72]). By composing q with g we obtain a degree 4 del Pezzo surface fibration $\pi : \mathcal{X} \to \mathbb{P}^1$. In this case IJ$(\mathcal{X})$ is the Jacobian of a genus 2 curve.

Note that q induces a map $h : \operatorname{Sec}(\mathcal{X}/\mathbb{P}^1) \to \operatorname{Sec}(S/\mathbb{P}^1)$. If C is a section of $g : S \to \mathbb{P}^1$ then C is not contained in the discriminant locus, so that its q-preimage is generically a rationally connected fibration over C. Thus the map $\operatorname{Sec}(\mathcal{X}/\mathbb{P}^1) \to \operatorname{Sec}(S/\mathbb{P}^1)$ is surjective.

Let C be a section of $g : S \to \mathbb{P}^1$ satisfying $-K_S \cdot C = 2$ and $C^2 = 0$. Note that the corresponding component N of $\mathrm{Sec}(S/\mathbb{P}^1)$ is an open subset of $\mathbb{P}^1$. Assuming C is general, it meets the discriminant locus transversally in 4 points. Choose any component M of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ mapping dominantly onto N and let $\overline{M}$ be a smooth projective variety birational to M for which the Abel-Jacobi map is a morphism. Denote by $h_1 : \overline{M} \to Z$ and $h_2 : Z \to \mathbb{P}^1$ the Stein factorization of the map $\overline{M} \to \mathbb{P}^1$ induced by h. We prove that Z is a smooth curve of genus at least 9.

Let $\gamma : S \to \mathbb{P}^1$ be the morphism which contracts the deformations of the conic C. The restriction of γ to the discriminant divisor D has degree 4. We claim that $\gamma|_D : D \to \mathbb{P}^1$ is simply branched. Indeed, by generality we may ensure that the branch points of $\gamma|_D$ avoid the singular fibers of $\gamma : S \to \mathbb{P}^1$. If we fix a fiber C of γ, then the locus in $|-2K_S|$ parametrizing curves D bitangent to C has codimension 2. Since there is only a 1-parameter family of deformations of C we conclude that a general D will not be bitangent to any of C.

We deduce that that $\gamma|_D$ has 12 ramification points each of which is contained in a smooth fiber of γ. This also shows that the monodromy of D over $\mathbb{P}^1$ is transitive and generated by transpositions, and is thus the entire symmetric group S_4.

Let I denote the divisor in $\mathcal{X}$ which is the preimage of D in S. Note that I is irreducible because it arises from the construction of [AM72]. Let I' denote a resolution of I and let B denote the Stein factorization of the map $I' \to \mathbb{P}^1$ induced by γ. Let G_{mon} denote the mondromy group of $\psi : B \to \mathbb{P}^1$. First of all note that ψ factors through D as a $2 : 1$ cover. Furthermore, the map $B \to D$ is étale: the discriminant divisor is smooth, so every fiber of $g|_S$ consists of two distinct lines and not a double line. Using this covering map to D, we get an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to G_{mon} \to S_4 \to 1.$$

From now on we will let $\sigma \in G_{mon}$ denote the involution coming from the subgroup $\mathbb{Z}/2\mathbb{Z}$. We claim this sequence does not split. If it did, we would obtain a map $B \to D' \to \mathbb{P}^1$ such that $D' \to \mathbb{P}^1$ has degree 2. However, since $B \to D$ is étale and $D \to \mathbb{P}^1$ is simply branched, we know that every fiber of $B \to \mathbb{P}^1$ is set-theoretically either 8 or 6 points, which proves the impossibility of factoring through such a D'.

Suppose we fix a conic C in our family on S. The g-preimage of C is a surface such that the map to C has 4 reducible fibers. Thus there are 16 possible intersection profiles for curves lying over C in a fixed numerical class on $\mathcal{X}$. We next study the action of G_{mon} on these 16 possible intersection profiles. Our goal is to prove that there is a unique orbit under the action of G_{mon} of size 16.

Suppose we fix a section in our family M. This section intersects the 4 reducible fibers along components Q_1, Q_2, Q_3, Q_4, and does not intersect the other 4 components Q'_1, Q'_2, Q'_3, Q'_4 of the reducible fibers. We can identify the action of G_{mon} on intersection profiles by analyzing how it permutes these 8 curves. Note that the quotient S_4 of G_{mon} acts on the set of pairs $P_i := \{Q_i, Q'_i\}$ via the natural symmetric group action. On the other hand σ acts trivially on the set of pairs $\{P_i\}$. Furthermore, σ lies in the center of G_{mon}. Since it has a non-trivial action on the set of intersection profiles, we deduce that the action of σ is given by $Q_i \leftrightarrow Q'_i$. In particular, σ is not contained in the stabilizer of any intersection profile.

Now the possible sizes of the G_{mon}-orbit of an intersection profile are $2, 4, 6, 8, 12, 16$. A combinatorial argument shows that if there is more than one orbit then there must be an orbit of size 2, 4, or 8, so it suffices to prove there is no orbit of these types. An orbit of size

2 is impossible: since σ is not contained in the stabilizer, the map $G_{mon} \to S_4$ would restrict to an isomorphism on the stabilizer group, contradicting the fact that there is no splitting of this map. If the size of the orbit of some intersection profile is 4, then the stabilizer G_{stab} is identified with A_4 under the quotient map $G_{mon} \to S_4$. Let τ_{ij} for $1 \le i < j \le 4$ be an element of G_{mon} mapping to the transposition (ij) in S_4. Any product of an even number of the τ_{ij} must either lie in G_{stab} or σG_{stab}. By writing down the possible actions of the τ_{ij} on the components of the singular fibers which satisfy this constraint, it is not hard to see that either some τ_{ij} or some $\sigma\tau_{ij}$ will also be contained in the stabilizer, a contradiction. Finally, if the size of the orbit of some intersection profile is 8, then the stabilizer is identified with S_3 under the quotient map $G_{mon} \to S_4$. Suppose that this S_3 acts on the first three indices. As before let τ_{ij} be elements of G_{mon} mapping to (ij). We may assume that for $j \ne 4$, τ_{ij} sends $Q_i \leftrightarrow Q_j$ and $Q'_i \leftrightarrow Q'_j$ and fixes every other component.

Suppose first that (perhaps after composing with σ) τ_{14} sends $Q_1 \leftrightarrow Q'_4$ and $Q'_1 \leftrightarrow Q_4$. Since $\tau_{14}\tau_{23}\tau_{14} = \tau_{23}$ or $\tau_{23}\sigma$, we conclude that τ_{14} fixes Q_2 and Q_3. Then using $\tau_{14}\tau_{1i}\tau_{14} = \tau_{i4}$ for $i = 2, 3$ one may compute τ_{i4} for $i = 2, 3$. But then $\tau_{14}\tau_{13}\tau_{12}\tau_{14}\tau_{24}\tau_{34}$ is an element in the kernel of $G_{mon} \to S_4$ that is not equal to σ, a contradiction.

Otherwise we may assume (perhaps after composing with σ) that τ_{14} maps $Q_1 \to Q'_4 \to Q'_1 \to Q_4 \to Q_1$. Note that $\tau_{14}\tau_{23}\tau_{14} = \tau_{23}$ or $\tau_{23}\sigma$. From this we may assume that τ_{14} fixes Q_2 and sends $Q_3 \to Q'_3$. As before we may compute τ_{i4} for $i = 2, 3$. But then $\tau_{24}\tau_{14}\tau_{24}\tau_{12}$ is an element in the kernel of $G_{mon} \to S_4$ that is not equal to σ, a contradiction.

To summarize, we have shown that the action of G_{mon} on the set of intersection profiles of sections lying above C has a unique orbit. This implies that there is a unique component M of $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ which maps to the component N and which represents a fixed numerical class on $\mathcal{X}$. The Stein factorization of a resolution of a projective closure of the map $M \to N$ has degree 16.

Consider again the conics C which are fibers of γ. If C is general, then the g-preimage Γ is birationally ruled over C with 4 singular fibers. But when C is tangent to D, the singular fibers of Γ behave differently. This will happen at the 12 simple ramification points of $\gamma|_D$. At each such point, formally locally in a neighborhood of the fiber over the tangency point Γ looks like the subset of $\mathbb{A}^2_{s,t} \times \mathbb{P}^3_{x,y,z}$ defined by the equations $s = 0$ and

$$(s - t^2)z^2 + xy.$$

Setting $s = 0$, we see that Γ has a single A_1-singularity. Its resolution has three singular fibers under the map to S which have 2, 2, and 3 components. We deduce that the fiber of h_2 corresponding to this surface has at most 12 components. Thus the contribution to the ramification divisor of h_2 at this point is at least 4. Altogether we see that the contribution of such fibers to the ramification divisor R for h_2 has degree at least 48, so that the genus of Z is at least 9.

We have a morphism $\mathrm{Alb}(\overline{M}) \to \mathrm{Jac}(Z)$ induced by h_1. Since $h_1 : \overline{M} \to Z$ is surjective, the image of $\mathrm{Alb}(\overline{M})$ generates $\mathrm{Jac}(Z)$. Thus the map from $\mathrm{Alb}(\overline{M}) \to \mathrm{Jac}(Z)$ is surjective. We conclude that the dimension of $\mathrm{Alb}(\overline{M})$ is at least 9, so it cannot coincide with $\mathrm{IJ}(\mathcal{X})$.

11. Gromov-Witten invariants

Let $\pi : \mathcal{X} \to \mathbb{P}^1$ denote a del Pezzo fibration. We will consider Gromov-Witten invariants on $\mathcal{X}$ with n point insertions, i.e. $\langle [pt]^n \rangle_{0,n}^{\mathcal{X},\beta}$ where β is a curve class satisfying $F \cdot \beta = 1$

for a general fiber F of π. In order to obtain the correct virtual dimension, we must have $-K_{\mathcal{X}/\mathbb{P}^1} \cdot \beta = 2n - 2$.

Proposition 11.1. *Let* $\pi : \mathcal{X} \to \mathbb{P}^1$ *be a del Pezzo fibration such that* $-K_{\mathcal{X}/\mathbb{P}^1}$ *is relatively ample. Suppose that*

$$n \geq \mathrm{maxdef}(\mathcal{X}) + 2 + \sup\{0, -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})\}.$$

Let $\beta \in N_1(\mathcal{X})$ *denote a curve class satisfying* $-K_{\mathcal{X}/\mathbb{P}^1} \cdot \beta = 2n - 2$ *and* $F \cdot \beta = 1$ *for a general fiber* F *of* π. *Then the GW invariant* $\langle [pt]^n \rangle_{0,n}^{\mathcal{X},\beta}$ *is enumerative: it counts the number of rational curves of class* β *through* n *general points of* $\mathcal{X}$.

This extends [Tia12, Theorem 4.1] which establishes the non-vanishing of certain GW-invariants for sections of a del Pezzo fibration. The argument is very similar to the proof of Theorem 1.2.

Proof. Fix n general points of $\mathcal{X}$. If a component $M \subset \overline{M}_{0,n}(\mathcal{X}, \beta)$ contributes to the GW invariant, then there must be a stable map parametrized by M whose image C contains all n points. By Proposition 6.4 we know that M must generically parametrize free curves. In particular, such components M have the expected dimension, are generically smooth, and do not admit a generic stabilizer. This suffices to yield the result. $\square$

By essentially the same argument, we have:

Proposition 11.2. *Let* $\pi : \mathcal{X} \to \mathbb{P}^1$ *be a del Pezzo fibration such that* $-K_{\mathcal{X}/\mathbb{P}^1}$ *is relatively ample. Suppose that*

$$n \geq \mathrm{maxdef}(\mathcal{X}) + \sup\{0, -\mathrm{neg}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{P}^1})\}.$$

Let $\beta \in N_1(\mathcal{X})$ *denote a curve class satisfying* $-K_{\mathcal{X}/\mathbb{P}^1} \cdot \beta = 2n - 1$ *and* $F \cdot \beta = 1$ *for a general fiber* F *of* π. *Fix a* π*-vertical curve* C *that is a general member of a basepoint free linear series in a general fiber. Then the GW invariant* $\langle [pt]^n, C \rangle_{0,n+1}^{\mathcal{X},\beta}$ *is enumerative: it counts the number of rational curves of class* β *that intersect* C *and contain* n *general points of* $\mathcal{X}$.

Finally, we prove an existence result which can be used to show that certain GW invariants are non-zero. Recall that a smooth rational curve C contained in a smooth threefold $\mathcal{X}$ is said to have balanced normal bundle if $N_{C/X} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ where $|b - a| \leq 1$.

Proposition 11.3. *Let* $\pi : \mathcal{X} \to \mathbb{P}^1$ *be a del Pezzo fibration such that* $-K_{\mathcal{X}/\mathbb{P}^1}$ *is relatively ample. Suppose that* $n \geq \frac{Q(\mathcal{X})+2}{2}$ *where* $Q(\mathcal{X})$ *is defined as in Theorem 7.2. There exists a free section with balanced normal bundle containing* n *general points of* $\mathcal{X}$.

If the general fiber of π *is not* $\mathbb{P}^2$ *or* $\mathbb{P}^1 \times \mathbb{P}^1$, *then for any* $a \geq \frac{Q(\mathcal{X})+8}{2}$ *there exists a free section with normal bundle* $\mathcal{O}(a) \oplus \mathcal{O}(a)$ *and a free section with normal bundle* $\mathcal{O}(a) \oplus \mathcal{O}(a+1)$. *In particular, the Gromov-Witten invariant as in Proposition 11.1 or Proposition 11.2 is positive for the corresponding curve class.*

[Tia12, Proof of Theorem 4.1] gives a similar statement with no explicit restriction on the size of n or a.

Proof. We know there exists a very free section C through n general points of $\mathcal{X}$. Let us choose such a curve of minimal height. Write $N_{C/\mathcal{X}} = \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a \leq b$. Furthermore since C contains n general points we know that if we replace C by a general deformation then $a \geq n - 1$ by Lemma 2.5.

Suppose for a contradiction that the normal bundle is not balanced, so that $a + 1 < b$. By repeating the arguments of [She12] as used in Theorem 7.2, we obtain a smooth model $\widetilde{\Sigma}$ of a surface Σ swept out by deformations of C through $a + 1$ general points. We further blow up the preimages of the general $a+1$ points so that the preimages are divisorial. There is an induced generically $\mathbb{P}^1$-fibration $\psi : \widetilde{\Sigma} \to \mathbb{P}^1$, and the preimage of each of the general points contains a component of a fiber of ψ that meets the strict transform $\widetilde{C}$ of C. Continuing the argument of Theorem 7.2, we can break the curve $\widetilde{C}$ on $\widetilde{\Sigma}$ into $\widetilde{C}_1 + F$ where F denotes a fiber of ψ. Note that $\widetilde{C}_1$ must have the same intersection against ψ-vertical curves in $\widetilde{\Sigma}$ as $\widetilde{C}$ does, and in particular, the image of $\widetilde{C}_1$ must also go through $a + 1$ general points of $\mathcal{X}$. This contradicts the minimality of C. Thus C must have balanced normal bundle, proving the first statement.

Now suppose that a general fiber of π is not $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, a general fiber F contains $-K_{\mathcal{X}/\mathbb{P}^1}$-cubics which are the strict transforms of lines on $\mathbb{P}^2$ and contains $-K_{\mathcal{X}/\mathbb{P}^1}$-quartics T such that $T_F|_T \cong \mathcal{O}(2) \oplus \mathcal{O}(2)$. Some elementary deformation arguments show:

- Suppose we take a general section C with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a)$, fix a general fiber F, and set T to be a curve in F through $C \cap F$. If T is a $-K_{\mathcal{X}/\mathbb{P}^1}$-cubic with a general tangent direction at $C \cap F$, then a general smoothing of $C \cup T$ has normal bundle $\mathcal{O}(a + 1) \oplus \mathcal{O}(a + 2)$. If T is a $-K_{\mathcal{X}/\mathbb{P}^1}$-quartic with $T_F|_T = \mathcal{O}(2) \oplus \mathcal{O}(2)$ and a general tangent direction at $C \cap F$, then a general smoothing of $C \cup T$ has normal bundle $\mathcal{O}(a + 2) \oplus \mathcal{O}(a + 2)$.
- Suppose we take a general section C with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a+1)$, fix a general fiber F, and set T to be a curve in F through $C \cap F$. If T is a $-K_{\mathcal{X}/\mathbb{P}^1}$-cubic with a general tangent direction at $C \cap F$, then a general smoothing of $C \cup T$ has normal bundle $\mathcal{O}(a + 2) \oplus \mathcal{O}(a + 2)$. If T is a $-K_{\mathcal{X}/\mathbb{P}^1}$-quartic with $T_F|_T = \mathcal{O}(2) \oplus \mathcal{O}(2)$ and a general tangent direction at $C \cap F$, then a general smoothing of $C \cup T$ has normal bundle $\mathcal{O}(a + 2) \oplus \mathcal{O}(a + 3)$.

By applying this gluing argument twice to a curve with balanced normal bundle, we obtain curves of the desired type with any height at least 6 more than the original curve. Since the balanced curve constructed in the first part has height $Q(\mathcal{X})$ or $Q(\mathcal{X}) + 1$, we obtain the desired statement. $\square$

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